UNIT-1

DC Circuit

Node-voltage analysis of resistive circuit in the context of dc voltages and currents

Objectives

- To provide a powerful but simple circuit analysis tool based on Kirchhoff's current law (KCL) only.

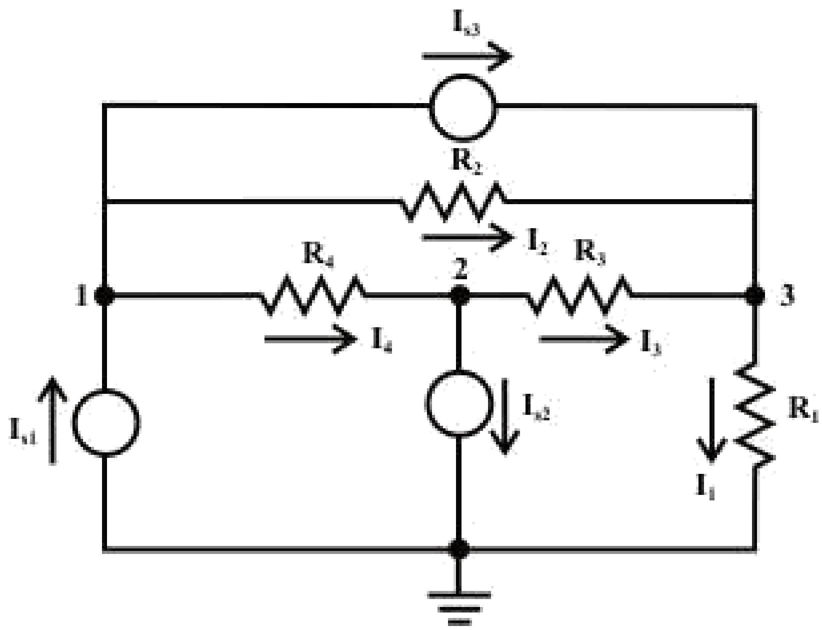
1 Node voltage analysis

In the previous lesson-4, it has been discussed in detail the analysis of a dc network by writing a set of simultaneous algebraic equations (based on KVL only) in which the variables are currents, known as mesh analysis or loop analysis. On the other hand, the node voltage analysis (Nodal analysis) is another form of circuit or network analysis technique, which will solve almost any linear circuit. In a way, this method completely analogous to mesh analysis method, writes KCL equations instead of KVL equations, and solves them simultaneously.

2.Solution of Electric Circuit Based on Node Voltage Method

In the node voltage method, we identify all the nodes on the circuit. Choosing one of them as the reference voltage (i.e., zero potential) and subsequently assign other node voltages (unknown) with respect to a reference voltage (usually ground voltage taken as zero (0) potential and denoted by ( ). If the circuit has "n" nodes there are "n-1" node voltages are unknown (since we are always free to assign one node to zero or ground potential). At each of these "n-1" nodes, we can apply KCL equation. The unknown node voltages become the independent variables of the problem and the solution of node voltages can be obtained by solving a set of simultaneous equations.

Let us consider a simple dc network as shown in Figure 5.1 to find the currents through different branches using **"Node voltage"** method.



Zero Potential
or
Ground Potential

Fig. 5.1

KCL equation at **"Node-1":**

$$I_{s1} - I_{s3} - \frac{V_1 - V_2}{R_4} - \frac{V_1 - V_3}{R_2} = 0 \; ; \rightarrow I_{s1} - I_{s3} - \left(\frac{1}{R_2} + \frac{1}{R_4}\right) V_1 - \frac{1}{R_4} V_2 - \frac{1}{R_2} V_3 = 0$$

$$I_{s1} - I_{s3} = G_{11} V_1 - G_{12} V_2 - G_{13} V_3 \qquad (5.1)$$

where G_{ii} = sum of total conductance (self conductance) connected to Node-1.

KCL equation at **"Node-2":**

$$\frac{V_1 - V_2}{R_4} - \frac{V_2 - V_3}{R_3} - I_{s2} = 0 \; ; \rightarrow -I_{s2} = -\frac{1}{R_4} V_1 + \left(\frac{1}{R_3} + \frac{1}{R_4}\right) V_2 - \frac{1}{R_3} V_3$$

$$-I_{s2} = -G_{21} V_1 + G_{22} V_2 - G_{23} V_3 \qquad (5.2)$$

KCL equation at **"Node-3":**

$$I_{s3} + \frac{V_2 - V_3}{R_3} + \frac{V_1 - V_3}{R_2} - \frac{V_3}{R_1} = 0 \; ; \rightarrow I_{s3} = -\frac{1}{R_2} V_1 - \frac{1}{R_3} V_2 + \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) V_3$$

$$I_{s3} = -G_{31} V_1 - G_{32} V_2 + G_{33} V_3 \qquad (5.3)$$

In general, for the i^{th} Node the KCL equation can be written as

$$\sum I_{ii} = -G_{i1} V_1 - G_{i2} V_2 - \cdots + G_{ii} V_i - \cdots - G_{iN} V_N$$

where,

$\sum I_{ii}$ = algebraic sum of all the current sourcesconnected to 'Node- i ',

$i = 1, 2, \quad N$. (Currents entering the node from current source is assigned as +ve sign and the current leaving the node from the current source is assigned as –ve sign).

G_{ii} = the sum of the values of conductance (reciprocal of resistance) connected to the node ' i ' .

G_{ij} = the sum of the values of conductance connected between the nodes ' i ' *and* ' j '.

Summarize the steps to analyze a circuit by node voltage method are as follows:

Step-1: Identify all nodes in the circuit. Select one node as the reference node (assign as ground potential or zero potential) and label the remaining nodes as unknown node voltages with respect to the reference node.

Step-2: Assign branch currents in each branch. (The choice of direction is arbitrary).

Step-3: Express the branch currents in terms of node assigned voltages**.**

Step-4: Write the standard form of node equations by inspecting the circuit. (No of node equations = No of nodes (N) – 1).

Step-5: Solve a set of simultaneous algebraic equation for node voltages and ultimately the branch currents.

Remarks:

- Sometimes it is convenient to select the reference node at the bottom of a circuit or the node that has the largest number of branches connected to it.
- One usually makes a choice between a mesh and a node equations based on the least number of required equations.

Example-L-5.1: Find the value of the current *I* flowing through the battery using 'Node voltage' method

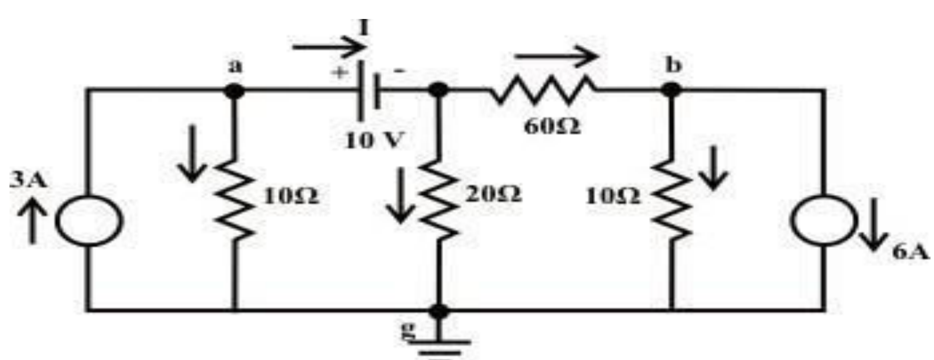


Fig. 5.2

Solution: All nodes are indicated in fig.5.2 and 'Node-g' is selected as reference voltage. If a voltage source is connected directly between the two nodes, the current flowing through the voltage source cannot be determined directly since the source voltage V_S is independent of current. Further to note that the source voltage V_S fixes the voltage between the nodes only. For the present example, the voltage of the central node is known since it is equal to $(V_a - 10)\ volt$.

KCL equation at node-a:

$$3 = \frac{V_a - 0}{10} + I \rightarrow 10I + V_a = 30 \tag{5.4}$$

KCL equation at node-b:

$$\frac{(V_a - 10) - V_b}{60} = 6 + \frac{V_b - 0}{10} \rightarrow V_a - 7V_b = 370 \tag{5.5}$$

To solve the equations (5.4)-(5.5), we need one more equation which can be obtained by applying KCL at the central node (note central node voltage is $(V_a - 10)$).

$$I = \frac{V_a - 10}{20} + \frac{(V_a - 10) - V_b}{60} \rightarrow 60I = 4V_a - V_b - 40 \rightarrow I = \frac{(4V_a - V_b - 40)}{60} \tag{5.6}$$

Substituting the current expression (5.6) in equation (5.4) we get,

$$\frac{(4V_a - V_b - 40)}{6} + V_a = 30 \rightarrow 10V_a - V_b = 220 \tag{5.7}$$

Equations (5.5) and (5.7) can be solved to find $V_b = -50.43V$ *and* $V_a = 16.99V$.
We can now refer to original circuit (fig.5.2) to find directly the voltage across every element and the current through every element. The value of current flowing through the voltage source can be computed using the equation (5.6) and it is given by $I = 1.307\ A$.
Note that the current I (+ve) is entering through the positive terminal of the voltage source and this indicates that the voltage source is absorbing the power, in other words this situation is observed when charging a battery or source.

Example-L-5.2: Find the **current** through 'ab-branch' (I_{ab}) and **voltage** **(**V_{cg} **)** across the current source using Node-voltage method.

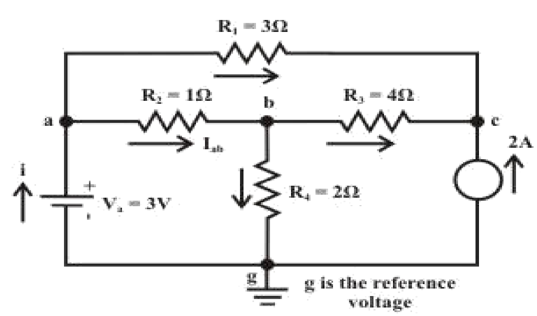


Fig. 5.3

Solution:

KCL at node-a: (note $V_a = 3V$)

$$i = \frac{V_a - V_b}{R_2} + \frac{V_a - V_c}{R_1} \rightarrow i = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_a - \frac{1}{R_2}V_b - \frac{1}{R_1}V_c \rightarrow i = 1.33V_a - V_b - \frac{1}{3}V_c \qquad (5.8)$$

KCL at node-b: (note $V_g = 0V$)

$$\frac{V_a - V_b}{R_2} = \frac{V_b - V_c}{R_3} + \frac{V_b - V_g}{R_4} \rightarrow \left(1 + \frac{1}{4} + \frac{1}{2}\right)V_b - V_a - \frac{1}{4}V_c = 0 \qquad (5.9)$$

KCL at node-c:

$$2 + \frac{V_b - V_c}{R_3} + \frac{V_a - V_c}{R_1} = 0 \qquad \rightarrow \left(\frac{1}{4} + \frac{1}{3}\right)V_c - \frac{1}{3}V_a - \frac{1}{4}V_b = 2 \qquad (5.10)$$

Using the value of $V_a = 3V$ in equations (5.8)-(5.10) we get the following equations:

$$V_b + \frac{1}{3}V_c = 3.99 - i \qquad (5.11)$$

$$1.75\,V_b - \frac{1}{4}V_c = 3 \qquad (5.12)$$

$$0.583\,V_c - \frac{1}{4}V_b = 3 \qquad (5.13)$$

Simultaneous solution of the above three equations, one can get $V_c = 6.26V$, $V_b = 2.61V$ and hence $I_{ab} = \frac{V_a - V_b}{R_2} = \frac{3 - 2.61}{1} = 0.39\,A$ (current flowing in the direction from 'a' to 'b').

Example-L-5.3 Determine the current, *i* shown in fig. 5.4 using node-voltage method --- (a) applying voltage to current source conversion (b) without any source conversion.

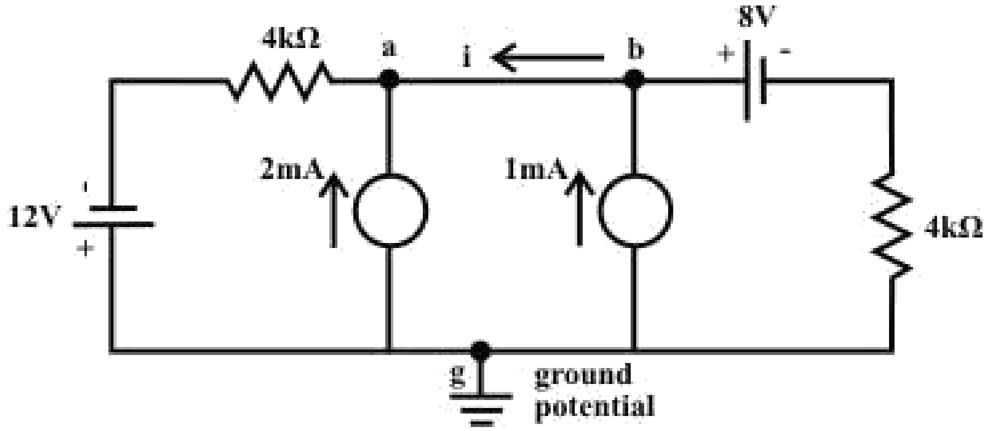


Fig. 5.4

Solution:

Part(a):

In node voltage analysis, sometimes the solution turns out to be very simple while we change all series branches containing voltage sources to their equivalent current sources. On the other hand, we observed in the loop analysis method that the conversion of current source to an equivalent voltage makes the circuit analysis very easy (see example-L4.2) and simple. For this example, both the practical voltage sources (one is left of 'node-a' and other is right of 'node-b') are converted into practical current sources. After transformation, the circuit is redrawn and shown in fig. 5.5(a).

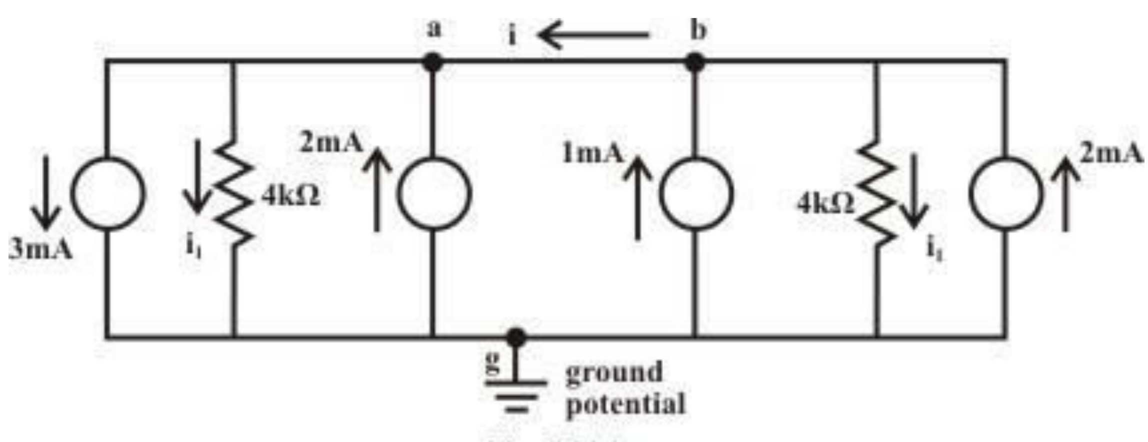


Fig. 5.5(a)

KCL at node 'b':

$i + i_1 = 2 + 1 = 3$ (5.14)

KCL at node 'a':

$i + 2 = 3 + i_1 \rightarrow i - i_1 = 1$ (5.15)

From equations (5.14)-(5.15), one can get $i = 2\ mA$ (current flows from 'b' to 'a') and $i_1 = 1\ mA$.

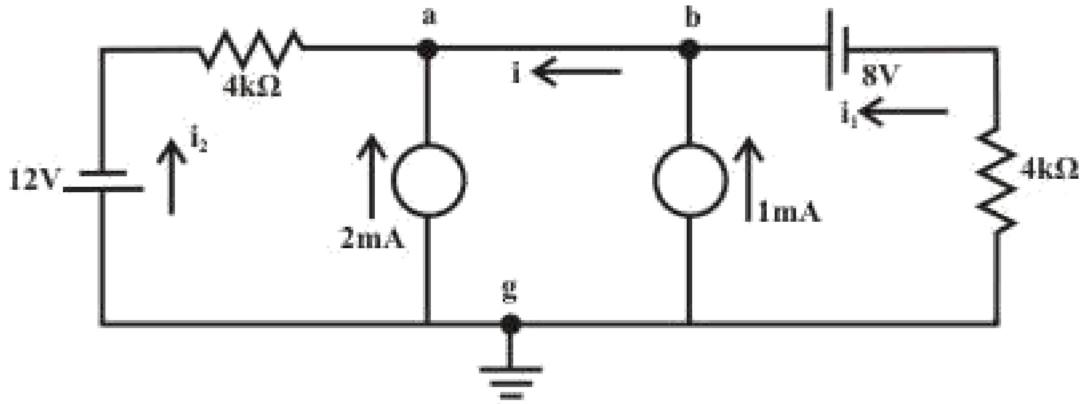


Fig. 5.5(b)

Part(b):

Let us assume i_1 is the current flowing through the $8V$ battery source from 'right to left' and i_2 is the current flowing through the $12V$ battery source from 'bottom to top'(see Fig.5.5(b)).

KCL at node 'b': It is assumed that the current flowing in 4 $k\Omega$ resistor from bottom to top terminal. This implies that the bottom terminal of 4 $k\Omega$ resistor is higher potential than the top terminal.(currents are in $m\,A$, note $V_a = V_b$)

$$i = 1 + i_1 \rightarrow i = 1 + \frac{0 - (V_a - 8)}{4} \qquad (5.16)$$

KCL at node 'a': (currents are in $m\,A$)

$$i + i_2 + 2 = 0 \rightarrow i = -i_2 - 2 \rightarrow i = -\frac{-12 - V_a}{4} - 2 \qquad (5.17)$$

From (6.16) and (5.17), we get $V_a = 4V$ and $i = 2\ mA$ (current flows from 'b' to 'a').

3 Test Your Understanding [Marks: 50]

T.5.1 Node analysis makes use of Kirchhoff's----------- law just as loop analysis makes use of Kirchhoff's --------- law. [1]

T.5.2 Describe a means of telling how many node voltage equations will be required for a given circuit. [1]

T.5.3 In nodal analysis how are voltage sources handled when (i) a voltage source in a circuit is connected between a non-reference node and the reference node (ii) a voltage source connected between two non-reference nodes in nodal analysis. [4]

T.5.4 A voltage in series with a resistance can be represented by an equivalent circuit that consists of ------------ in parallel with that -------------. [2]

T.5.5 The algebraic sum of the currents ----------- in a node must be equal to the algebraic sum of currents --------- the node. [2]

T.5.6 Apply node voltage analysis to find i_0 and the power dissipated in each resistor in the circuit of Fig.5.6. [10]

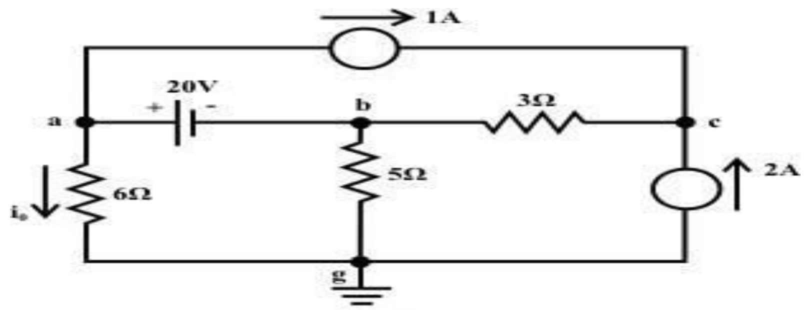


Fig. 5.6

(Ans. $i_0 = 2.73\ A$, $P_6 = 44.63W$, $P_5 = 3.8W$, $P_3 = 0.333W$ ($note \rightarrow V_c = 5.36V$, $V_b = 4.36V$)

T.5.7 For the circuit shown in fig. 5.7, find V_a using the node voltage method. Calculate power delivered or absorbed by the sources. [10]

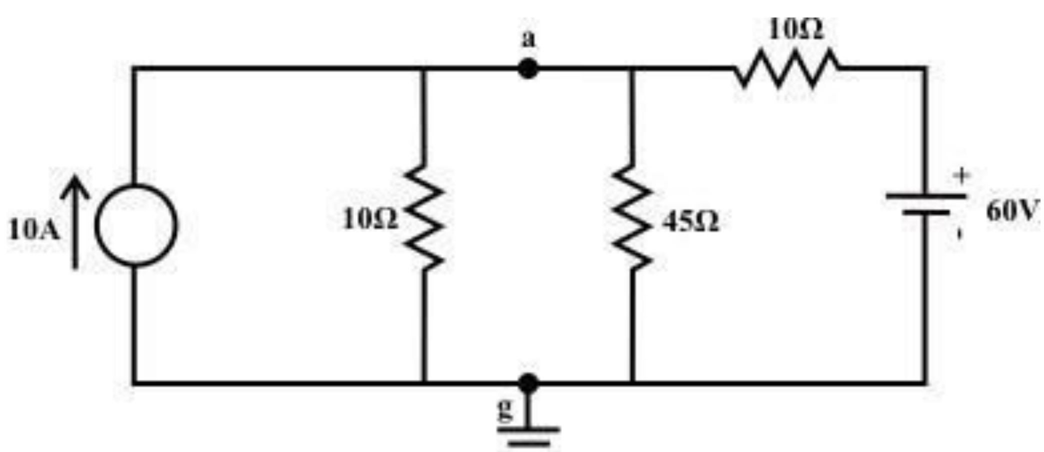


Fig. 5.7

(Answer: $V_a = 72\ V$, $P_{(\ voltage\ source)} = 72\ W$ ($absorbed$), $P_{(\ current\ source)} =$ 201.8 W ($delivered$))

T.5.8 Using nodal analysis, solve the voltage (V_x) across the $6A$ current source for the circuit of fig. 5.8. Calculate power delivered or absorbed by the sources [10]

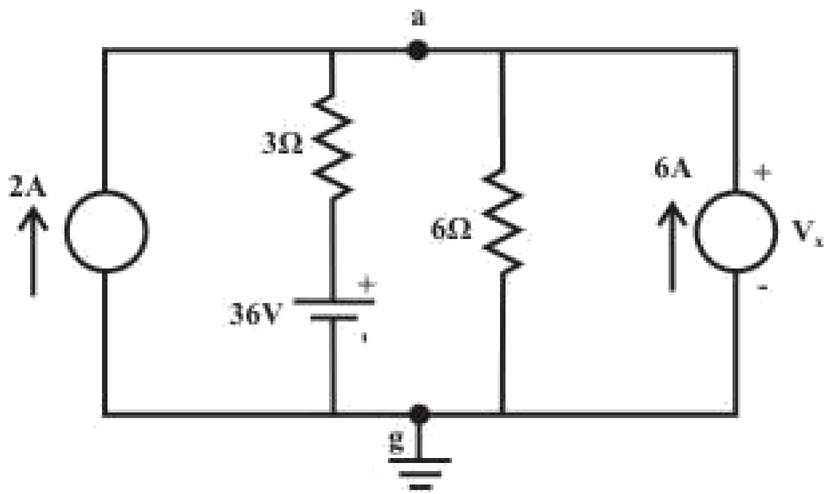


Fig. 5.8

(Answer: $V_a = 60V, P_{(12\,A\ ideal\ current\ source)} = 720W(delivered),$
$P_{(\ ideal\ voltage\ source)} = 288W(absorbed), P_{(6\,A\ ideal\ current\ source)} = 360W(delivered)$)

T.8 Determine the voltage across the 10 Ω resistor of fig. 5.9 using nodal analysis. [10]

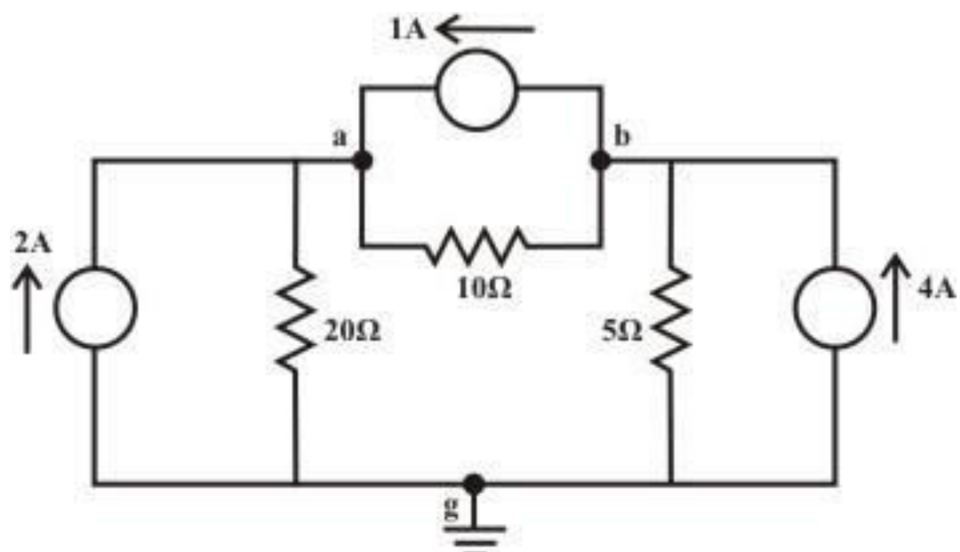


Fig. 5.9

(Answer: V_{ab} =34.29V (*a is higher potential than b*)

DC Circuit

Introduction of Electric Circuit

Objectives

- Familiarity with and understanding of the basic elements encountered in electric networks.
- To learn the fundamental differences between linear and nonlinear circuits.
- To understand the Kirchhoff's voltage and current laws and their applications to circuits.
- Meaning of circuit ground and the voltages referenced to ground.
- Understanding the basic principles of voltage dividers and current dividers.
- Potentiometer and loading effects.
- To understand the fundamental differences between ideal and practical voltage and current sources and their mathematical models to represent these source models in electric circuits.
- Distinguish between independent and dependent sources those encountered in electric circuits.
- Meaning of delivering and absorbing power by the source.

L.3.1 Introduction

The interconnection of various electric elements in a prescribed manner comprises as an electric circuit in order to perform a desired function. The electric elements include controlled and uncontrolled source of energy, resistors, capacitors, inductors, etc. Analysis of electric circuits refers to computations required to determine the unknown quantities such as voltage, current and power associated with one or more elements in the circuit. To contribute to the solution of engineering problems one must acquire the basic knowledge of electric circuit analysis and laws. Many other systems, like mechanical, hydraulic, thermal, magnetic and power system are easy to analyze and model by a circuit. To learn how to analyze the models of these systems, first one needs to learn the techniques of circuit analysis. We shall discuss briefly some of the basic circuit elements and the laws that will help us to develop the background of subject.

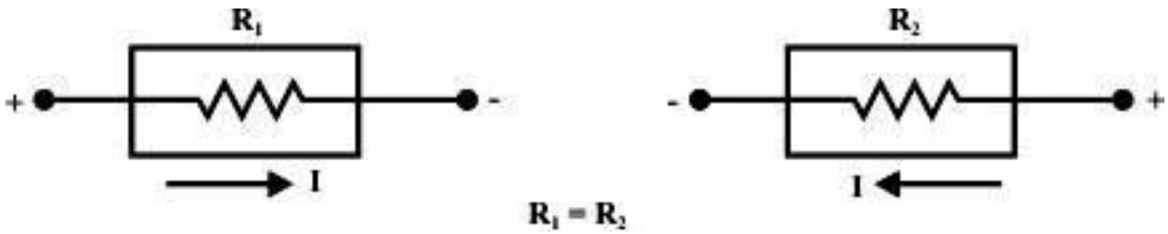
L-3.1.1 Basic Elements & Introductory Concepts

Electrical Network: A combination of various electric elements (Resistor, Inductor, Capacitor, Voltage source, Current source) connected in any manner what so ever is called an electrical network. We may classify circuit elements in two categories, passive and active elements.

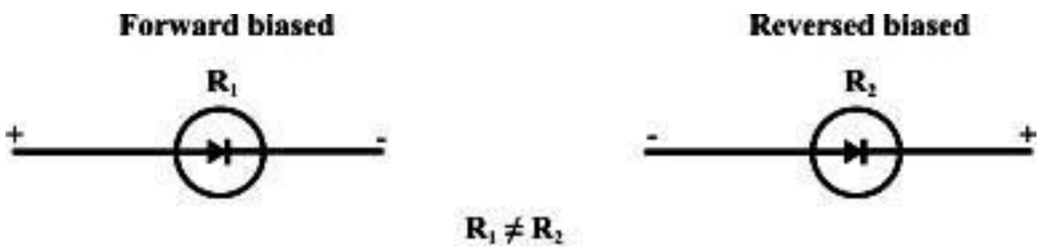
Passive Element: The element which receives energy (or absorbs energy) and then either converts it into heat (R) or stored it in an electric (C) or magnetic (L) field is called passive element.

Active Element: The elements that supply energy to the circuit is called active element. Examples of active elements include voltage and current sources, generators, and electronic devices that require power supplies. A transistor is an active circuit element, meaning that it can amplify power of a signal. On the other hand, transformer is not an active element because it does not amplify the power level and power remains same both in primary and secondary sides. Transformer is an example of passive element.

Bilateral Element: Conduction of current in both directions in an element (example: Resistance; Inductance; Capacitance) with same magnitude is termed as bilateral element.



Unilateral Element: Conduction of current in one direction is termed as unilateral (example: Diode, Transistor) element.



Meaning of Response: An application of input signal to the system will produce an output signal, the behavior of output signal with time is known as the response of the system.

L-3.2 Linear and Nonlinear Circuits

Linear Circuit: Roughly speaking, a linear circuit is one whose parameters do not change with voltage or current. More specifically, a linear system is one that satisfies (i) homogeneity property [response of $\alpha u(t)$ equals α times the response of $u(t)$, $S(\alpha u(t)) = \alpha S(u(t))$ for all α; and $u(t)$] (ii) additive property [that is the response of system due to an input $(\alpha_1 u_1(t) + \alpha_2 u_2(t))$ equals the sum of the response of input $\alpha_1 u_1(t)$ and the response of input $\alpha_2 u_2(t)$, $S(\alpha_1 u_1(t) + \alpha_2 u_2(t)) = \alpha_1 S(u_1(t)) + \alpha_2 S(u_2(t))$.] When an input $u_1(t)$ *or* $u_2(t)$ is applied to a system " S ", the corresponding output response of the system is observed as $S(u_1(t)) = y_1(t)$ *or* $S(u_2(t)) = y_2(t)$ respectively. Fig. 3.1 explains the meaning of homogeneity and additive properties of a system.

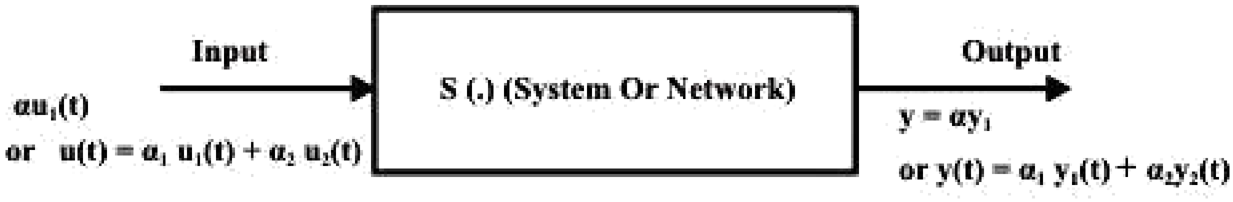


Fig. 3.1: Input output behavior of a system

Non-Linear Circuit: Roughly speaking, a non-linear system is that whose parameters change with voltage or current. More specifically, non-linear circuit does not obey the homogeneity and additive properties. Volt-ampere characteristics of linear and non-linear elements are shown in figs. 3.2 - 3.3. In fact, a circuit is linear if and only if its input and output can be related by a straight line passing through the origin as shown in fig.3.2. Otherwise, it is a nonlinear system.

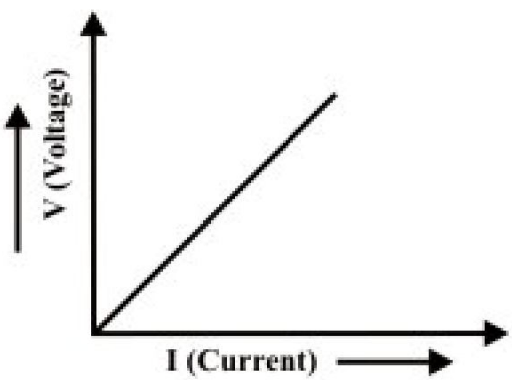


Fig. 3.2: V-I characteristics of linear element.

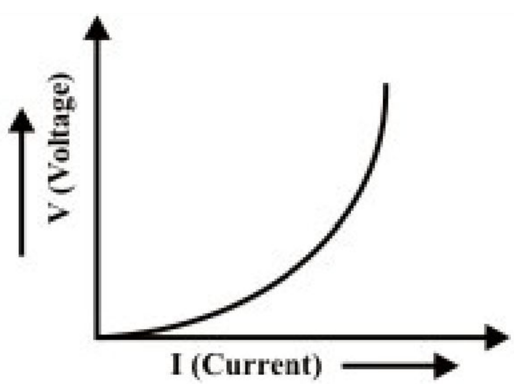


Fig. 3.3: V-I characteristics of non-linear element.

Potential Energy Difference: The voltage or potential energy difference between two points in an electric circuit is the amount of energy required to move a unit charge between the two points.

3 Kirchhoff's Laws

Kirchhoff's laws are basic analytical tools in order to obtain the solutions of currents and voltages for any electric circuit; whether it is supplied from a direct-current system or an alternating current system. But with complex circuits the equations connecting the currents and voltages may become so numerous that much tedious algebraic work is involve in their solutions.

Elements that generally encounter in an electric circuit can be interconnected in various possible ways. Before discussing the basic analytical tools that determine the currents and voltages at different parts of the circuit, some basic definition of the following terms are considered.

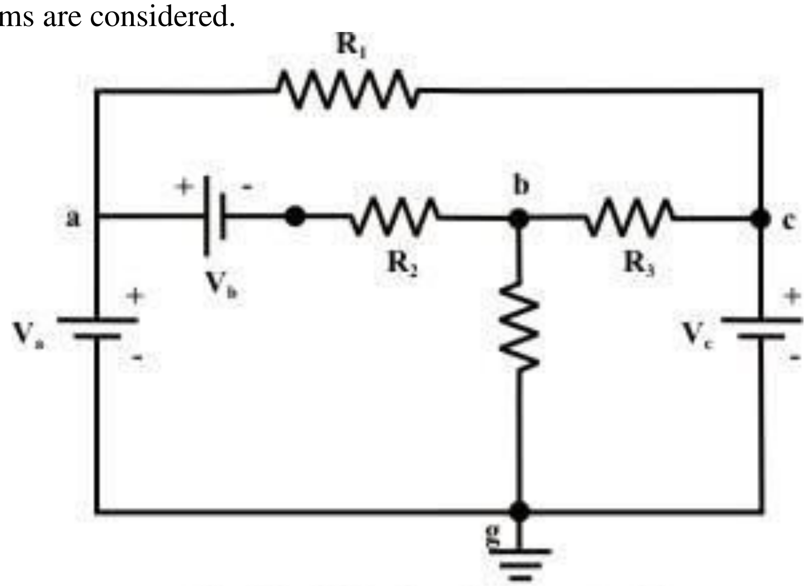
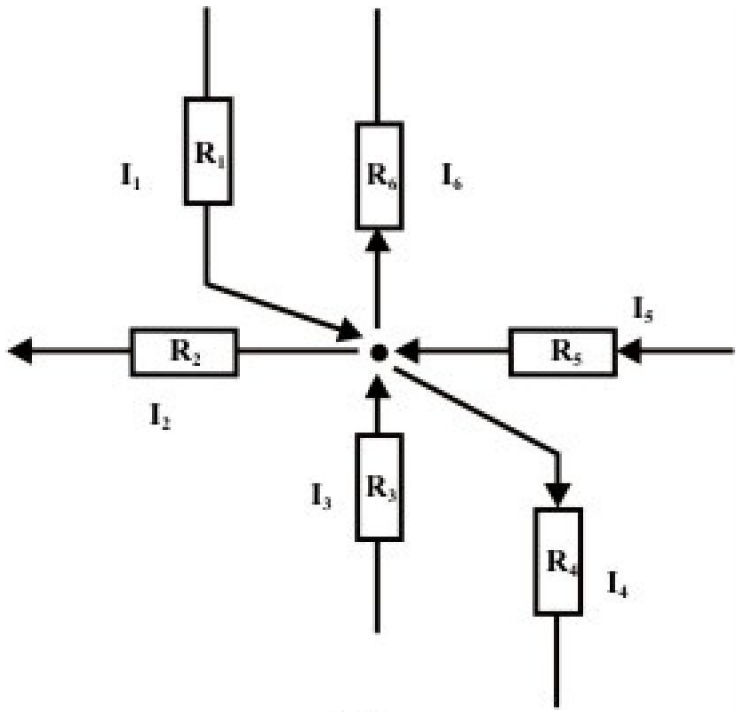


Fig. 3.4: A simple resistive network

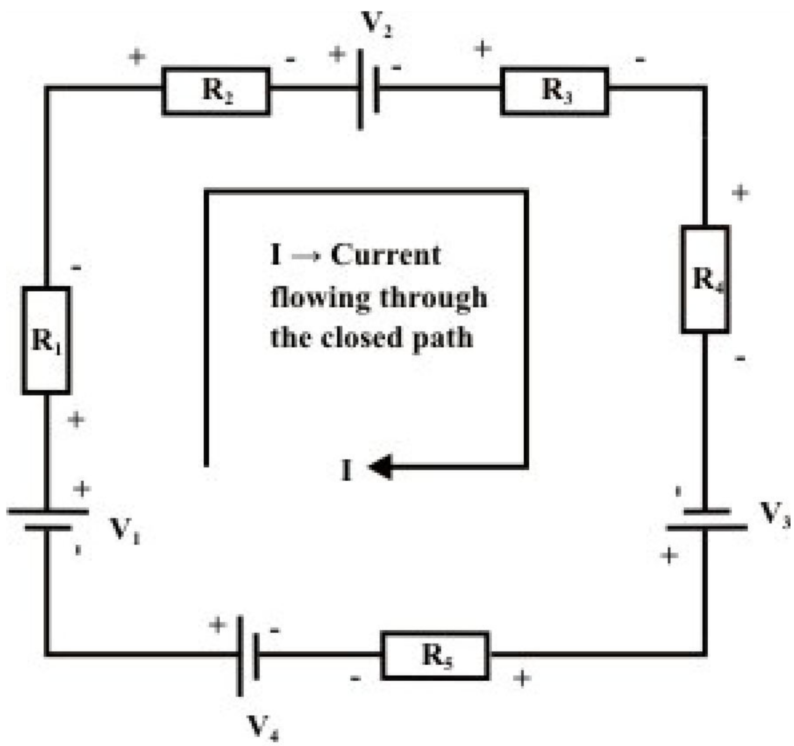
- Node- A node in an electric circuit is a point where two or more components are connected together. This point is usually marked with dark circle or dot. The circuit in fig. 3.4 has nodes a, b, c, and g. Generally, a point, or a node in an circuit specifies a certain voltage level with respect to a reference point or node.
- Branch- A branch is a conducting path between two nodes in a circuit containing the electric elements. These elements could be sources, resistances, or other elements. Fig.3.4 shows that the circuit has six branches: three resistive branches (a-c, b-c, and b-g) and three branches containing voltage and current sources (a-, a-, and c-g).
- Loop- A loop is any closed path in an electric circuit i.e., a closed path or loop in a circuit is a contiguous sequence of branches which starting and end points for tracing the path are, in effect, the same node and touches no other node more than once. Fig. 3.4 shows three loops or closed paths namely, a-b-g-a; b-c-g-b; and a-c-b-a. Further, it may be noted that the outside closed paths a-c-g-a and a-b-c-g-a are also form two loops.
- Mesh- a mesh is a special case of loop that does not have any other loops within it or in its interior. Fig. 3.4 indicates that the first three loops (a-b-g-a; b-c-g-b; and a-c-b-a) just identified are also 'meshes' but other two loops (a-c-g-a and a-b-c-g-

a) are not.

With the introduction of the Kirchhoff's laws, a various types of electric circuits can be analyzed.



(a)



(b)

Fig. 3.5: Illustrates the Kirchhoff's laws

Kirchhoff's Current Law (KCL): KCL states that at any node (junction) in a circuit the algebraic sum of currents entering and leaving a node at any instant of time must be equal to zero. Here currents entering(+ve sign) and currents leaving (- ve sign) the node must be assigned opposite algebraic signs (see fig. 3.5 (a), $I_1 - I_2 + I_3 - I_4 + I_5 - I_6 = 0$).

Kirchhoff's Voltage Law (KVL): It states that in a closed circuit, the algebraic sum of all source voltages must be equal to the algebraic sum of all the voltage drops. Voltage drop is encountered when current flows in an element (resistance or load) from the higher-potential terminal toward the lower potential terminal. Voltage rise is encountered when current flows in an element (voltage source) from lower potential terminal (or negative terminal of voltage source) toward the higher potential terminal (or positive terminal of voltage source). Kirchhoff's voltage law is explained with the help of fig. 3.5(b).

KVL equation for the circuit shown in fig. 3.5(b) is expressed as (we walk in clockwise direction starting from the voltage source V_1 and return to the same point)

$$V_1 - IR_1 - IR_2 - V_2 - IR_3 - IR_4 + V_3 - IR_5 - V_4 = 0$$

$$V_1 - V_2 + V_3 - V_4 = IR_1 + IR_2 + IR_3 + IR_4 + IR_5$$

Example: L-3.1 For the circuit shown in fig. 3.6, calculate the potential of points A, B , C , *and* E with respect to point D . Find also the value of voltage source V_1 .

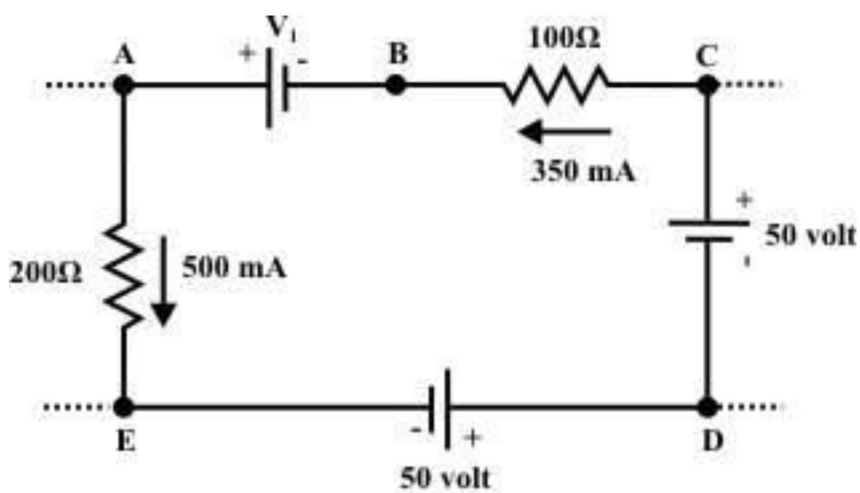


Fig. 3.6: A part of dc resistive circuit is presented

Solution Let us assume we move in clockwise direction around the close path D-E-A-B-C-D and stated the following points.

- 50 *volt* source is connected between the terminals D & E and this indicates that the point E is lower potential than D. So, V_{ED} (i.e., it means potential of E with respect to D) is -50 *volt* and similarly $V_{CD} = 50\ volt$ or $V_{DC} = -50\ volt$.
- 500 *mA* current is flowing through 200 Ω resistor from A *to* E and this implies that point A is higher potential than E . If we move from lower potential (E) to higher potential (A), this shows there is a rise in potential. Naturally, $V_{AE} = 500 \cdot 10^{-3} \cdot 200 = 100\ volt$ and $V_{AD} = -50 + 100 = 50 volt$.

Similarly, $V_{CB} = 350 \cdot 10^{-3} \cdot 100 = 35 volt$

- V_1 voltage source is connected between A & B and this indicates that the terminal B is lower potential than A i.e., $V_{AB} = V_1\ volt$ or $V_{BA} = -V_1\ volt.$. One can write the voltage of point B with respect to D is $V_{BD} = 50 - V_1\ volt.$
- One can write KVL law around the closed-loop D-E-A-B-C-D as $V_{ED} + V_{AE} + V_{BA} + V_{CB} + V_{DC} = 0$

$-50 + 100 - V_1 + 35 - 50 = 0 \Rightarrow V_1 = 35\ volt.$

Now we have $V_{ED} = -50\ volt$, $V_{AD} = -50 + 100 = 50\ volt$, $V_{BD} = 50 - 35 = 15\ volt$, $V_{CD} = 15 + 35 = 50\ volt.$

L-3.4 Meaning of Circuit Ground and the Voltages referenced to Ground

In electric or electronic circuits, usually maintain a reference voltage that is named "ground voltage" to which all voltages are referred. This reference voltage is thus at ground potential or zero potential and each other terminal voltage is measured with respect to ground potential, some terminals in the circuit will have voltages above it (positive) and some terminals in the circuit will have voltages below it (negative) or in other words, some potential above or below ground potential or zero potential.
Consider the circuit as shown in fig. 3.7 and the common point of connection of elements V_1 & V_3 is selected as ground (or reference) node. When the voltages at different nodes are referred to this ground (or reference) point, we denote them with double subscripted voltages V_{ED} , V_{AD} , V_{BD} , *and* V_{CD} . Since the point D is selected as ground potential or zero potential, we can write V_{ED} as V_E , V_{AD} as V_A and so on.

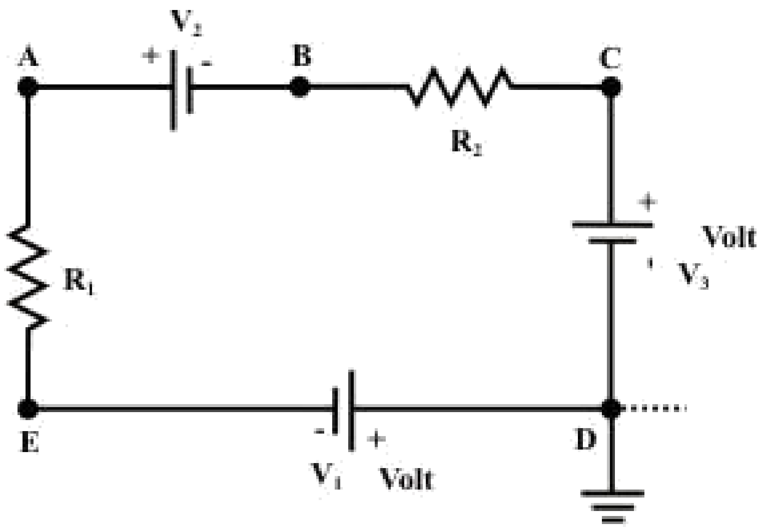


Fig. 3.7: A Simple dc resistive circuit

In many cases, such as in electronic circuits, the chassis is shorted to the earth itself for safety reasons.

L-3.5 Understanding the Basic Principles of Voltage Dividers and Current dividers

L-3.5.1 Voltage Divider

Very often, it is useful to think of a series circuit as a voltage divider. The basic idea behind the voltage divider is to assign a portion of the total voltage to each resistor. In Figure 3.8 (a), suppose that the source voltage is E. By the circuit configuration shown one can divide off any voltage desired (V_{out}), less than the supply voltage E, by adjusting R_1, R_2 *and* R_3.

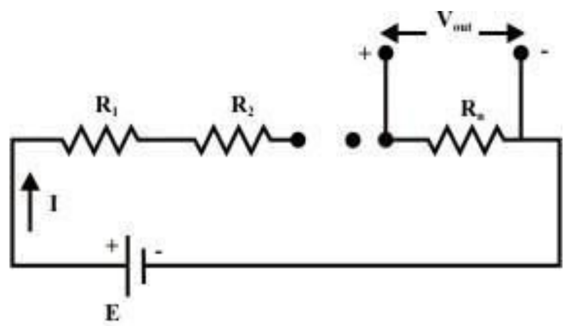


Fig. 3.8(a): Voltage Divider

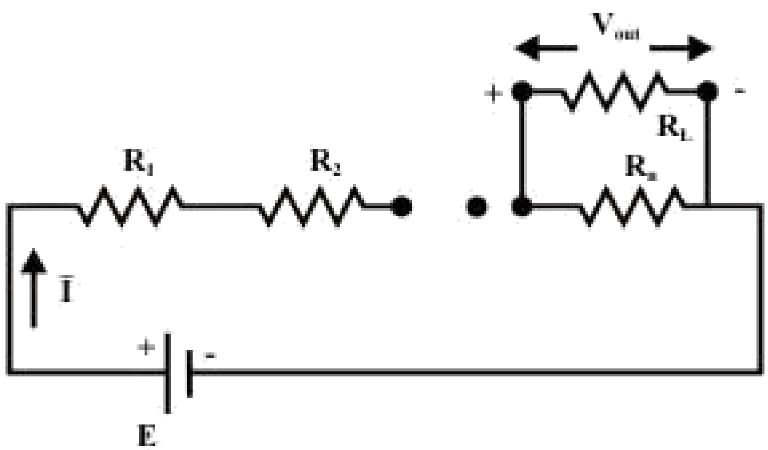


Fig. 3.8(b): Voltage Divider With Load

From figure 3.8(a) the output of the voltage divider V_{out} is computed by the relation

$$V_{out} = I\,R_n = \frac{E}{R_1 + R_2 + ... + R_n} R_n \tag{3.1}$$

Equation (3.1) indicates that the voltage across any resistor R_i (R_i $i = 1,2,.....n$) in a series circuit is equal to the applied voltage (E) across the circuit multiplied by a factor $\frac{R_i}{\sum_{j=1}^{n} R_j}$. It should be noted that this expression is only valid if the same current I flows through all the resistors. If a load resistor R_L is connected to the voltage divider (see figure 3.8(b)), one can easily modify the expression (3.1) by simply combining R_L & R_n in parallel to find a new $\overline{R_n}$ and replacing R_n by $\overline{R_n}$ in equation (3.1).

Example: L-3.2 For the circuit shown in Figure 3.9,

(i) Calculate V_{out}, ignoring the internal resistance R_s of the source E. Use voltage division.

(ii) Recalculate V_{out} taking into account the internal resistance R_s of the source. What percent error was introduced by ignoring R_s in part (i)?

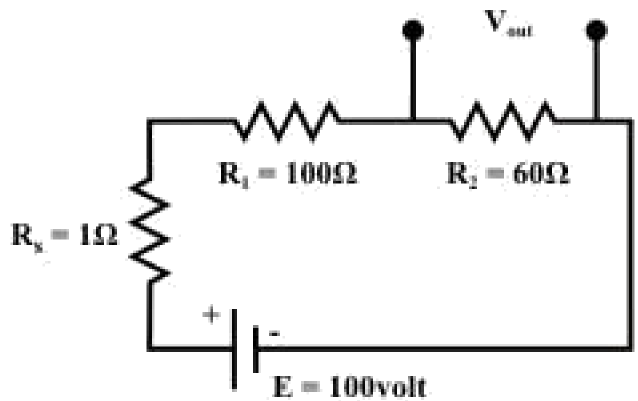


Fig. 3.9

Solution: Part (i): From equation (3.1) the output voltage V_{out} across the resistor $R_2 = \frac{E}{R_1 + R_2} R_2 = \frac{100}{100+60} \cdot 60 = 37.9\ volt$ (when the internal resistance R_s of the source is considered zero.) Similarly, $V_{out} = 37.27\ volt$ when R_s is taken into account for calculation. Percentage error is computed as = $\frac{37.9 - 37.27}{37.27} \cdot 100 = 1.69\%$

L-3.5.2 Current divider

Another frequently encountered in electric circuit is the current divider. Figure 3.10 shows that the current divider divides the source current I_s between the two resistors.

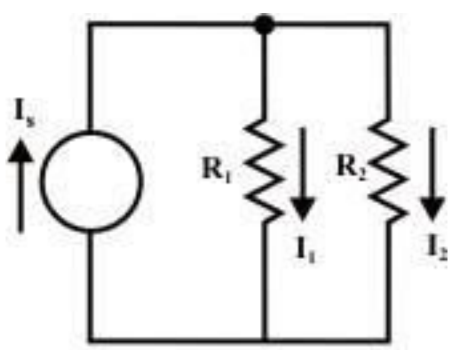


Fig. 3.10: Current Divider

The parallel combination of two resistors is sometimes termed as current divider, because the supply current is distributed between the two branches of the circuit. For the circuit, assume that the voltage across the branch is V and the current expression in R_1 resistor can be written as

$\frac{I_1}{I_s} = \frac{\frac{1}{R_1}}{\frac{1}{R_1}+\frac{1}{R_2}} = \frac{R_2}{R_1+R_2}$ or $I_1 = \frac{R_2}{R_1+R_2} \cdot I_s$. Similarly, the current flowing through the R_2 can be obtained as $I_2 = \frac{R_1}{R_1+R_2} \cdot I_s$. It can be noted that the expression for I_1 has R_2 on its top line, that for I_2 has R_1 on its top line.

Example: L-3.3 Determine I_1, I_2, I_3 & I_5 using only current divider formula when $I_4 = 4$A.

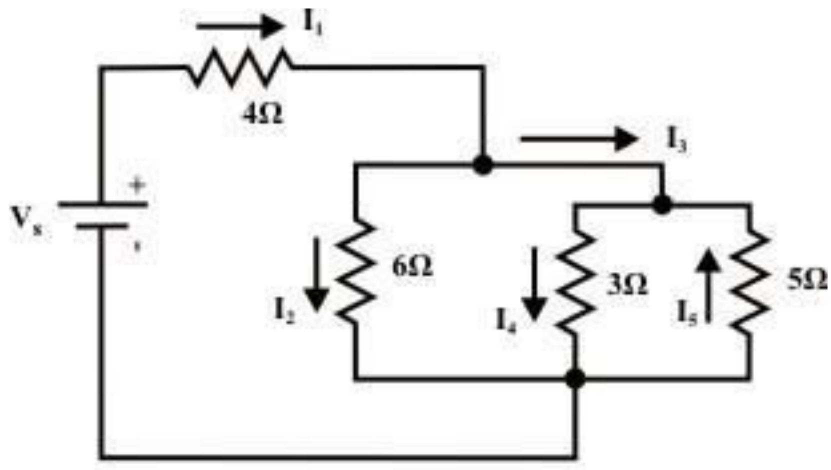


Fig. 3.11

Solution- Using the current division formula we can write $I_4 = \frac{5}{5+3} I_3 = \frac{5}{8} I_3 \rightarrow I_3 = \frac{4 \cdot 8}{5} = 6.4\,A$. Similarly, $-I_5 = \frac{3}{8} \cdot I_3 \rightarrow I_5 = \frac{3}{8} \cdot 6.4 = 2.4\,A$.

Furthermore, we can write $I_3 = \frac{6}{6+(3\|5)} I_1 = \frac{6}{6+1.879} I_1 \rightarrow I_1 = \frac{7.879}{6} \cdot 6.4 = 8.404\,A$ and

$I_2 = \frac{1.879}{6+1.879} \cdot I_1 = 2.004\,A$.

6 Potentiometer and its function

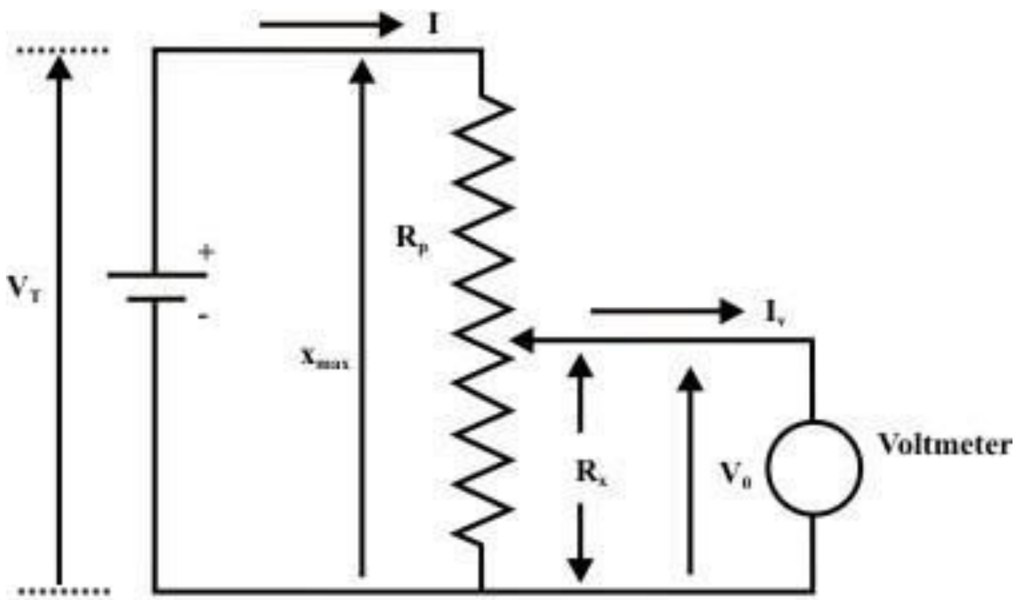


Fig. 3.12: A voltmeter is connected across the output terminals of potentiometer

The potentiometer has a resistance R_p and its wiper can move from top position $x = x_{max}$ to bottom position $x = 0$. The resistance R_x corresponds to the position x of the wiper such that

$$\frac{R_x}{x} = \frac{R_p}{x_{max}} \Rightarrow R_x = \frac{R_p}{x_{max}} x$$

(assumed that the per unit length resistance of the potentiometer is same through out its length). Figure 3.12 represents a potentiometer whose output is connected to a voltmeter. In true sense, the measurement of the output voltage V_o with a voltmeter is affected by the voltmeter resistance R_V and the relationship between V_O and x (x = wiper distance from the bottom position) can easily be established. We know that the voltmeter resistance is very high in $M\,\Omega$ range and practically negligible current is flowing through the voltmeter. Under this condition, one can write the expression for voltage between the wiper and the bottom end terminal of the potentiometer as

$$V_{out} (= I R_x) = \frac{V_T (= I R_p)}{x_{max}} \cdot x \Rightarrow V_{out} = V_T \cdot \frac{x}{x_{max}} = V_{out} = V_T \cdot \frac{R_x}{R_p}$$

It may be noted that depending on the position of movable tap terminal the output voltage (V_{out}) can be controlled. By adjusting the wiper toward the top terminal, we can increase V_{out} . The opposite effect can be observed while the movable tap moves toward the bottom terminal. A simple application of potentiometer in real practice is the volume control of a radio receiver by adjusting the applied voltage to the input of audio amplifier of a radio set. This audio amplifier boosts this voltage by a certain fixed factor and this voltage is capable of driving the loudspeaker.

Example- L-3.4 A 500 $- k\ \Omega$ potentiometer has 110 V applied across it. Adjust the position of R_{bot} such that 47.5 V appears between the movable tap and the bottom end terminal (refer fig.3.12).

Solution- Since the output voltage (V_{bot}) is not connected to any load, in turn, we can write the following expression

$$V_{out} = V_T \cdot \frac{x}{x_{max}} \quad \frac{V_{bot}}{V_T} = \frac{R_{bot}}{R_T} \to R_{bot} = \frac{V_{bot}}{V_T} \cdot R_T = \frac{47.5}{110} \cdot 500000 = 216 - k\Omega.$$

L-3.7 Practical Voltage and Current Sources

L-3.7.1 Ideal and Practical Voltage Sources

- An ideal voltage source, which is represented by a model in fig.3.13, is a device that produces a constant voltage across its terminals ($V = E$) no matter what current is drawn from it (terminal voltage is independent of load (resistance) connected across the terminals)

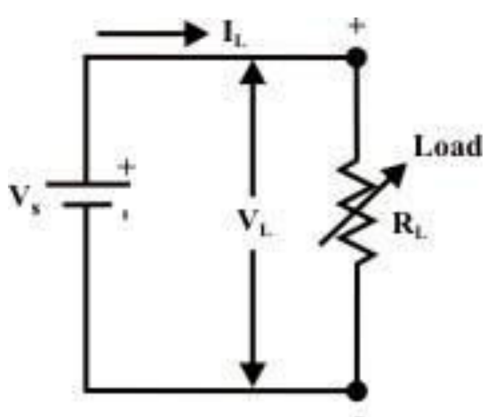


Fig. 3.13: Ideal dc voltage source

For the circuit shown in fig.3.13, the upper terminal of load is marked plus (+) and its lower terminal is marked minus (-). This indicates that electrical potential of upper terminal is V_L volts higher than that of lower terminal. The current flowing through the load R_L is given by the expression $V_s = V_L = I_L R_L$ and we can represent the terminal $V - I$ characteristic of an ideal dc voltage as a straight line parallel to the x-axis. This means that the terminal voltage V_L remains constant and equal to the source voltage V_s irrespective of load current is small or large. The $V - I$ characteristic of ideal voltage source is presented in Figure 3.14.

- However, real or practical dc voltage sources do not exhibit such characteristics (see fig. 3.14) in practice. We observed that as the load resistance R_L connected across the source is decreased, the corresponding load current I_L increases while the terminal voltage across the source decreases (see eq.3.1). We can realize such voltage drop across the terminals with increase in load current provided a resistance element (R_s) present inside the voltage source. Fig. 3.15 shows the model of practical or real voltage source of value V_s .

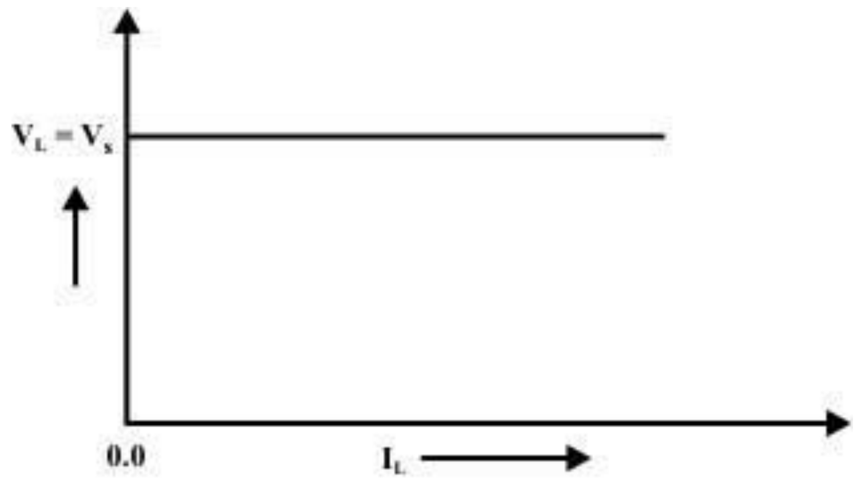


Fig. 3.14: V-I characteristics of ideal voltage source

The terminal $V - I$ characteristics of the practical voltage source can be described by an equation

$$V_L = V_s - I_L R_s \qquad (3.1)$$

and this equation is represented graphically as shown in fig.3.16. In practice, when a load resistance R_L more than 100 times larger than the source resistance R_s , the source can be considered approximately ideal voltage source. In other words, the internal resistance of the source can be omitted. This statement can be verified using the relation $R_L = 100R_s$ in equation (3.1). The practical voltage source is characterized by two parameters namely known as (i) Open circuit voltage (V_s) (ii) Internal resistance in the source's circuit model. In many practical situations, it is quite important to determine the source parameters experimentally. We shall discuss briefly a method in order to obtain source parameters.

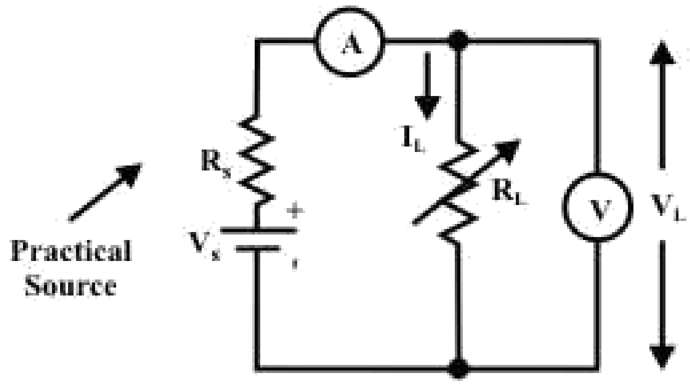


Fig. 3.15: Practical dc voltage source model

Method-: Connect a variable load resistance across the source terminals (see fig. 3.15). A voltmeter is connected across the load and an ammeter is connected in series with the load resistance. Voltmeter and Ammeter readings for several choices of load resistances are presented on the graph paper (see fig. 3.16). The slope of the line is $-R_s$, while the curve intercepts with voltage axis (at $I_L = 0$) is the value of V_s .

The $V - I$ characteristic of the source is also called the source's **<u>"regulation curve" or "load line"</u>**. The open-circuit voltage is also called the "no-load" voltage, V_{oc} . The maximum allowable load current (rated current) is known as full-load current I_{Fl} and the corresponding source or load terminal voltage is known as "full-load" voltage V_{FL} . We know that the source terminal voltage varies as the load is varied and this is due to internal voltage drop inside the source. The percentage change in source terminal voltage from no- load to full-load current is termed the "voltage regulation" of the source. It is defined as

$$\textit{Voltage regulation}\ (\%) = \frac{V_{oc} - V_{FL}}{V_{FL}} \cdot 100$$

For ideal voltage source, there should be no change in terminal voltage from no-load to full-load and this corresponds to "zero voltage regulation". For best possible performance, the voltage source should have the lowest possible regulation and this indicates a smallest possible internal voltage drop and the smallest possible internal resistance.

Example:-L-3.5 A practical voltage source whose short-circuit current is 1.0A and open-circuit voltage is 24 Volts. What is the voltage across, and the value of power dissipated in the load resistance when this source is delivering current 0.25A?

Solution: From fig. 3.10, $I_{sc} = \frac{V_s}{R_s} = 1.0\,A$ (short-circuit test) $V_{oc} = V_s = 24\ volts$ (open-circuit test). Therefore, the value of internal source resistance is obtained as $R_s = \frac{V_s}{I_{sc}} = 24\ \Omega$. Let us assume that the source is delivering current $I_L = 0.25$A when the load resistance R_L is connected across the source terminals. Mathematically, we can write the following expression to obtain the load resistance R_L .

$$\frac{24}{24 + R_L} = 0.25 \rightarrow R_L = 72\ \Omega .$$

Now, the voltage across the load $R_L = I_L R_L = 0.25 \cdot 72 = 18\ volts.$, and the power consumed by the load is given by $P_L = I_L^{\,2} R_L = 0.0625 \cdot 72 = 4.5\ watts.$

Example-L-3.6 (Refer fig. 3.15) A certain voltage source has a terminal voltage of 50 V when I= 400 mA; when I rises to its full-load current value 800 mA the output voltage is recorded as 40 V. Calculate (i) Internal resistance of the voltage source (R_s). (ii) No-load voltage (open circuit voltage V_s). (iii) The voltage Regulation.

Solution- From equation (3.1) ($V_L = V_s - I_L R_s$) one can write the following expressions under different loading conditions.

$50 = V_s - 0.4\, R_s$ & $40 = V_s - 0.8\, R_s \rightarrow$ solving these equations we get, $V_s = 60V$ & $R_s = 25\Omega$.

$$Voltage\ regulation\ (\%) = \frac{V_{oc} - V_{FL}}{V_{FL}} \cdot 100 = \frac{60 - 40}{60} \cdot 100 = 33.33\%$$

L-3.7.2 Ideal and Practical Current Sources

- Another two-terminal element of common use in circuit modeling is `current source` as depicted in fig.3.17. An ideal current source, which is represented by a model in fig. 3.17(a), is a device that delivers a constant current to any load resistance connected across it, no matter what the terminal voltage is developed across the load (i.e., independent of the voltage across its terminals across the terminals).

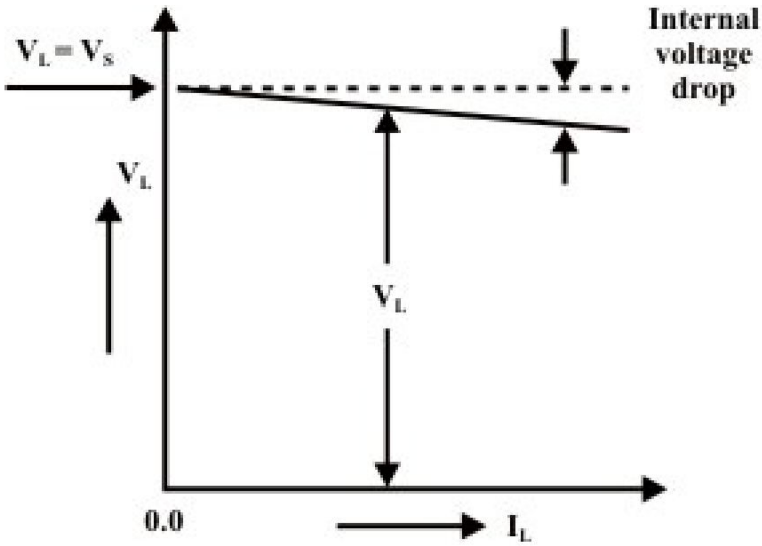


Fig. 3.16: V-I characteristics of practical voltage source

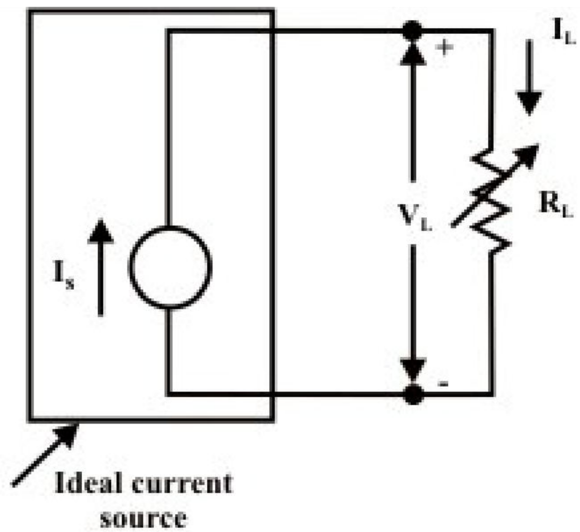


Fig. 3.17(a): Ideal current source with variable load

It can be noted from model of the current source that the current flowing from the source to the load is always constant for any load resistance (see fig. 3.19(a)) i.e. whether R_L is small (V_L is small) or R_L is large (V_L is large). The vertical dashed line in fig. 3.18 represents the $V - I$ characteristic of ideal current source. In practice, when a load R_L is connected across a practical current source, one can observe that the current flowing in load resistance is reduced as the voltage across the current source's terminal is increased, by increasing the load resistance R_L . Since the distribution of source current in two parallel paths entirely depends on the value of external resistance that connected across the source (current source) terminals. This fact can be realized by introducing a parallel resistance R_s in parallel with the practical current source I_s , as shown in fig. 3.17(b). The dark lines in fig. 3.18 show

the $V - I$ characteristic (load-line) of practical current source. The slope of the curve represents the internal resistance of the source. One can apply KCL at the top terminal of the current source in fig. 3.17(b) to obtain the following expression.

$$I_L = I_s - \frac{V_L}{R_s} \text{ Or } V_L = I_s R_s - R_s I_L = V_{oc} - R_s I_L \tag{3.2}$$

The open circuit voltage and the short-circuit current of the practical current source are given by $V_{oc} = I_s R_s$ and $I_{short} = I_s$ respectively. It can be noted from the fig.3.18 that source 1 has a larger internal resistance than source 2 and the slope the curve indicates the internal resistance R_s of the current source. Thus, source 1 is closer to the ideal source. More specifically, if the source internal resistance $R_s \geq 100$ R_L then source acts nearly as an ideal current source.

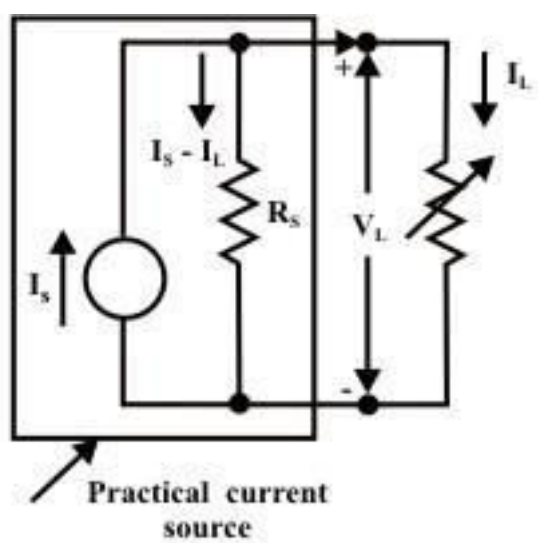


Fig. 3.17(b): Practical current source with variable load

L-3.7.3 Conversion of a Practical Voltage Source to a Practical Current source and vise-versa

- Voltage Source to Current Source

For the practical voltage source in fig. 3.19(a), the load current is calculated as

$$I_L = \frac{V_s}{R_s + R_L} \tag{3.3}$$

Note that the maximum current delivered by the source when $R_L = 0$ (under short-circuit condition) is given by $I_{max} = I_s = \frac{V_s}{R_s}$. From eq.(3.3) one can rewrite the expression for load current as

$$I_L = \frac{I_s \cdot R_s}{R_s + R_L} \tag{3.4}$$

A simple current divider circuit having two parallel branches as shown in fig.3.19 (b) can realize by the equation (3.4).

Note: A practical voltage source with a voltage V_s and an internal source resistance R_s can be replaced by an equivalent practical current source with a current $I_s = \frac{V_s}{R_s}$ and a source internal resistance R_s (see fig. 3.19(b)).

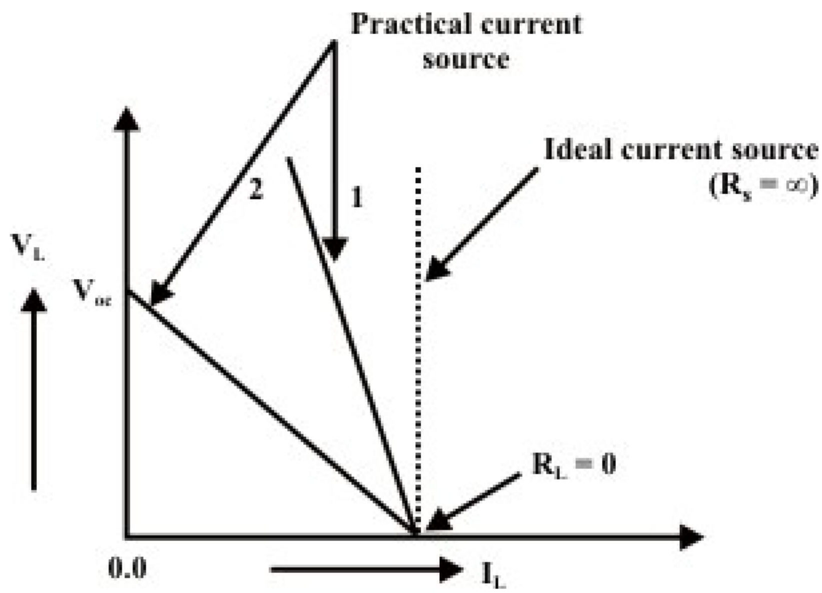


Fig. 3.18: V-I characteristic of practical current source

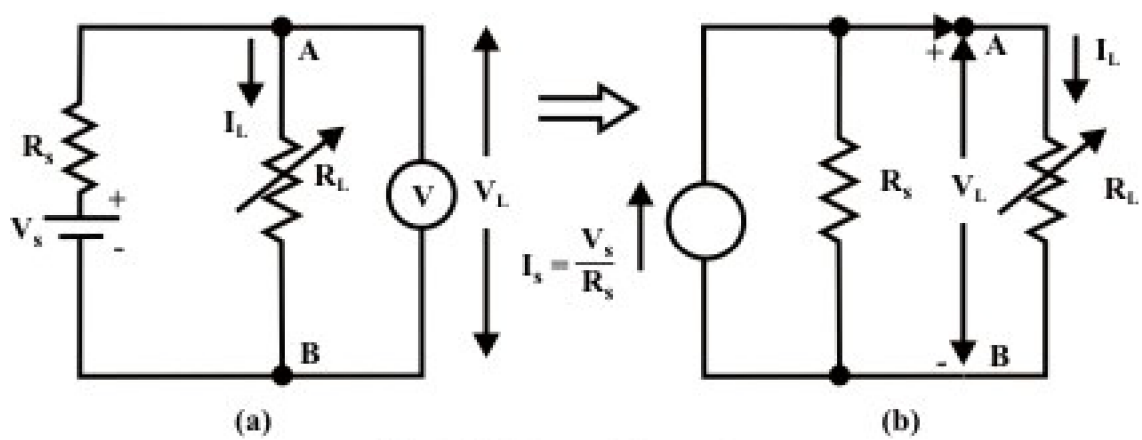


Fig. 3.19: Source Conversions

- Current source to Voltage Source

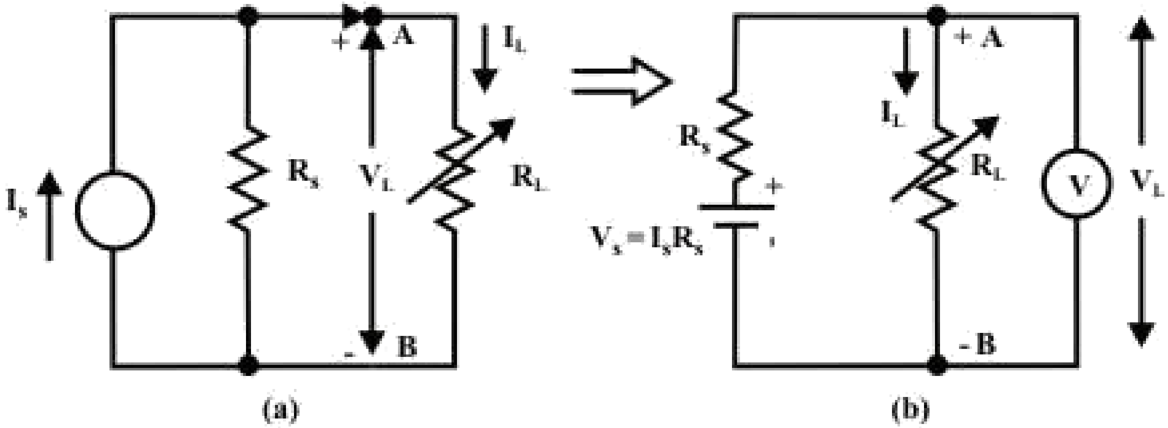


Fig. 3.20: Current Source to Voltage source conversion

For the circuit in fig. 3.15(a), the load voltage V_L is given by

$$V_L = I_L R_L = \frac{R_s}{R_s + R_L} \cdot I_s R_L = I_s R_s \frac{R_L}{R_s + R_L} = V_s \frac{R_L}{R_s + R_L} \qquad (3.5)$$

Equation (3.5) represents output from the voltage source across a load resistance and this act as a voltage divider circuit. Figure 3.20(b) describes the situation that a voltage source with a voltage value $V_s = I_s R_s$ and an internal source resistance R_s has an equivalent effect on the same load resistor as the current source in figure 3.20(a).
Note: A current source with a magnitude of current I_s and a source internal resistance R_s can be replaced by an equivalent voltage source of magnitude $V_s = I_s R_s$ and an internal source resistance R_s (see fig. 3.20(b)).

Remarks on practical sources: (i) The open circuit voltage that appears at the terminals A & B for two sources (voltage & current) is same (i.e., V_s).
(ii) When the terminals A & B are shorted by an ammeter, the shot-circuit results same in both cases (i.e., I_s).
(iii) If an arbitrary resistor (R_L) is connected across the output terminals A & B of either source, the same power will be dissipated in it.
(iv) The sources are equivalent only as concerns on their behavior at the external terminals.
(v) The internal behavior of both sources is quite different (i.e., when open circuit the voltage source does not dissipate any internal power while the current source dissipates. Reverse situation is observed in short-circuit condition).

8 Independent and Dependent Sources that encountered in electric circuits

- Independent Sources

So far the voltage and current sources (whether ideal or practical) that have been discussed are known as independent sources and these sources play an important role to drive the circuit in order to perform a specific job. The internal values of these sources (either voltage source or current source) – that is, the generated voltage V_s or the generated current I_s (see figs. 3.15 & 3.17) are not affected by the load connected across the source terminals or across any other element that exists elsewhere in the circuit or external to the source.

- Dependent Sources

Another class of electrical sources is characterized by dependent source or controlled source. In fact the source voltage or current depends on a voltage across or a current through some other element elsewhere in the circuit. Sources, which exhibit this dependency, are called dependent sources. Both voltage and current types of sources may be dependent, and either may be controlled by a voltage or a current. In general, dependent source is represented by a diamond (◇)-shaped symbol as not to confuse it with an independent source. One can classify dependent voltage and current sources into four types of sources as shown in fig.3.21. These are listed below:

(i) Voltage-controlled voltage source (VCVS) (ii) Current-controlled voltage source (ICVS) (iii) Voltage-controlled current source(VCIS) (iv) Current-controlled current source(ICIS)

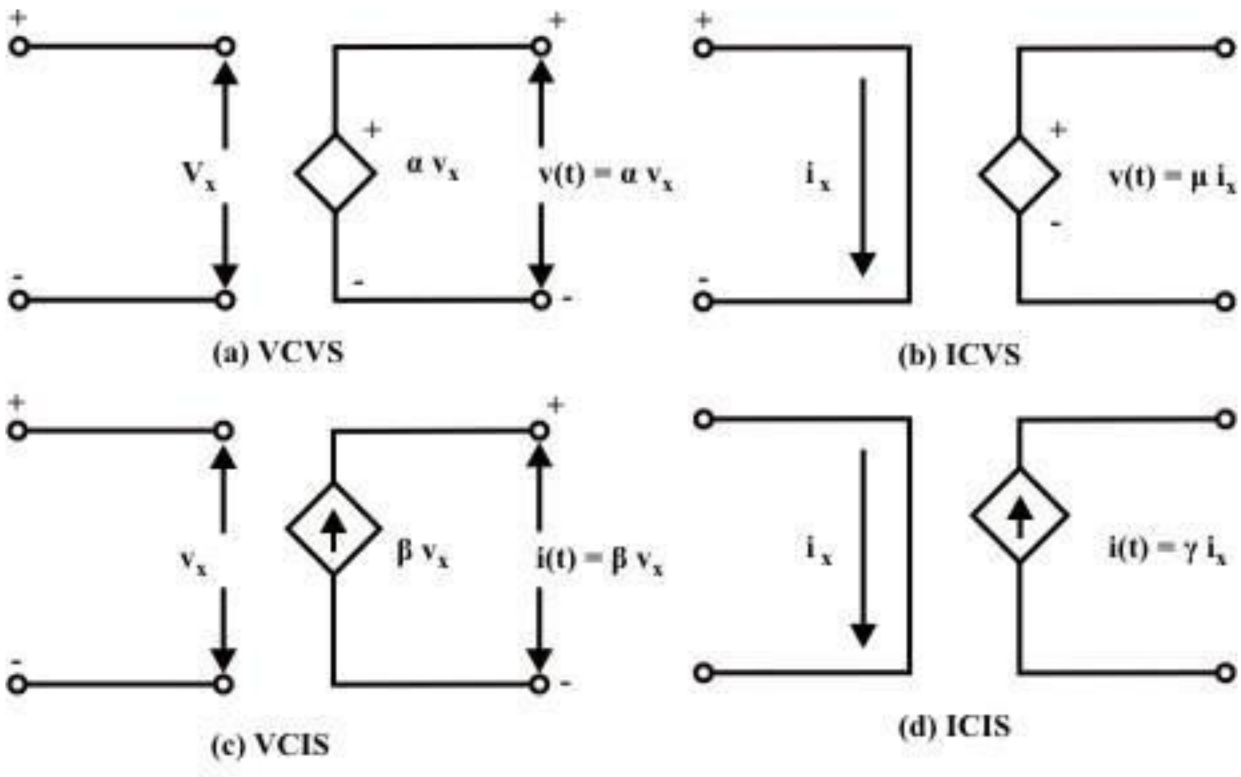
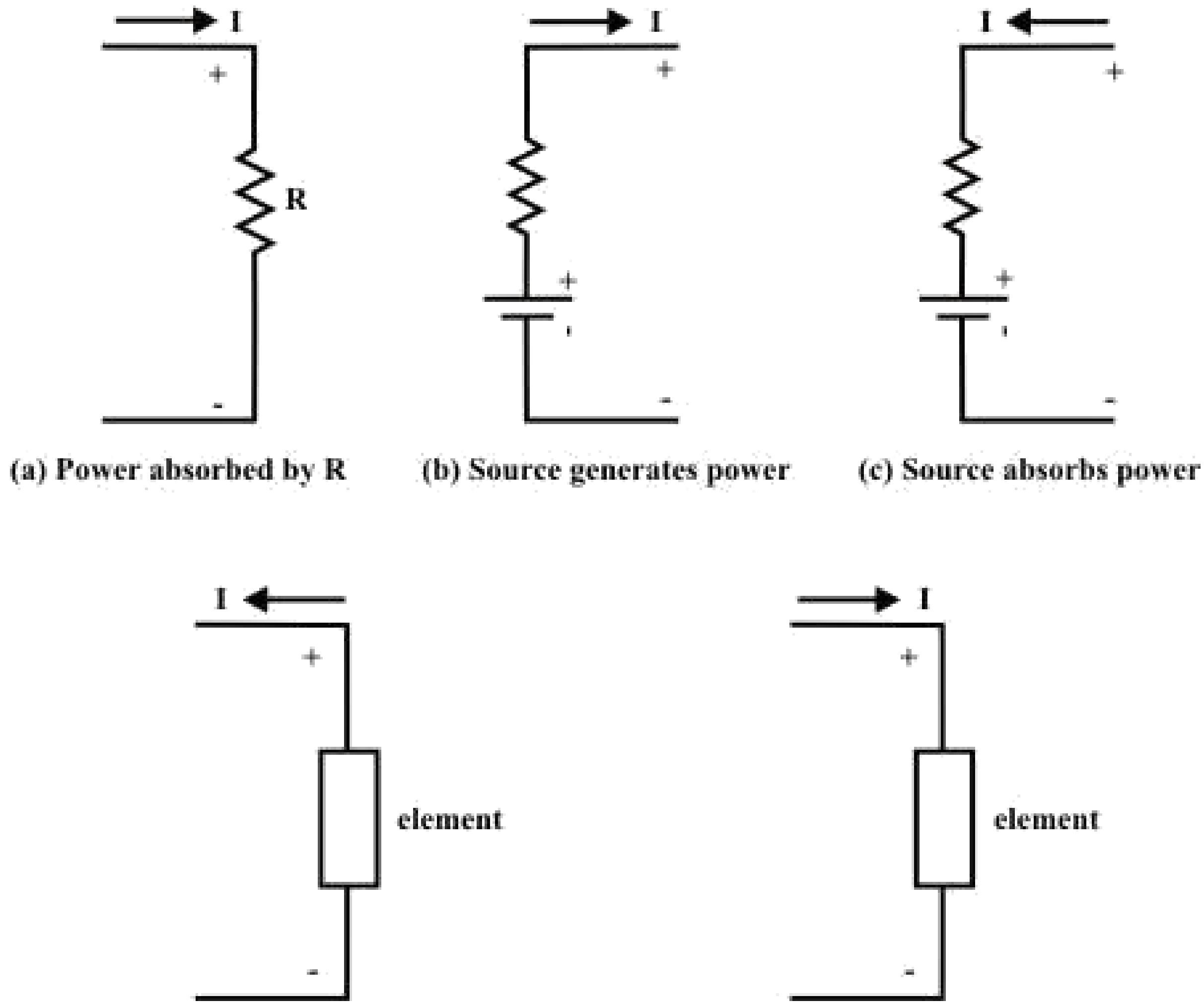


Fig. 3.21: Ideal dependent (controlled) sources

Note: When the value of the source (either voltage or current) is controlled by a voltage (v_x) somewhere else in the circuit, the source is said to be voltage-controlled source. On the other hand, when the value of the source (either voltage or current) is controlled by a current (i_x) somewhere else in the circuit, the source is said to be current-controlled source. KVL and KCL laws can be applied to networks containing such dependent sources. Source conversions, from dependent voltage source models to dependent current source models, or visa-versa, can be employed as needed to simplify the network. One may come across with the dependent sources in many equivalent-circuit models of electronic devices (transistor, BJT(bipolar junction transistor), FET(field-effect transistor) etc.) and transducers.

9 Understanding Delivering and Absorbing Power by the Source.

It is essential to differentiate between the absorption of power (or dissipating power) and the generating (or delivering) power. The power absorbed or dissipated by any circuit element when flows in a load element from higher potential point (i.e +ve terminal) toward the lower terminal point (i.e., - ve terminal). This situation is observed when charging a battery or source because the source is absorbing power. On the other hand, when current flows in a source from the lower potential point (i.e., -ve terminal) toward the higher potential point (i.e., +ve terminal), we call that source is generating power or delivering power to the other elements in the electric circuit. In this case, one can note that the battery is acting as a "source" whereas the other element is acting as a "sink". Fig.3.22 shows mode of current entering in a electric element and it behaves either as source (delivering power) or as a sink (absorbing or dissipating power).



(a) Power absorbed by R (b) Source generates power (c) Source absorbs power

(d) Power generated by this element (e) Power absorbed by this element

Fig. 3.22: Source and sink configurations

L.3.10 <u>**Test Your Understanding**</u> [marks distribution shown inside the bracket]

T.1 If a 30 V source can force 1.5 A through a certain linear circuit, how much current can 10 V force through the same circuit? (Ans. 500 mA.) [1]

T.2 Find the source voltage V_s in the circuit given below [1]

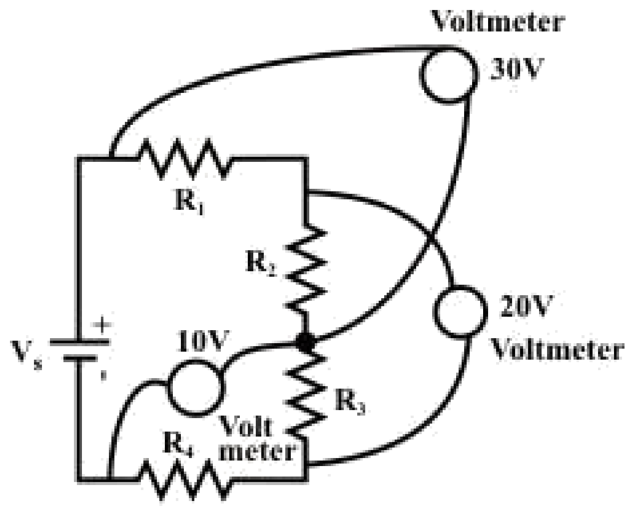


Fig. 3.33

(Ans. 40 V)

T.3 For the circuit shown in Figure T.3 [1x4]

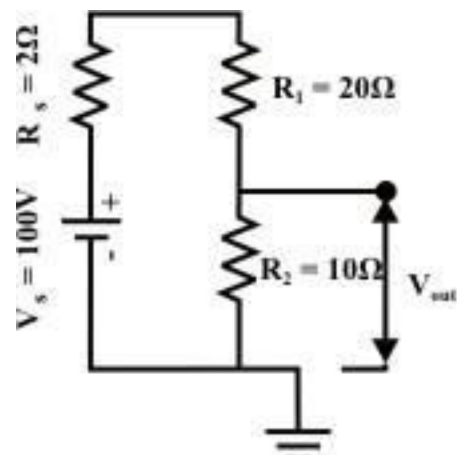


Fig. 3.34

(a) Calculate V_{out} , ignoring the internal resistance of the source R_s (assuming it's zero). Use Voltage division method. (Ans.33.333 V)

(b) Recalculate V_{out} , taking into account R_s . What percentage error was introduced by ignoring R_s in part (a). (Ans. 31.29 V , 6.66%)

(c) Repeat part (a) & (b) with the same source and replacing $R_1 = 20\ \Omega$ *by* $20k\Omega$ & $R_2 = 10\ \Omega$ *by* $1k\ \Omega$. Explain why the percent error is now so much less than in part (b). (Ans. 33.333 V, 33.331 V, 0.006%)

T.4 For the circuit shown in figure T.4 [1x6]

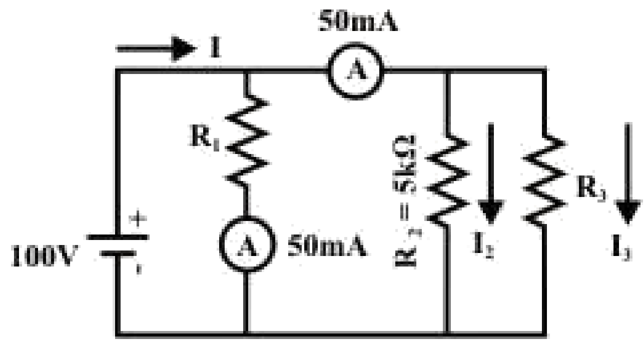


Fig. 3.35

(a) Find , in any order, I_2 , I_3 , $and\ I$ (b) Find, in any order, R_1 , R_3 , $and\ R_{eq}$.
(Ans. (a) 20 mA, 30 mA and 100 mA (b) 2 $k\Omega$, 3.33 $k\Omega$ and 1$k\Omega$.)

T.5 Refer to the circuit shown in Figure T.5 [1x4]

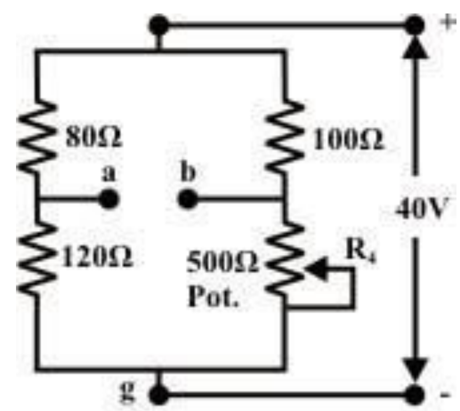


Fig. 3.36

(a) What value of R_4 will balance the bridge (i.e., V_{ab} = 0.0) (b) At balanced condition, find the values of V_{ag} & V_{bg} . (Ans. 150Ω , $24V$ (a is higher potential than 'g', since current is flowing from 'a' to 'b'), $24V$ (b is higher potential than 'g')
(b) Does the value of V_{ag} depend on whether or not the bridge is balanced? Explain this. (Ans. No., since flowing through the 80 Ω branch will remain same and hence potential drop across the resistor remains same.)
(c) Repeat part (b) for V_{bg} . (Ans. Yes . Suppose the value of $R4$ is increased from its balanced condition, this in turn decreases the value of current in that branch and subsequently voltage drop across the 100 Ω is also decreases. The indicates that the voltage across V_{bg} will increase to satisfy the KVL.)
(d) If the source voltage is changed to 50 V will the answer to part (a) change? Explain this. (Ans. No.)

T.6 If an ideal voltage source and an ideal current source are connected in parallel, then the combination has exactly the same properties as a voltage source alone. Justify this statement. [1]

T.7 If an ideal voltage source and an ideal current source are connected in series, the combination has exactly the same properties as a current source alone. Justify this statement. [1]

T.8 When ideal arbitrary voltage sources are connected in parallel, this connection violates KVL. Justify. [1]

T.9 When ideal arbitrary current sources are connected in series, this connection violates KCL. Justify.
[1]

T.10 Consider the nonseries-parallel circuit shown in figure T.10. Determine R and the equivalent resistance R_{eq} between the terminals "a" & "b" when $v_1 = 8V$.
(Appling basic two Kirchhoff's laws) (Ans. $R = 4\ \Omega$ & $R_{eq} = 4\Omega$) [3]

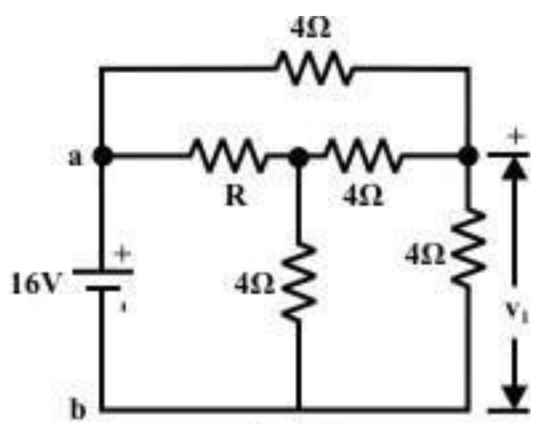


Fig. 3.37

T.11 A 20 V voltage source is connected in series with the two series-resistors $R_1 = 5\ \Omega$ & $R_2 = 10\Omega$. (a) Find I, V_{R1} , V_{R2} . (Ans. 1.333 A, 6.6667 V, 13.33 V)

(b) Find the power absorbed or generated by each of the three elements. (8.88 W (absorbed), 17.76 W (absorbed), 26.66 W delivered or generated (since current is leaving the plus terminal of that source.) [2]

T.12 Consider the circuit of figure T.12

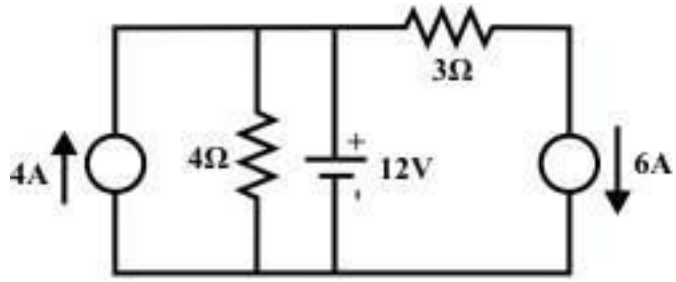


Fig. 3.38

Find powers involved in each of the five elements and whether absorbed or generated. (Ans. 48 W (G), 36 W (A), 60 W (G), 108 W (A) and 36 W (G). (results correspond to elements from left to right, CS, R, VS, R, CS). [4]

T.13 For the circuit of Figure T.13 Suppose $V_{in} = 20V$.

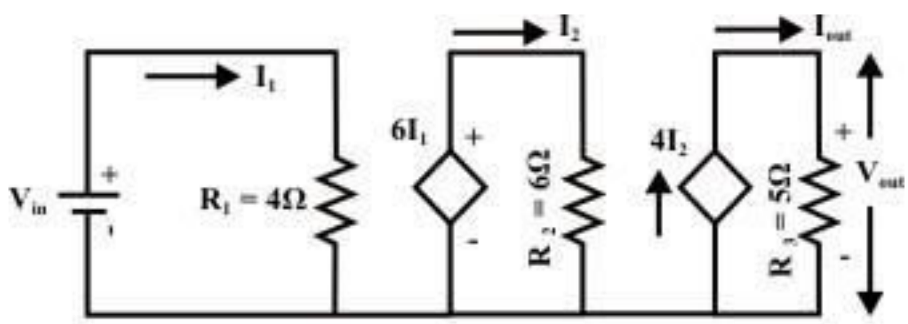


Fig. 3.39

(a) Find the output voltage and output current. [2]

(b) Find the ratio of output voltage (V_{out}) to input voltage (V_{in}) i.e. $\frac{V_{out}}{V_{in}}$ = voltage gain. [1]

(c) Find the power delivered by each source(dependent & independent sources).[2]

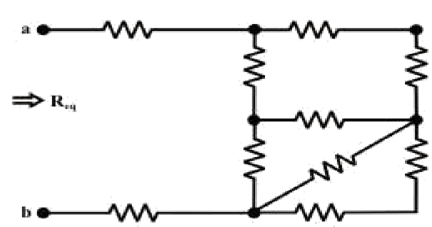


Fig. 3.40

(Ans. (a) 100 V, 20 A (note that 6 I_1 is the value of dependent voltage source with the polarity as shown in fig. T.13 whereas 4 I_2 represents the value of dependent current source) (b) 5 (voltage gain). (c) 100 W (VS), 150 W (DVS), 2000 W (DCS)).

T.14 Find the choice of the resistance R_2 (refer to Fig. T.13) so that the voltage gain is 30. (Ans. $R_2 = 1\Omega$ [1]

T.15 Find equivalent resistance between the terminals 'a' & 'b' and assume all resistors values are 1Ω . [2]

DC Transient

Study of DC transients in R-L-C Circuits

Objectives

- Be able to write differential equation for a dc circuits containing two storage elements in presence of a resistance.
- To develop a thorough understanding how to find the complete solution of second order differential equation that arises from a simple $R - L - C$ circuit.
- To understand the meaning of the terms (i) overdamped (ii) criticallydamped, and (iii) underdamped in context with a second order dynamic system.
- Be able to understand some terminologies that are highly linked with the performance of a second order system.

L.11.1 Introduction

In the preceding lesson, our discussion focused extensively on dc circuits having resistances with either inductor (L) or capacitor (C) (i.e., single storage element) but not both. Dynamic response of such first order system has been studied and discussed in detail. The presence of resistance, inductance, and capacitance in the dc circuit introduces at least a second order differential equation or by two simultaneous coupled linear first order differential equations. We shall see in next section that the complexity of analysis of second order circuits increases significantly when compared with that encountered with first order circuits. Initial conditions for the circuit variables and their derivatives play an important role and this is very crucial to analyze a second order dynamic system.

L.11.2 Response of a series R-L-C circuit due to a dc voltage source

Consider a series $R - L - C$ circuit as shown in fig.11.1, and it is excited with a dc voltage source V_s. Applying KVL around the closed path for $t > 0$,

$$L\frac{di(t)}{dt} + R\,i(t) + v_c(t) = V_s \tag{11.1}$$

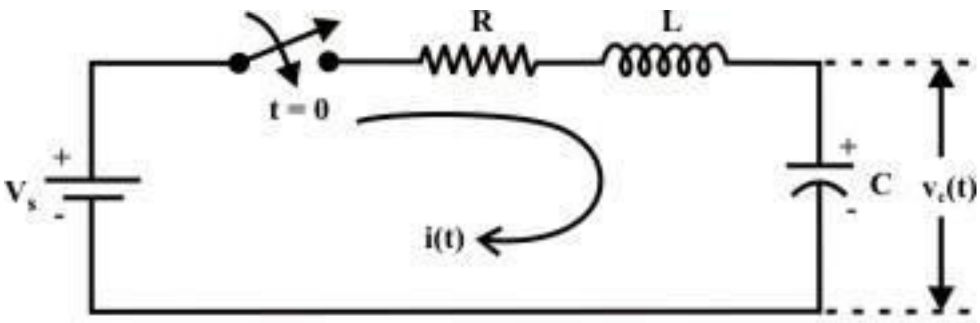


Fig. 11.1: A Simple R-L-C circuit excited with a dc voltage source

The current through the capacitor can be written as

$$i(t) = C\,\frac{dv_c(t)}{dt}$$

Substituting the current ' $i(t)$ 'expression in eq.(11.1) and rearranging the terms,

$$LC\,\frac{d^2 v_c(t)}{dt^2} + RC\,\frac{dv_c(t)}{dt} + v_c(t) = V_s \tag{11.2}$$

The above equation is a 2nd-order linear differential equation and the parameters associated with the differential equation are constant with time. The complete solution of the above differential equation has two components; the transient response $v_{cn}(t)$ and the steady state response $v_{cf}(t)$. Mathematically, one can write the complete solution as

$$v_c(t) = v_{cn}(t) + v_{cf}(t) = \left(A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}\right) + A \tag{11.3}$$

Since the system is linear, the nature of steady state response is same as that of forcing function (input voltage) and it is given by a constant value A. Now, the first part $v_{cn}(t)$ of the total response is completely dies out with time while $R > 0$ and it is defined as a transient or natural response of the system. The natural or transient response (see Appendix in Lesson-10) of second order differential equation can be obtained from the homogeneous equation (i.e., from force free system) that is expressed by

$$LC\,\frac{d^2 v_c(t)}{dt^2} + RC\,\frac{dv_c(t)}{dt} + v_c = 0 \qquad \frac{d^2 v_c(t)}{dt^2} + \frac{R}{L}\frac{dv_c(t)}{dt} + \frac{1}{LC}v_c(t) = 0$$

$$a\,\frac{d^2 v_c(t)}{dt^2} + b\,\frac{dv_c(t)}{dt} + c\,v_c(t) = 0 \quad (\text{where } a = 1,\ b = \frac{R}{L} \text{ and } c = \frac{1}{LC}) \tag{11.4}$$

The characteristic equation of the above homogeneous differential equation (using the operator $\alpha = \frac{d}{dt}$, $\alpha^2 = \frac{d^2}{dt^2}$ and $v_c(t) \neq 0$) is given by

$$\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0 \qquad a\,\alpha^2 + b\,\alpha + c = 0 \quad (\text{where } a = 1,\ b = \frac{R}{L} \text{ and } c = \frac{1}{LC}) \tag{11.5}$$

and solving the roots of this equation (11.5) one can find the constants α_1 *and* α_2 of the exponential terms that associated with transient part of the complete solution (eq.11.3) and they are given below.

$$\alpha_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\frac{b}{2a} + \frac{1}{a}\sqrt{\left(\frac{b}{2}\right)^2 - a\,c}\ ;$$

$$\alpha_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\frac{b}{2a} - \frac{1}{a}\sqrt{\left(\frac{b}{2}\right)^2 - a\,c} \tag{11.6}$$

where, $b = \frac{R}{L}$ *and* $c = \frac{1}{LC}$.

The roots of the characteristic equation (11.5) are classified in three groups depending upon the values of the parameters R , L , and C of the circuit.

Case-A (overdamped response): When $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} > 0$, this implies that the roots are distinct with negative real parts. Under this situation, the natural or transient part of the complete solution is written as

$$v_{cn}(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} \tag{11.7}$$

and each term of the above expression decays exponentially and ultimately reduces to zero as $t \to \infty$ and it is termed as overdamped response of input free system. A system that is overdamped responds slowly to any change in excitation. It may be noted that the exponential term $A_1 e^{\alpha_1 t}$ takes longer time to decay its value to zero than the term $A_1 e^{\alpha_2 t}$. One can introduce a factor ξ that provides an information about the speed of system response and it is defined by damping ratio

$$(\xi) = \frac{Actual\ damping}{critical\ damping} = \frac{b}{2\sqrt{ac}} = \frac{R/L}{2/\sqrt{LC}} > 1 \tag{11.8}$$

Case-B (critically damped response): When $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0$, this implies that the roots of eq.(11.5) are same with negative real parts. Under this situation, the form of the natural or transient part of the complete solution is written as

$$v_{cn}(t) = \left(A_1 t + A_2\right) e^{\alpha t} \quad (\text{where } \alpha = -\frac{R}{2L}) \tag{11.9}$$

where the natural or transient response is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term. The expression (11.9) that arises from the natural solution of second order differential equation having the roots of characteristic equation are same value can be verified following the procedure given below.

The roots of this characteristic equation (11.5) are same $\alpha = \alpha_1 = \alpha_2 = \frac{R}{2L}$ when $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0 \quad \left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$ and the corresponding homogeneous equation (11.4) can be rewritten as

$$\frac{d^2 v_c(t)}{dt^2} + 2\frac{R}{2L}\frac{dv_c(t)}{dt} + \frac{1}{LC} v_c(t) = 0$$

$$or \quad \frac{d^2 v_c(t)}{dt^2} + 2\alpha \frac{dv_c(t)}{dt} + \alpha^2 v_c(t) = 0$$

$$or \quad \frac{d}{dt}\left(\frac{dv_c(t)}{dt} + \alpha v_c(t)\right) + \alpha\left(\frac{dv_c(t)}{dt} + \alpha v_c(t)\right) = 0$$

$$or \quad \frac{df}{dt} + \alpha f = 0 \qquad \text{where } f = \frac{dv_c(t)}{dt} + \alpha v_c(t)$$

The solution of the above first order differential equation is well known and it is given by $f = A_1 e^{\alpha t}$

Using the value of f in the expression $f = \frac{dv_c(t)}{dt} + \alpha v_c(t)$ we can get,

$$\frac{dv_c(t)}{dt} + \alpha v_c(t) = A_1 e^{-\alpha t} e^{\alpha t} \qquad \frac{dv_c(t)}{dt} + e^{\alpha t} \alpha v_c(t) = A_1 \qquad \frac{d}{dt}\left(e^{\alpha t} v_c(t)\right) = A_1$$

Integrating the above equation in both sides yields,

$$v_{cn}(t) = \left(A_1 t + A_2\right) e^{\alpha t}$$

In fact, the term $A_2 e^{\alpha t}$ (with $\alpha = -\frac{R}{2L}$) decays exponentially with the time and tends to zero as $t \to \infty$. On the other hand, the value of the term $A_1 t e^{\alpha t}$ (with $\alpha = -\frac{R}{2L}$) in equation (11.9) first increases from its zero value to a maximum value $A_1 \frac{2L}{R} e^{-1}$ at a time $t = -\frac{1}{\alpha} = -- \frac{2L}{R} = \frac{2L}{R}$ and then decays with time, finally reaches to zero. One can easily verify above statements by adopting the concept of maximization problem of a single valued function. The second order system results the speediest response possible without any overshoot while the roots of characteristic equation (11.5) of system having the same negative real parts. The response of such a second order system is defined as a critically damped system's response. In this case damping ratio

$$(\xi) = \frac{Actual\ damping}{critical\ \ damping} = \frac{b}{2\sqrt{ac}} = \frac{\cancel{R}/L}{\cancel{2}/\sqrt{LC}} = 1 \tag{11.10}$$

Case-C (underdamped response): When $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} < 0$, this implies that the roots of eq.(11.5) are complex conjugates and they are expressed as

$$\alpha_1 = -\frac{R}{2L} + j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = \beta + j\gamma; \quad \alpha_2 = -\frac{R}{2L} - j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = \beta - j\gamma.$$ The

form of the natural or transient part of the complete solution is written as

$$v_{cn}(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} = A_1 e^{(\beta + j\gamma)} + A_2 e^{(\beta - j\gamma)}$$

$$= e^{\beta t}\left[\left(A_1 + A_2\right)\cos\left(\gamma t\right) + j\left(A_1 - A_2\right)\sin\left(\gamma t\right)\right] \tag{11.11}$$

$$= e^{\beta t}\left[B_1 \cos\left(\gamma t\right) + B_2 \sin\left(\gamma t\right)\right] \text{ where } B_1 = A_1 + A_2\ ;\ B_2 = j\left(A_1 - A_2\right)$$

For real system, the response $v_{cn}(t)$ must also be real. This is possible only if A_1 *and* A_2 conjugates. The equation (11.11) further can be simplified in the following form:

$$e^{\beta t} K \sin\left(\gamma t + \theta\right) \tag{11.12}$$

where β = real part of the root, γ = complex part of the root, $K=\sqrt{B_1^2+B_2^2}$ and $\theta=\tan^{-1}\frac{B_1}{B_2}$. Truly speaking the value of K and θ can be calculated using the initial conditions of the circuit. The system response exhibits oscillation around the steady state value when the roots of characteristic equation are complex and results an under -damped system's response. This oscillation will die down with time if the roots are with negative real parts. In this case the damping ratio

$$(\xi) = \frac{\text{Actual damping}}{\text{critical damping}} = \frac{b}{2\sqrt{ac}} = \frac{R/L}{2/\sqrt{LC}} < 1 \qquad (11.13)$$

Finally, the response of a second order system when excited with a dc voltage source is presented in fig.L.11.2 for different cases, i.e., (i) under-damped (ii) over-damped (iii) critically damped system response.

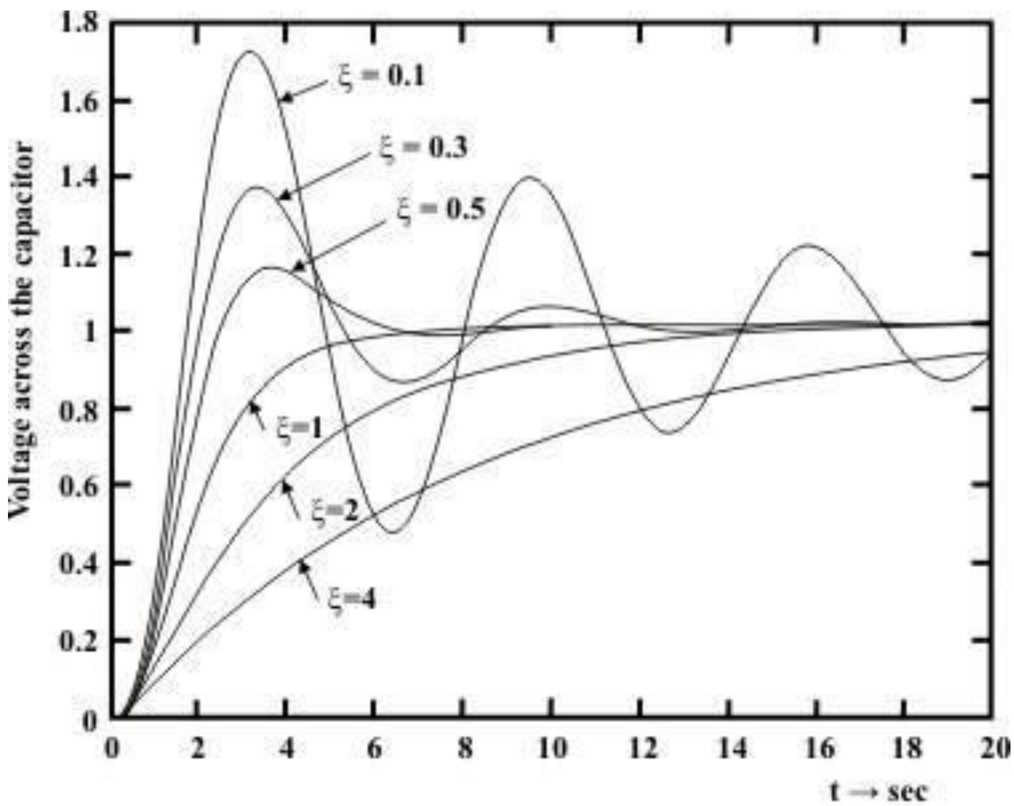


Fig. 11.2: System response for series R-L-C circuit:
(a) underdamped
(b) critically damped
(c) overdamped system

Example: L.11.1 The switch $S1$ was closed for a long time as shown in fig.11.3. Simultaneously at $t = 0$, the switch $S1$ is opened and $S2$ is closed Find $(a)\, i_L(0_+)$; $(b)\, v_c(0_+)$; $(c)\, i_R(0_+)$; $(d)\, v_L(0_+)$; $(e)\, i_c(0_+)$; $(f)\, \frac{dv(0^+)}{dt}$.

Solution: When the switch $S1$ is kept in position '1' for a sufficiently long time, the circuit reaches to its steady state condition. At time $t = 0^-$, the capacitor is completely charged and it acts as a open circuit. On other hand,

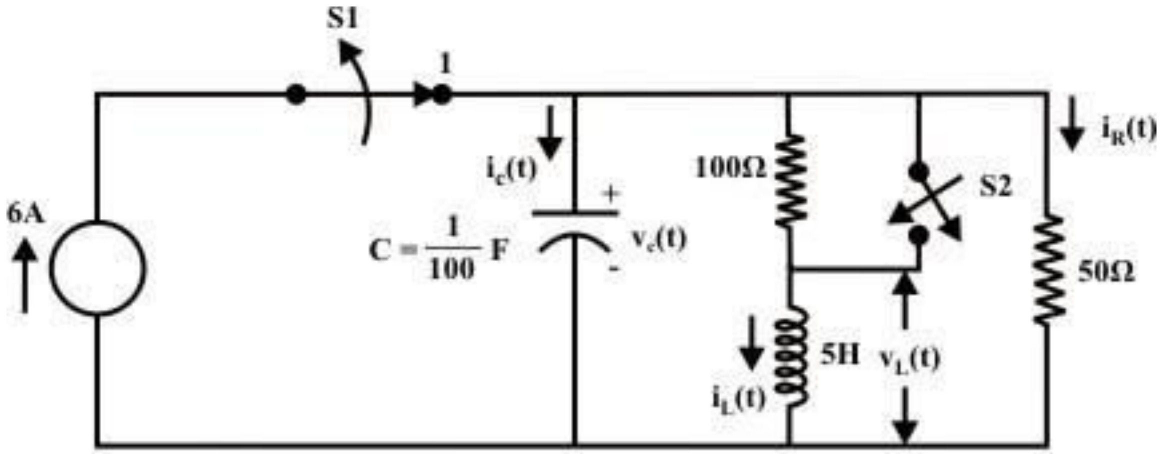


Fig. 11.3

the inductor acts as a short circuit under steady state condition, the current in inductor can be found as

$$i_L(0^-) = \frac{50}{100+50} \cdot 6 = 2\,A$$

Using the KCL, one can find the current through the resistor $i_R(0^-) = 6 - 2 = 4A$ and subsequently the voltage across the capacitor $v_c(0^-) = 4 \cdot 50 = 200\ volt$.

Note at $t = 0^+$ not only the current source is removed, but 100 Ω resistor is shorted or removed as well. The continuity properties of inductor and capacitor do not permit the current through an inductor or the voltage across the capacitor to change instantaneously. Therefore, at $t = 0^+$ the current in inductor, voltage across the capacitor, and the values of other variables at $t = 0^+$ can be computed as

$i_L(0^+) = i_L(0^-) = 2\,A$; $v_c(0^+) = v_c(0^-) = 200\ volt.$

Since the voltage across the capacitor at $t = 0^+$ is 200 $volt$, the same voltage will appear across the inductor and the 50 Ω resistor. That is, $v_L(0^+) = v_R(0^+) = 200\ volt$. and hence, the current $(i_R(0^+))$ in 50 Ω resistor $= \frac{200}{50} = 4\,A$. Applying KCL at the bottom terminal of the capacitor we obtain $i_c(0^+) = -(4+2) = -6\,A$ and subsequently, $\dfrac{dv_c(0^+)}{dt} = \dfrac{i_c(0^+)}{C} = \dfrac{-6}{0.01} = -600\ volt./\sec.$

Example: L.11.2 The switch 'S' is closed sufficiently long time and then it is opened at time '$t = 0$' as shown in fig.11.4. Determine $(i)\ v_0(0^+)\ (ii)\left.\dfrac{dv_c(t)}{dt}\right|_{t=0^+}\ (iii)\ i_L(0^+),\ and\ (iv)\left.\dfrac{di_L(t)}{dt}\right|_{t=0^+}\ (v)\left.\dfrac{dv_0(t)}{dt}\right|_{t=0}$ when $R_1 = R_2 = 3\Omega$.

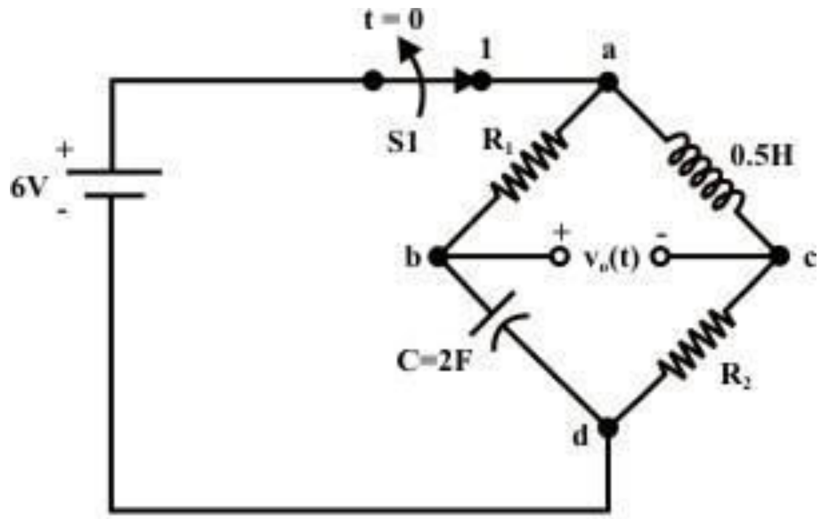


Fig. 11.4

Solution: At $t = 0^-$ (just before opening the switch), the capacitor is fully charged and current flowing through it totally blocked i.e., capacitor acts as an open circuit). The voltage across the capacitor is $v_c(0^-) = 6\ V = v_c(0^+) = v_{bd}(0^+)$ and terminal 'b' is higher potential than terminal 'd'. On the other branch, the inductor acts as a short circuit (i.e., voltage across the inductor is zero) and the source voltage $6V$ will appear across the resistance R_2. Therefore, the current through inductor $i_L(0^-) = \dfrac{6}{3} = 2\,A = i_L(0^+)$. Note at $t = 0^+$, $v_{ad}(0^+) = 0$ (since the voltage drop across the resistance $R_1 = 3\Omega = v_{ab} = -6V$) and $v_{cd}(0^+) = 6V$ and this implies that $v_{ca}(0^+) = 6V$ = voltage across the inductor (note, terminal 'c' is + ve terminal and inductor acts as a source of energy).

Now, the voltage across the terminals 'b' and 'c' ($v_0(0^+)$) $= v_{bd}(0^+) - v_{cd}(0^+)$ $= 0V$. The following expressions are valid at $t = 0^+$

$C\left.\dfrac{dv_c}{dt}\right|_{t=0^+} = i_c(0^+) = 2A \qquad \left.\dfrac{dv_c}{dt}\right|_{t=0^+} = 1\ volt/\sec.$ (note, voltage across the capacitor will decrease with time i.e., $\left.\frac{dv_c}{dt}\right|_{t=0^+} = -1\ volt/\sec$). We have just calculated the voltage across the inductor at $t = 0^+$ as

$$v_{ca}(0^+) = L\left.\frac{di_L(t)}{dt}\right|_{t=0^+} = 6V \qquad \left.\frac{di_L(t)}{dt}\right|_{t=0^+} = \frac{6}{0.5} = 12\ A/\sec.$$

Now, $\frac{dv_0(0^+)}{dt} = \frac{dv_c(0^+)}{dt} - R_2\frac{di_L(0^+)}{dt} = 1 - (12 \cdot 3) = -35\ volt/\sec.$

Example: L.11.3 Refer to the circuit in fig.11.5(a). Determine,

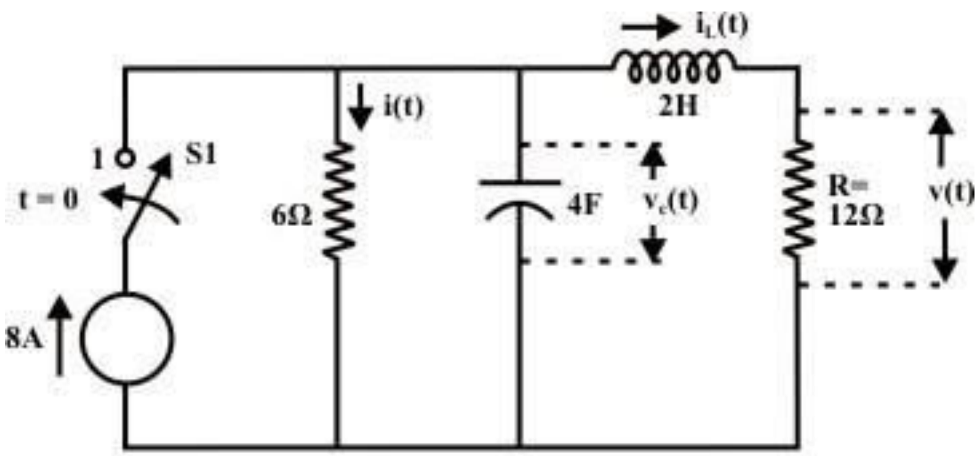


Fig. 11.5(a)

(i) $i(0^+), i_L(0^+)$ *and* $v(0^+)$ (ii) $\frac{di(0^+)}{dt}$ *and* $\frac{dv(0^+)}{dt}$ **(iii)** $i(\infty), i_L(\infty)$ *and* $v(\infty)$

(assumed $v_c(0) = 0$; $i_L(0) = 0$)

Solution: When the switch was in 'off' position i.e., t < 0

$i(0^-) = i_L(0^-) = 0,\ v(0^-) = 0$ and $v_C(0^-) = 0$

The switch '$S1$' was closed in position '1' at time t = 0 and the corresponding circuit is shown in fig 11.5 (b).

(i) From continuity property of inductor and capacitor, we can write the following expression for $t = 0^+$

$$i_L(0^+) = i_L(0^-) = 0,\ v_c(0^+) = v_c(0^-) = 0 \quad i(0^+) = \frac{1}{6}v_c(0^+) = 0$$

$$v(0^+) = i_L(0^+) \cdot 6 = 0 \text{ volt}.$$

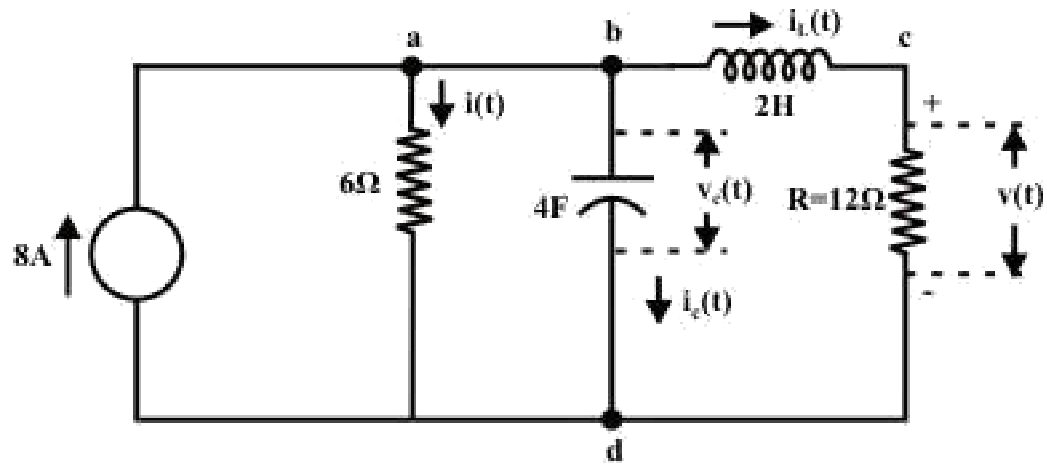


Fig. 11.5(b)

(ii) KCL at point ‘a’

$$8 = i(t) + i_c(t) + i_L(t)$$

At $t = 0^+$, the above expression is written as

$$8 = i(0^+) + i_c(0^+) + i_L(0^+) \quad i_c(0^+) = 8\ A$$

We know the current through the capacitor $i_c(t)$ can be expressed as

$$i_c(t) = C\frac{dv_c(t)}{dt}$$

$$i(0^+) = C\frac{dv_c(0^+)}{dt}$$

$$\frac{dv_c(0^+)}{dt} = 8 \times \frac{1}{4} = 2 \text{ volt./sec.}.$$

Note the relations

$\frac{dv_c(0^+)}{dt}$ = change in voltage drop in 6 Ω resistor = change in current through 6 Ω

resistor·6 $= 6\cdot\frac{di(0^+)}{dt} \qquad \frac{di(0^+)}{dt} = \frac{2}{6} = \frac{1}{3}\ amp./$ sec.

Applying KVL around the closed path ‘b-c-d-b’, we get the following expression.

$v_c(t) = v_L(t) + v(t)$

At, $t = 0^+$ the following expression

$$v_c(0^+) = v_L(0^+) + i_L(0^+)\cdot 12$$

$$0 = v_L(0^+) + 0\cdot 12 v_L(0^+) = 0 \qquad L\frac{di(0^+)}{dt} = 0 \qquad \frac{di(0^+)}{dt} = 0$$

$\frac{di_L(0^+)}{dt} = 0$ and this implies $12\frac{di_L(0^+)}{dt} = 12\cdot 0 = 0$ v/sec $= \frac{dv(0^+)}{dt} = 0$

Now, $v(t) = R\, i_L(t)$ *also* at $t = 0^+$

$$\frac{dv(0^+)}{dt} = R\frac{di_L(0^+)}{dt} = 12\frac{di_L(0^+)}{dt} = 0\, volt/\text{sec.}$$

(iii) At $t = \alpha$, the circuit reached its steady state value, the capacitor will block the flow of dc current and the inductor will act as a short circuit. The current through 6 Ω and 12 Ω resistors can be formed as

$$i(\infty) = \frac{12\times 8}{18} = \frac{16}{3} = 5.333\text{A}, \quad i_L(\infty) = 8 - 5.333 = 2.667 \text{ A}$$

$$v_c(\infty) = 32\ volt.$$

Example: L.11.4 The switch $S1$ has been closed for a sufficiently long time and then it is opened at $t = 0$ (see fig.11.6(a)). Find the expression for (a) $v_c(t)$, (b) $i_c(t)$, $t > 0$ for inductor values of $(i)\ L = 0.5\ H\ (ii)\ L = 0.2\ H\ (iii)\ L = 1.0\ H$ and plot $v_c(t) - vs - t$ and $i(t) - vs - t$ for each case.

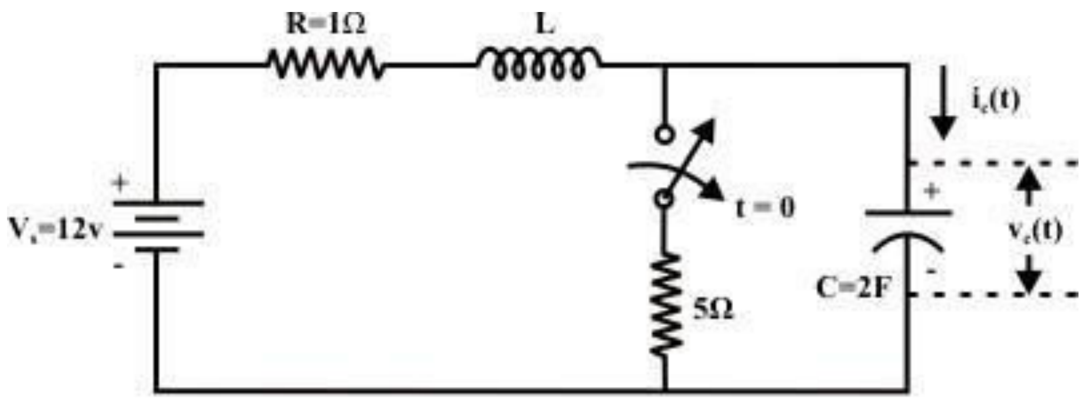


Fig. 11.6(a)

Solution: At $t = 0^-$ (before the switch is opened) the capacitor acts as an open circuit or block the current through it but the inductor acts as short circuit. Using the properties of inductor and capacitor, one can find the current in inductor at time $t = 0^+$ as $i_L(0^+) = i_L(0^-) = \frac{12}{1+5} = 2A$ (note inductor acts as a short circuit) and voltage across the 5 Ω resistor = $2 \cdot 5 = 10\ volt$. The capacitor is fully charged with the voltage across the 5 Ω resistor and the capacitor voltage at $t = 0^+$ is given by $v_c(0^+) = v_c(0^-) = 10\ volt$. The circuit is opened at time $t = 0$ and the corresponding circuit diagram is shown in fig. 11.6(b).

Case-1: $L = 0.5H$, $R = 1\Omega$ *and* $C = 2F$

Let us assume the current flowing through the circuit is $i(t)$ and apply KVL equation around the closed path is

$+ v_c(t)$ (note, $i(t) = C \frac{dv_c(t)}{dt}$)

$$V_s = R\,i(t) + L\frac{di(t)}{dt} + v_c(t) \quad V_s = RC\frac{dv_c(t)}{dt} + LC\frac{d^2 v_c(t)}{dt^2} \qquad (11.14)$$

$$V_s = \frac{d^2 v_c(t)}{dt^2} + \frac{R}{L}\frac{dv_c(t)}{dt} + \frac{1}{LC} v_c(t) \qquad (11.15)$$

The solution of the above differential equation is given by

$v_c(t) = v_{cn}(t) + v_{cf}(t)$

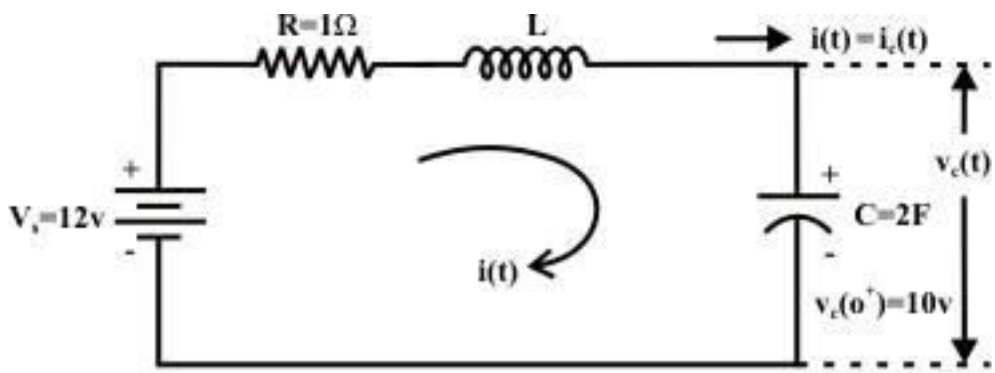


Fig. 11.6(b)

The solution of natural or transient response $v_{cn}(t)$ is obtained from the force free equation or homogeneous equation which is

$$\frac{d^2 v_c(t)}{dt^2} \; \frac{R}{} \frac{dv_c(t)}{} + \frac{1}{} \qquad (11.16)$$

The characteristic equation of the above homogeneous equation is written as

$$\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0 \qquad (11.17)$$

The roots of the characteristic equation are given as

$$\alpha_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -1.0\,; \quad \alpha_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -1.0$$

and the roots are equal with negative real sign. The expression for natural response is given by

$$v_{cn}(t) = \left(A_1 t + A_2\right)e^{\alpha t} \quad \text{(where } \alpha = \alpha_1 = \alpha_2 = -1\text{)} \qquad (11.18)$$

The forced or the steady state response $v_{cf}(t)$ is the form of applied input voltage and it is constant 'A'. Now the final expression for $v_c(t)$ is

$$v_c(t) = \left(A_1 t + A_2\right)e^{\alpha t} + A = \left(A_1 t + A_2\right)e^{-t} + A \qquad (11.19)$$

The initial and final conditions needed to evaluate the constants are based on $v_c(0^+) = v_c(0^-) = 10\ volt$; $i_L(0^+) = i_L(0^-) = 2\,A$ (Continuity property).

At $t = 0^+$;

$$v_c(t)\Big|_{t=0^+} = A_2 e^{-1 \cdot 0} + A = A_2 + A \quad A + A_2 = 10 \tag{11.20}$$

Forming $\frac{dv_c(t)}{dt}$ (from eq.(11.19)as

$$\frac{dv_c(t)}{dt} = \alpha \left(A_1 t + A_2 \right) e^{\alpha t} + A_1 e^{\alpha t} = -\left(A_1 t + A_2 \right) e^{-t} + A_1 e^{-t}$$

$$\frac{dv_c(t)}{dt}\Bigg|_{t=0^+} = A_1 - A_2 \quad A_1 - A_2 = 1 \tag{11.21}$$

(note, $C \frac{dv_c(0^+)}{dt} = i_c(0^+) = i_L(0^+) = 2 \quad \frac{dv_c(0^+)}{dt} = 1 \; volt/\sec.$)

It may be seen that the capacitor is fully charged with the applied voltage when $t = \infty$ and the capacitor blocks the current flowing through it. Using $t = \infty$ in equation (11.19) we get,

$v_c(\infty) = A \quad A = 12$

Using the value of A in equation (11.20) and then solving (11.20) and (11.21) we get, $A_1 = -1$; $A_2 = -2$.

The total solution is

$$\begin{aligned} v_c(t) &= -\left(t + 2 \right) e^{-t} + 12 = 12 - \left(t + 2 \right) e^{-t} ; \\ i(t) &= C \frac{dv_c(t)}{dt} = 2 \cdot \left[\left(t + 2 \right) e^{-t} - e^{-t} \right] = 2 \cdot \left(t + 1 \right) e^{-t} \end{aligned} \tag{11.22}$$

The circuit responses (critically damped) for $L = 0.5\,H$ are shown fig.11.6 (c) and fig.11.6(d).

Case-2: $L = 0.2\,H$, $R = 1\Omega$ *and* $C = 2F$

It can be noted that the initial and final conditions of the circuit are all same as in case-1 but the transient or natural response will differ. In this case the roots of characteristic equation are computed using equation (11.17), the values of roots are

$\alpha_1 = -0.563$; $\alpha_2 = -4.436$

The total response becomes

$$v_c(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + A = A_1 e^{-4.436t} + A_2 e^{-0.563t} + A \tag{11.23}$$

$$\frac{dv_c(t)}{dt} = \alpha_1 A_1 e^{\alpha_1 t} + \alpha_2 A_2 e^{\alpha_2 t} = -4.435 A_1 e^{-4.436t} - 0.563 A_2 e^{-0.536t} \tag{11.24}$$

Using the initial conditions($v_c(0^+) = 10$, $\frac{dv_c(0^+)}{dt} = 1 \; volt/\sec.$) that obtained in case-1 are used in equations (11.23)-(11.24) with $A = 12$ (final steady state condition) and simultaneous solution gives

$A_1 = 0.032$; $A_2 = -2.032$

The total response is

$$v_c(t) = 0.032\, e^{-4.436t} - 2.032\, e^{-0.563t} + 12$$

$$i(t) = C\frac{dv_c(t)}{dt} = 2\left(1.14\, e^{-0.563t} - 0.14\, e^{-4.436t}\right) \tag{11.25}$$

The system responses (overdamped) for $L = 0.2\ H$ are presented in fig.11.6(c) and fig.11.6 (d).

Case-3: $L = 8.0\ H$, $R = 1\Omega$ *and* $C = 2F$

Again the initial and final conditions will remain same and the natural response of the circuit will be decided by the roots of the characteristic equation and they are obtained from (11.17) as

$\alpha_1 = \beta + j\gamma = -0.063 + j\,0.243$; $\alpha_2 = \beta - j\gamma = -0.063 - j$ 0.242 The expression for the total response is

$$v_c(t) = v_{cn}(t) + v_{cf}(t) = e^{\beta t}\, K\sin\left(\gamma t + \theta\right) + A \tag{11.26}$$

(note, the natural response $v_{cn}(t) = e^{\beta t}\quad K\sin\left(\gamma t + \theta\right)$ is written from eq.(11.12) when roots are complex conjugates and detail derivation is given there.)

$$\frac{dv_c(t)}{dt} = K\, e^{\beta t}\left[\beta\sin\left(\gamma t + \theta\right) + \gamma\cos\left(\gamma t + \theta\right)\right] \tag{11.27}$$

Again the initial conditions ($v_c(0^+) = 10$, $\frac{dv_c(0^+)}{dt} = 1\ volt/\sec.$) that obtained in case-1 are used in equations (11.26)-(11.27) with $A = 12$ (final steady state condition) and simultaneous solution gives

$K = 4.13$; $\theta = -28.98^0$ (deg *ree*)

The total response is

$$v_c(t) = e^{\beta t}\, K\sin\left(\gamma t + \theta\right) + 12 = e^{-0.063t}\, 4.13\sin\left(0.242\, t - 28.99^0\right) + 12$$

$$v_c(t) = 12 + 4.13\, e^{-0.063t}\sin\left(0.242\, t - 28.99^0\right) \tag{11.28}$$

$$i(t) = C\frac{dv_c(t)}{dt} = 2e^{-0.063t}\left[0.999*\cos\left(0.242\, t - 28.99^0\right) - 0.26\sin\left(0.242\, t - 28.99^0\right)\right]$$

The system responses (under-damped) for $L = 8.0\ H$ are presented in fig.11.6(c) and fig. 11.6(d).

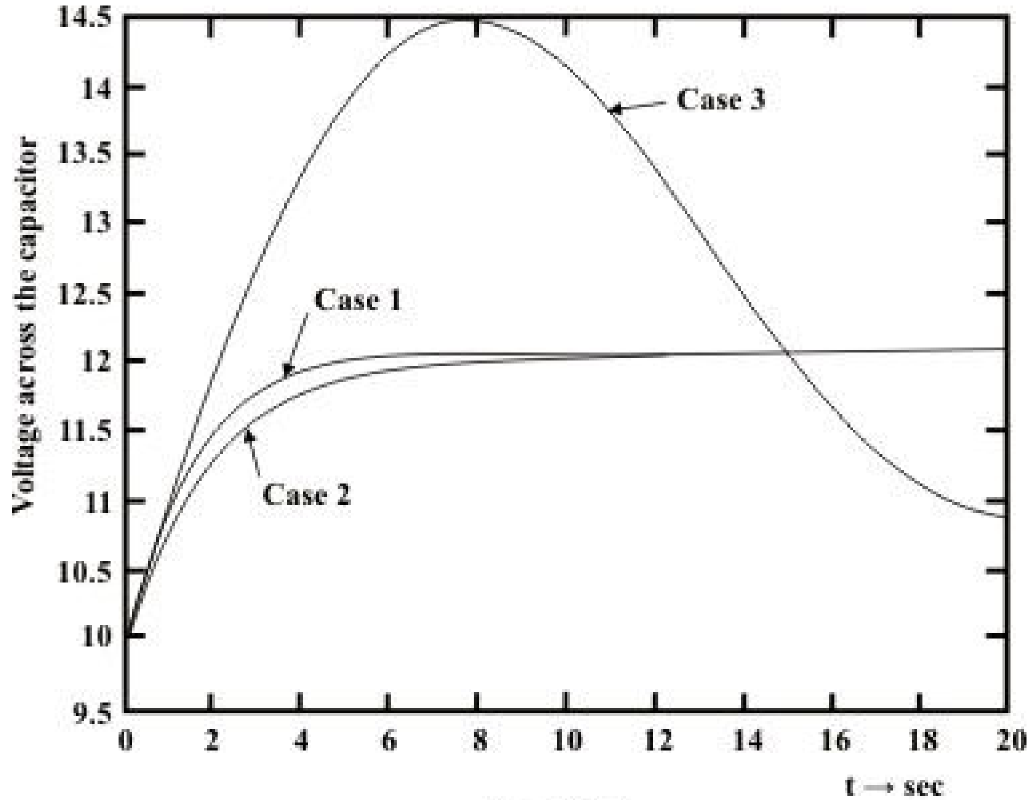


Fig. 11.6(c)

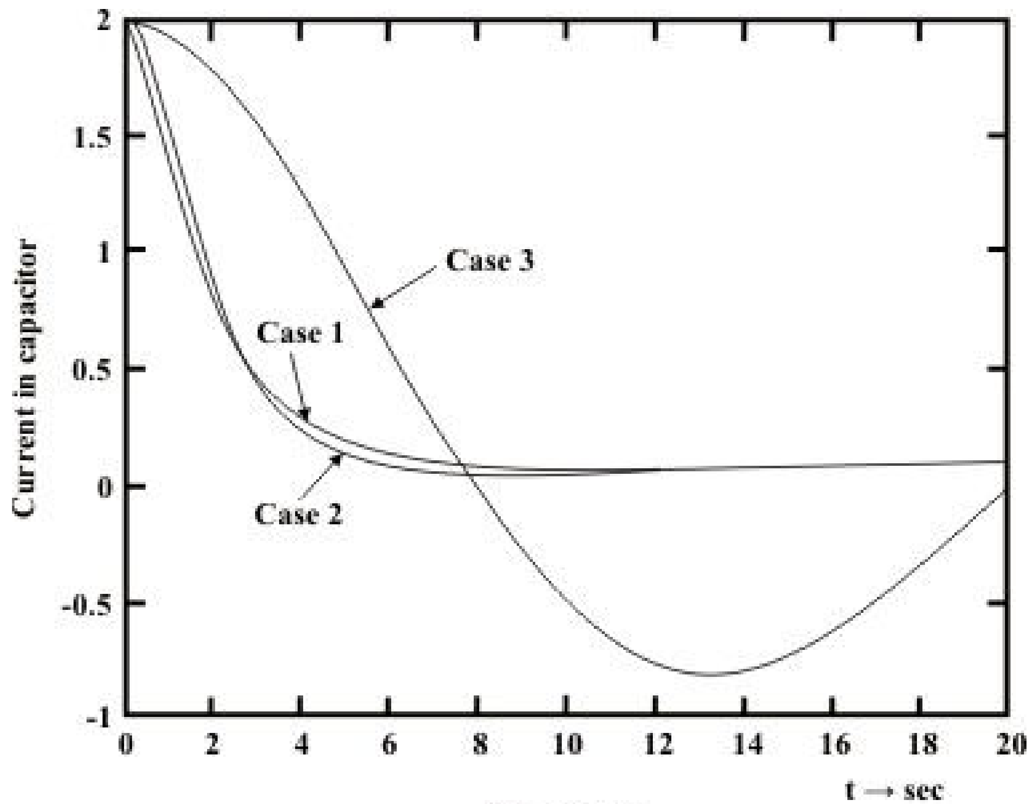


Fig. 11.6(d)

Remark: One can use $t = 0$ *and* $t = \infty$ in eq. 11.22 or eq. 11.25 or eq. 11.28 to verify whether it satisfies the initial and final conditions (i.e., initial capacitor voltage $v_c(0^+) = 10$ *volt.* , and the steady state capacitor voltage $v_c(\infty) = 12$ *volt.*) of the circuit.

Example: L.11.5 The switch ' $S1$' in the circuit of Fig. 11.7(a) was closed in position '1' sufficiently long time and then kept in position '2'. Find (i) $v_c(t)$ (ii) $i_c(t)$ for $t \geq 0$ if C is (a) $\frac{1}{9}F$ (b) $\frac{1}{4}F$ (c) $\frac{1}{8}F$.

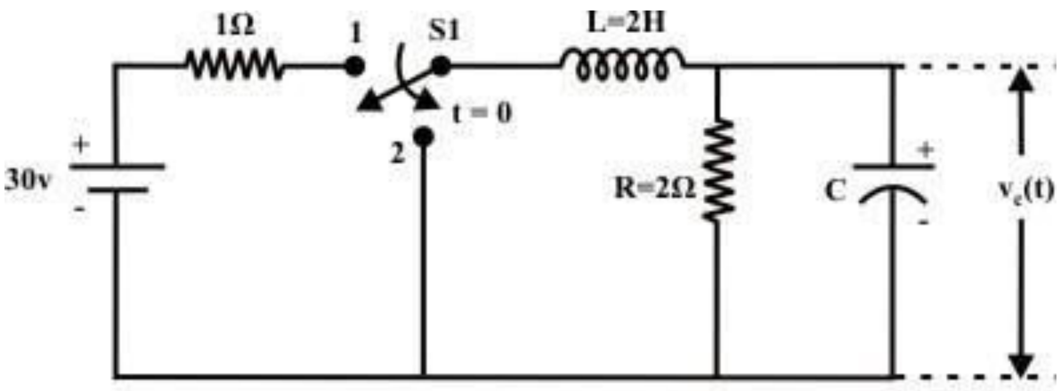


Fig. 11.7(a)

Solution: When the switch was in position '1', the steady state current in inductor is given by

$i_L(0^-) = \frac{30}{1+2} = 10A$, $v_c(0^-) = i_L(0^-)\,R = 10 \times 2 = 20$ volt.

Using the continuity property of inductor and capacitor we get

$i_L(0^+) = i_L(0^-) = 10$, $v_c(0^+) = v_c(0^-) = 20$ volt.

The switch ' $S1$' is kept in position '2' and corresponding circuit diagram is shown in Fig.11.7 (b)

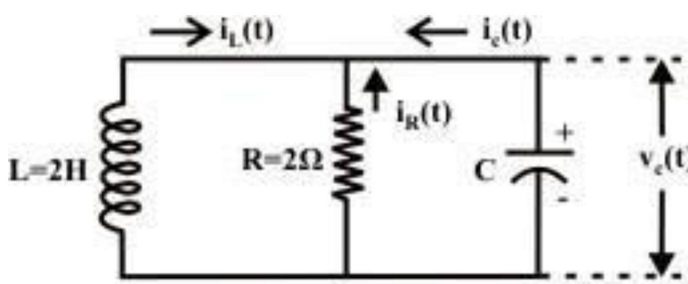


Fig. 11.7(b)

Applying KCL at the top junction point we get,

$$\frac{v_c(t)}{R} + i_c(t) + i_L(t) = 0$$

$$\frac{v_C(t)}{R} + C\frac{dv_C(t)}{dt} + i_L(t) = 0$$

$$\frac{L}{R}\frac{di_L(t)}{dt} + C.L\frac{d^2 i_L(t)}{dt^2} + i_L(t) = 0 \quad \left[\text{note: } v_c(t) = L\frac{di_L(t)}{dt}\right]$$

or $$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{1}{LC}\, i_L(t) = 0 \tag{11.29}$$

The roots of the characteristics equation of the above homogeneous equation can obtained for $C = \frac{1}{9}F$

$$\alpha_1 = \frac{-\frac{1}{RC} + \sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}}{2} = \frac{-\frac{9}{2} + \sqrt{\left(\frac{9}{2}\right)^2 - \frac{4\times 9}{2}}}{2} = -1.5$$

$$\alpha_2 = \frac{-\frac{1}{RC} - \sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}}{2} = \frac{-\frac{9}{2} - \sqrt{\left(\frac{9}{2}\right)^2 - \frac{4\times 9}{2}}}{2} = -3.0$$

Case-1 (ξ=1.06, *over damped system*) **:** $C = \frac{1}{9}F$, the values of roots of characteristic equation are given as

$\alpha_1 = -1.5$, $\alpha_2 = -3.0$

The transient or neutral solution of the homogeneous equation is given by

$$i_L(t) = A_1 e^{-1.5t} + A_2 e^{-3.0t} \tag{11.30}$$

To determine A_1 and A_2, the following initial conditions are used.

At $t = 0^+$;

$$i_L(0^+) = i_L(0^-) = A_1 + A_2$$
$$10 = A_1 + A_2 \tag{11.31}$$

$$v_c(0^+) = v_c(0^-) = v_L(0^+) = L\left.\frac{di_L(t)}{dt}\right|_{t=0^+}$$

$$20 = 2\times \left[A_1\cdot -1.5\, e^{-1.5t} - 3.0\cdot A_2 e^{-3.0t}\right] \tag{11.32}$$
$$= 2\left[-1.5A_1 - 3A_2\right] = -3A_1 - 6A_2$$

Solving equations (11.31) and (11,32) we get , $A_2 = -16.66$, $A_1 = 26.666$.

The natural response of the circuit is

$$i_L = \frac{80}{3}e^{-1.5t} - \frac{50}{3}e^{-3.0t} = 26.66e^{-1.5t} - 16.66e^{-3.0t}$$

$$L\frac{di_L}{dt} = 226.66 \cdot -1.5e^{-1.5t} - 16.66 \cdot -3.0\, e^{-3.0t}$$

$v_{Lc}(t) = v\,(t) = 100e^{-3.0t} - 80e^{-1.5t}$

$$i_c(t) = c\frac{dv_c(t)}{dt} = \frac{1}{9}\left(-300.0e^{-3.0t} + 120e^{-1.5t}\right) = \left(13.33e^{-1.5t} - 33.33e^{-3.0t}\right)$$

Case-2 $(\xi = 0.707, \textit{under damped system})$**:** For $C = \frac{1}{4}$ F , the roots of the characteristic equation are

$$\alpha_1 = -1.0 + j1.0 = \beta + j\gamma$$
$$\alpha_2 = -1.0 - j1.0 = \beta - j\gamma$$

The natural response becomes 1

$$i_L(t) = k\,e^{\beta t}\sin(\gamma t + \theta) \qquad (11.33)$$

Where k and θ are the constants to be evaluated from initial condition.

At $t = 0^+$, from the expression (11.33) we get,

$$i_L\left(0^+\right) = k\sin\theta$$
$$10 = k\sin\theta \qquad (11.34)$$

$$L\left.\frac{di(t)}{dt}\right|_{t=0^+} = 2\cdot k\,\beta\, e^{\beta t}\sin(\gamma t + \theta) + e^{\beta t}\gamma\cos(\gamma t + \theta)\Big|_{t=0^+} \qquad (11.35)$$

Using equation (11.34) and the values of β *and* γ in equation (11.35) we get,

$$20 = 2\,k\,(\beta sn\theta + \gamma\cos\theta) = k\cos\theta \text{ (note: } \beta = -1,\ \gamma = 1 \textit{ and } k\sin\theta = 10\text{)} \qquad (11.36)$$

From equation (11.34) and (11.36) we obtain the values of θ and k as

$$\tan\theta = \frac{1}{2} \qquad \theta = \tan^{-1}\frac{1}{2} = 26.56^o \text{ and } k = \frac{10}{\sin\theta} = 22.36$$

The natural or transient solution is

$$i_L(t) = 22.36\, e^{-t}\sin\left(t + 26.56^o\right)$$

$$L\frac{di(t)}{dt} = v_c(t) = 2\cdot k\cdot\left[\beta\sin(\gamma t + \theta) + \gamma\cos(\gamma t + \theta)\right]e^{\beta t}$$

$$= 44.72\cos(t + 26.56^o) - \sin(t + 26.56^o)\cdot e^{-t}$$

$$i_c(t) = c\frac{dv_c(t)}{dt} = \frac{1}{4}\cdot 44.72\frac{d}{dt}\left\{\cos(t + 26.56^o) - \sin(t + 26.56^o)\ e^{-t}\right.$$

$$= -22.36\cos(t + 26.56)\, e^{-t}$$

Case-3 $\left(\xi=1,\ \textit{critically damped system}\right)$**:** For $C=\frac{1}{8}$ F; the roots of characteristic equation are $\alpha_1=-2;\ \alpha_2=-2$ respectively. The natural solution is given by

$$i_L(t)=\left(A_1\,t+A_2\right)e^{\alpha t} \qquad (11.37)$$

where constants are computed using initial conditions.

At $t=0^+$; from equation (11.37) one can write

$$i_L(0^+)=A_2 \quad A_2=10$$

$$L\left.\frac{\text{di(t)}}{\text{dt}}\right|_{t=0^+}=2\cdot A_2\,\alpha e^{\alpha t}+\alpha A_1\,t\,e^{\alpha t}+A_1\,e^{\alpha t}\Big|_{t=0^+}$$

$$=2\cdot\left(A_1+A_2\alpha\right)e^{\alpha t}+\alpha A_1\,t\,e^{\alpha t}\Big|_{t=0^+}$$

$$L\left.\frac{\text{di(t)}}{\text{dt}}\right|_{t=0^+}=v_c(0^+)=20=2\left(A_1-2A_2\right)\quad A_1=30$$

The natural response is then

$$i_L(t)=\left(10+30t\right)e^{-2t}$$

$$L\frac{di_L(t)}{dt}=2\cdot\frac{d}{dt}\left(10+30t\right)e^{-2t}$$

$$L\frac{di_L(t)}{dt}=v_c(t)=2\left[10-60t\right]e^{-2t}$$

$$i_c(t)=c\frac{dv_c(t)}{dt}=\frac{1}{8}\cdot 2\cdot\frac{d}{dt}\left(10-60t\right)e^{-2t}=-20\,e^{-2t}+30t\,e^{-2t}$$

Case-4 $\left(\xi=2,\ \textit{over damped system}\right)$: For $C=\frac{1}{32}F$

Following the procedure as given in case-1 one can obtain the expressions for (i) current in inductor $i_L(t)$ (ii) voltage across the capacitor $v_c(t)$

$$i_L(t)=11.5\,e^{-1.08t}-1.5e^{-14.93t}$$

$$L\frac{di(t)}{dt}=v_c(t)=44.8\,e^{-14.93t}-24.8\,e^{-1.08t}$$

$$i_c(t)=c\frac{dv_c(t)}{dt}=\frac{1}{32}\cdot\frac{d}{dt}\,44.8\,e^{-14.93t}-24.8\,e^{-1.08t}$$

$$=0.837\,e^{-1.08t}-20.902\,e^{-14.93t}$$

L.11.3 Test your understanding

(Marks: 80)

T.11.1 Transient response of a second-order ------------------ dc network is the sum of two real exponentials. [1]

T.11.2 The complete response of a second order network excited from dc sources is the sum of -------- response and ---------------- response. [2]

T.11.3 Circuits containing two different classes of energy storage elements can be described by a ------------------- order differential equations. [1]

T.11.4 For the circuit in fig.11.8, find the following [6]

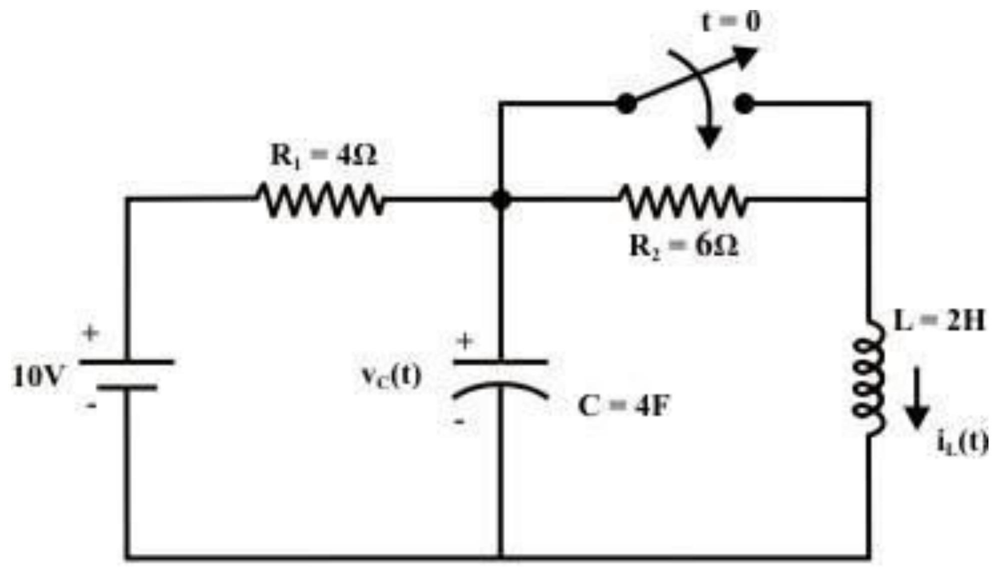


Fig. 11.8

$(a)\ v_c(0^-)\ \ (b)\ v_c(0^+)\ \ (c)\ \frac{dv_c(0^-)}{dt}\ \ (d)\ \frac{dv_c(0^+)}{dt}\ \ (e)\ \frac{di_L(0^-)}{dt}\ \ (f)\ \frac{di_L(0^+)}{dt}$

(Ans. (*a*) 6 *volt* . (*b*) 6 *volt* . (*c*) 0 *V* / sec. (*d*) 0 *V* / sec. (*e*) 0 *amp* / sec. (*f*) 3 *amp.* / sec.)

T.11.5 In the circuit of Fig. 11.9,

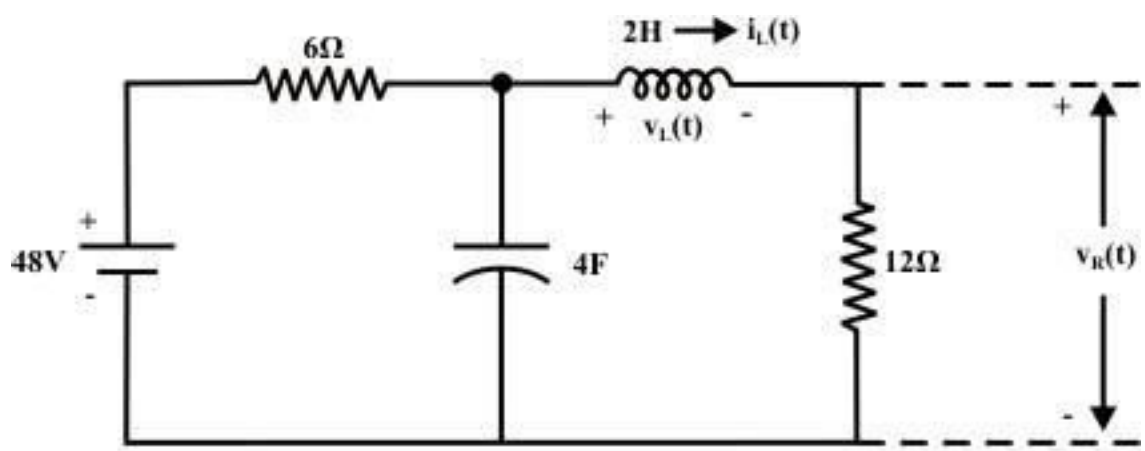


Fig. 11.9

Fin

$(a)\ v_R(0^+)\ and\ v_L(0^+)\ (b)\ \frac{dv_R(0^+)}{dt}\ \ and\ \ \frac{dv_L(0^+)}{dt}\ \ (c)\ v_R(\infty)\ \ and\ v_L(\infty)$ [8]

(Assume the capacitor is initially uncharged and current through inductor is zero).
(Ans. $(a)\ 0\,V, 0\,V\ \ (b)\ 0\,V, 2\ Volt./Sec.\ (c)\ 32\,V, 0V$)

T.11.6 For the circuit shown in fig.11.10, the expression for current through inductor

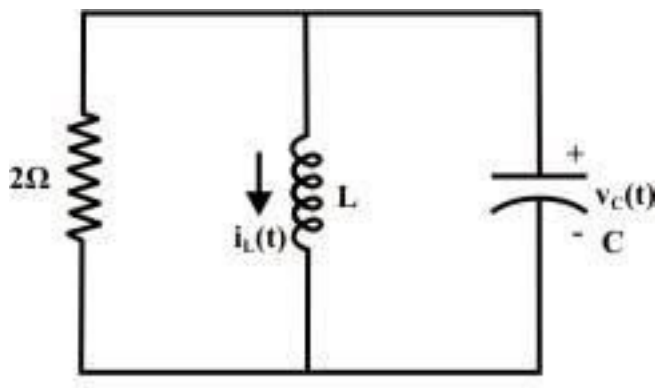


Fig. 11.10

is given by $i_L(t) = (10 + 30t)\,e^{-2t}\ \ for\ \ t \geq 0$
Find, (a) the values of L, C (b) initial condition $v_c(0^-)$ (c) the expression for $v_c(t) > 0$.

(Ans. $(a)\ L = 2H,\ C = \frac{1}{8}F\ (b)\ v_c(0^-) = 20V\ \ (c)\ v_c(t) = (20 - 120t)\,e^{-2t}\ V.$) [8]

T.11.7 The response of a series RLC circuit are given by

$$v_c(t) = 12 + 0.032\,e^{-4.436t} - 2.032\,e^{-0.563t}$$
$$i_L(t) = 2.28\,e^{-0.563t} - 0.28e^{-4.436t}$$

where $v_c(t)\ and\ i_L(t)$ are capacitor voltage and inductor current respectively. Determine (a) the supply voltage (b) the values R, L, C of the series circuit. [4+4]

(Ans. $(a)\ 12\ V\ (b)\ R = 1\ \Omega,\ L = 0.2\ H\ and\ C = 2\ F$)

T.11.8 For the circuit shown in Fig. 11.11, the switch ‘ S ’was in position ‘1’ for a long time and then at $t = 0$ it is kept in position ‘2’.

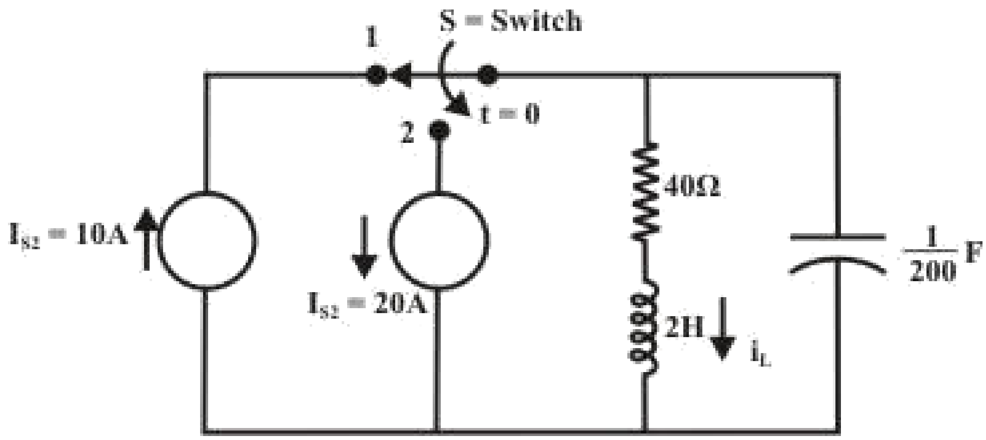


Fig. 11.11

Find,
$(a)\ i_L(0^-)$; $(b)\ v_c(0^+)$; $(c)\ v_R(0^+)$; $(d)\ i_L(\infty)$; [8]

Ans.
$(a)\ i_L(0^-) = 10\,A$; $(b)\ v_c(0^+) = 400\,V$;
$(c)\ v_R(0^+) = 400\,V$ $(d)\ i_L(\infty) = -20\,A$

T.11.9 For the circuit shown in Fig.11.12, the switch ' S ' has been in position '1' for a long time and at $t = 0$ it is instantaneously moved to position '2'.

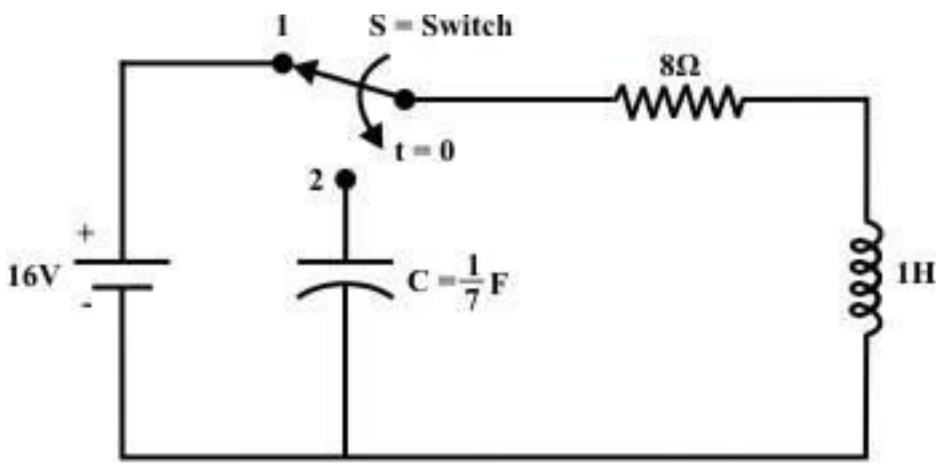


Fig. 11.12

Determine $i(t)$ *for* $t \geq 0$ and sketch its waveform. Remarks on the system's response. [8]

(Ans. $i(t) = \frac{7}{3}e^{-7t} - \frac{1}{3}e^{-t}$ *amps.*)

T.11.10 The switch ' S ' in the circuit of Fig.11.13 is opened at $t = 0$ having been closed for a long t

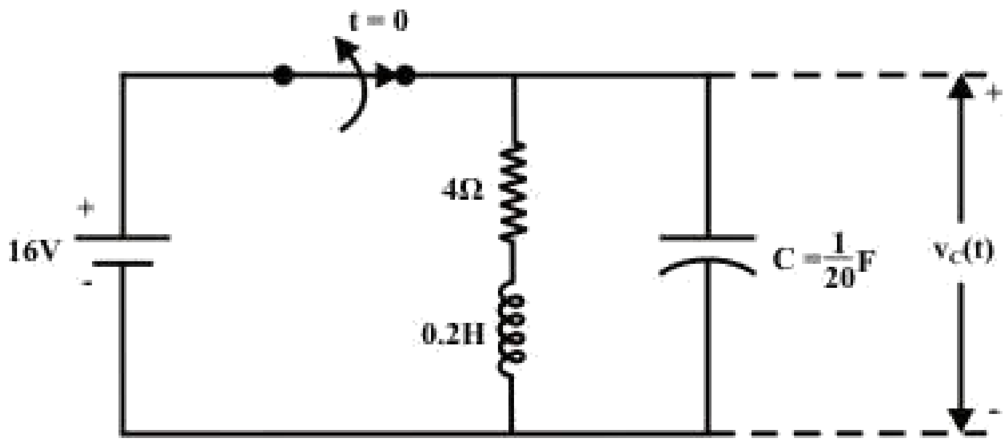


Fig. 11.13

Determine (i) $v_c(t)$ *for* $t \geq 0$ (ii) how long must the switch remain open for the voltage $v_c(t)$ to be less than 10% ot its value at $t = 0$? [10]

(Ans. (i) $(i)\ v_c(t) = \left(16 + 240t\right) e^{-10t}$ (ii) 0.705sec.)

T.11.11 For the circuit shown in Fig.11.14, find the capacitor voltage $v_c(t)$ and inductor current $i_L(t)$ for all t ($t < 0$ *and* $t \geq 0$). [10]

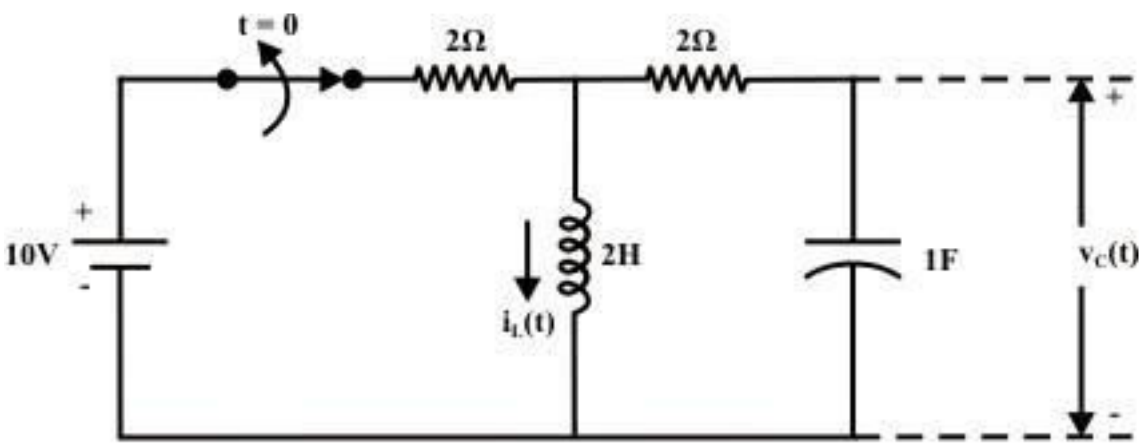


Fig. 11.14

Plot the wave forms $v_c(t)$ and $i_L(t)$ for $t \geq 0$.

(Ans. $v_{c(t)} = 10\, e^{-0.5t} \sin(0.5t)$; $i_L(t) = 5\left(\cos(0.5t) - \sin(0.5t)\right) e^{-0.5t}$)

T.11.12 For the parallel circuit *RLC* shown in Fig.11.15, Find the response

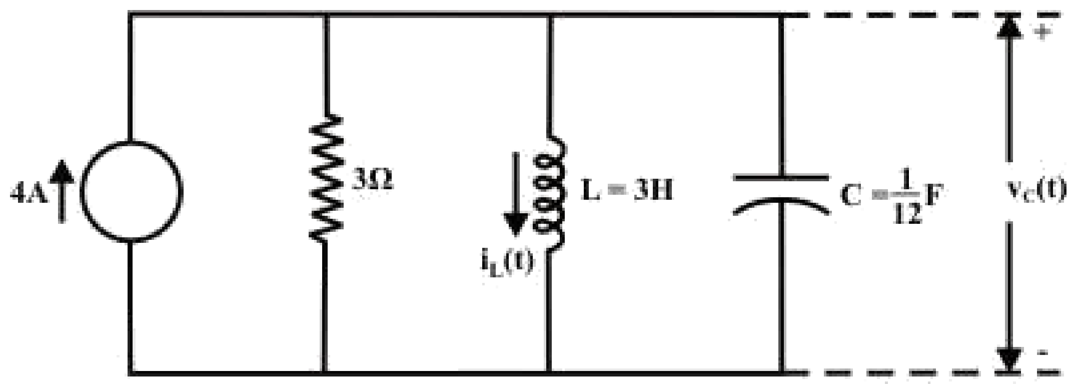


Fig. 11.15

of $i_L(t)$ *and* $v_c(t)$ respectively. [10]

(Ans. $i_L(t) = 4 - 4e^{-2t}(1+2t)$ *amps.* ; $v_c(t) = 48\,t\,e^{-2t}$ *volt.*)

UNIT-1

DC Circui

Superposition Theorem in the context of dc voltage and current sources acting in a resistive network

Objectives

- Statement of superposition theorem and its application to a resistive d.c network containing more than one source in order to find a current through a branch or to find a voltage across the branch.

1 Introduction

If the circuit has more than one independent (voltage and/or current) sources, one way to determine the value of variable (voltage across the resistance or current through a resistance) is to use nodal or mesh current methods as discussed in detailed in lessons 4 and 5. Alternative method for any linear network, to determine the effect of each independent source (whether voltage or current) to the value of variable (voltage across the resistance or current through a resistance) and then the total effects simple added. This approach is known as the superposition. In lesson-3, it has been discussed the properties of a linear circuit that satisfy (i) homogeneity property [response of output due to input= $\alpha u(t)$ equals to α times the response of output due to input= $u(t)$, $S(\alpha u(t)) = \alpha S(u(t))$ for all α; and $u(t)$ = input to the system] (ii) additive property [that is the response of $u_1(t) + u_2(t)$ equals the sum of the response of $u_1(t)$ and the response of $u_2(t)$, $S(u_1(t) + u_2(t)) = S(u_1(t)) + S(u_2(t))$]. Both additive and multiplicative properties of a linear circuit help us to analysis a complicated network. The principle of superposition can be stated based on these two properties of linear circuits.

1.1 Statement of superposition theorem

In any linear bilateral network containing two or more independent sources (voltage or current sources or combination of voltage and current sources), the resultant current / voltage in any branch is the algebraic sum of currents / voltages caused by each independent sources acting along, with all other independent sources being replaced meanwhile by their respective internal resistances.

Superposition theorem can be explained through a simple resistive network as shown in fig.7.1 and it has two independent practical voltage sources and one practical current source.

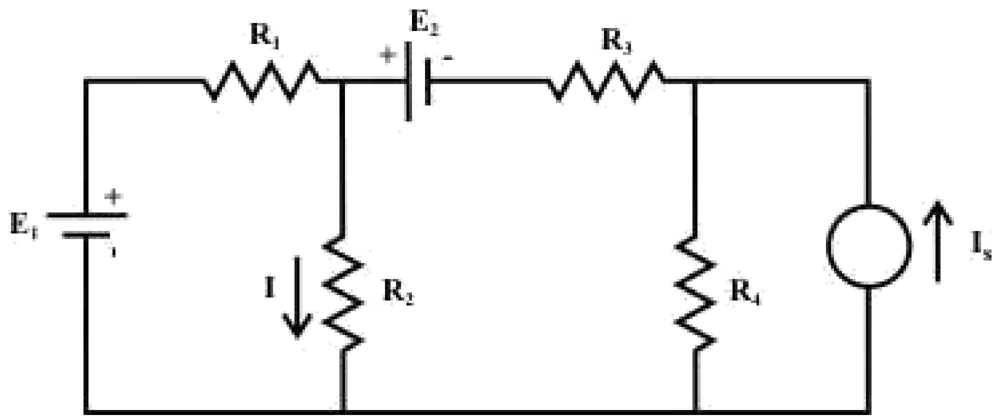


Fig. 7.1

One may consider the resistances R_1 *and* R_3 are the internal resistances of the voltage sources whereas the resistance R_4 is considered as internal resistance of the current source. The problem is to determine the response I in the in the resistor R_2 . The current I can be obtained from

$$I = I'\Big|_{\text{due to } E_1 (\text{ alone })} + I''\Big|_{\text{due to } E_2 (\text{ alone })} + I'''\Big|_{\text{due to } I_s (\text{ alone})}$$

according to the application of the superposition theorem. It may be noted that each independent source is considered at a time while all other sources are turned off or killed. To kill a voltage source means the voltage source is replaced by its internal resistance (i.e. R_1 or R_3 ; in other words E_1 *or* E_2 should be replaced temporarily by a short circuit) whereas to kill a current source means to replace the current source by its internal resistance (i.e. R_4 ; in other words I_s should be replaced temporarily by an open circuit).

Remarks: Superposition theorem is most often used when it is necessary to determine the individual contribution of each source to a particular response.

1.2 Procedure for using the superposition theorem

Step-1: Retain one source at a time in the circuit and replace all other sources with their internal resistances.

Step-2: Determine the output (current or voltage) due to the single source acting alone using the techniques discussed in lessons 3 and 4.

Step-3: Repeat steps 1 and 2 for each of the other independent sources.

Step-4: Find the total contribution by adding algebraically all the contributions due to the independent sources.

2 Application of superposition theorem

Example- L.7.1 Consider the network shown in fig. 7.2(a). Calculate I_{ab} *and* V_{cg} using superposition theorem.

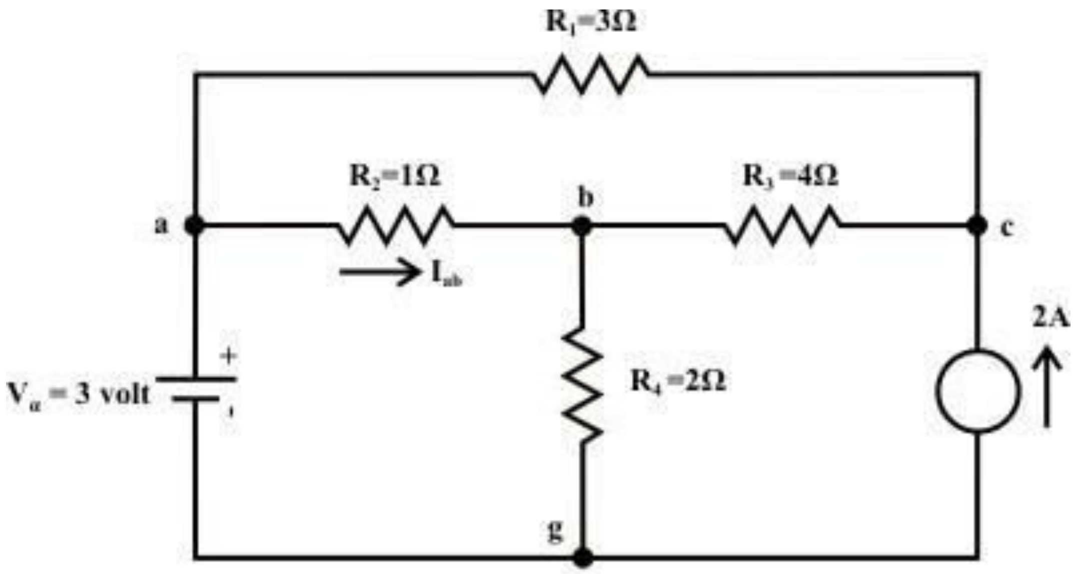


Fig. 7.2(a)

Solution: Voltage Source Only (retain one source at a time):

First consider the voltage source V_a that acts only in the circuit and the current source is replaced by its internal resistance (in this case internal resistance is infinite (∞)). The corresponding circuit diagram is shown in fig.7.2(b) and calculate the current flowing through the 'a-b' branch.

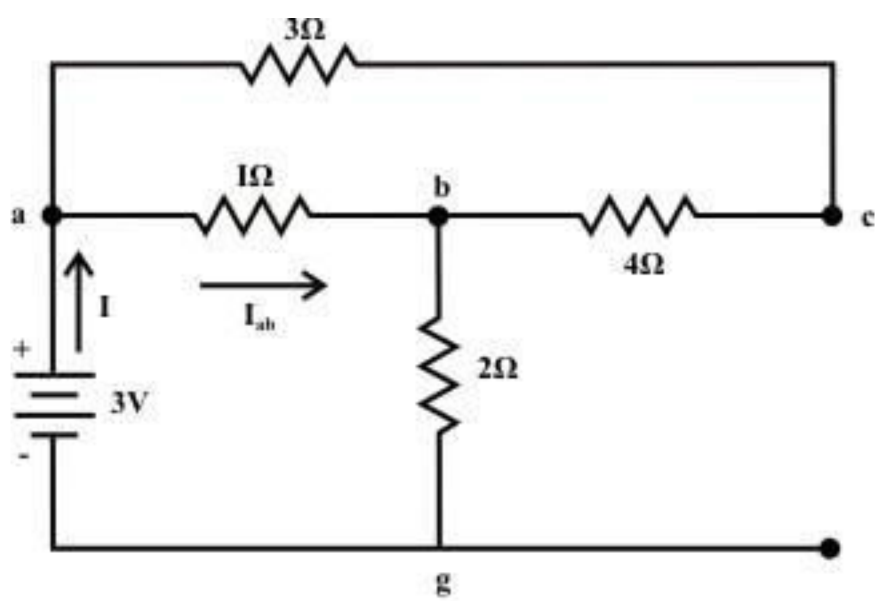


Fig. 7.2(b)

$$R_{eq} = [(R_{ac} + R_{cb}) \parallel R_{ab}] + R_{bg} = \frac{7}{8} + 2 = \frac{23}{8}\,\Omega$$

$I = \frac{3}{\frac{23}{8}} A = 1.043A$; Now current through a to b, is given by

$$I_{ab} = \frac{7}{8} \cdot \frac{24}{23} = 0.913\text{A} \text{ (a to b)}$$

$$I_{acb} = 1.043 - 0.913 = 0.13A$$

Voltage across c-g terminal :

$V_{cg} = V_{bg} + V_{cb} = 2 \cdot 1.043 + 4 \cdot 0.13 = 2.61 volts$ (Note: we are moving opposite to the direction of current flow and this indicates there is rise in potential). Note ' c ' is higher potential than ' g '.

Current source only (retain one source at a time):

Now consider the current source $I_s = 2\ A$ only and the voltage source V_a is replaced by its internal resistance which is zero in the present case. The corresponding the simplified circuit diagram is shown below (see fig.7.2(c)& fig.7.2(d)).

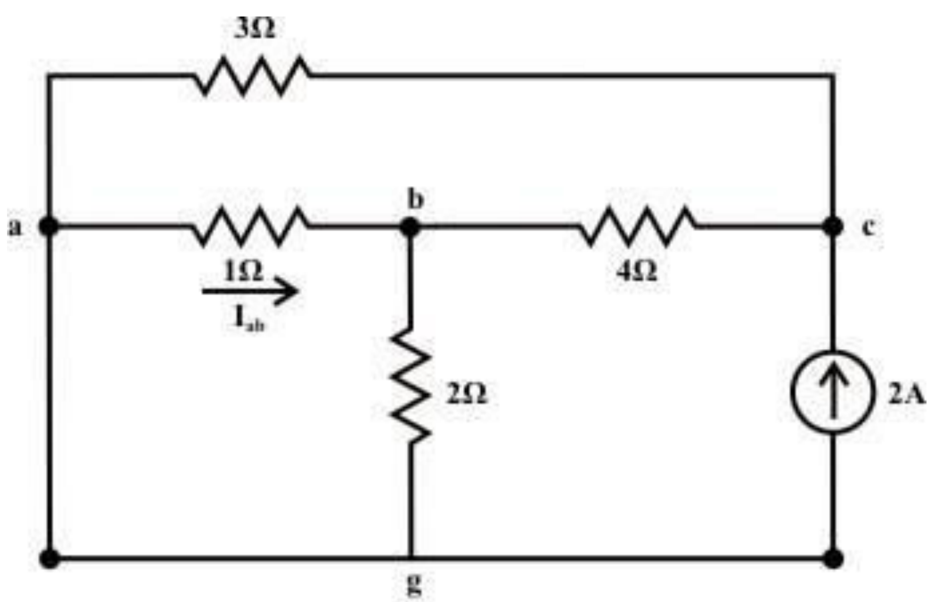


Fig. 7.2(c)

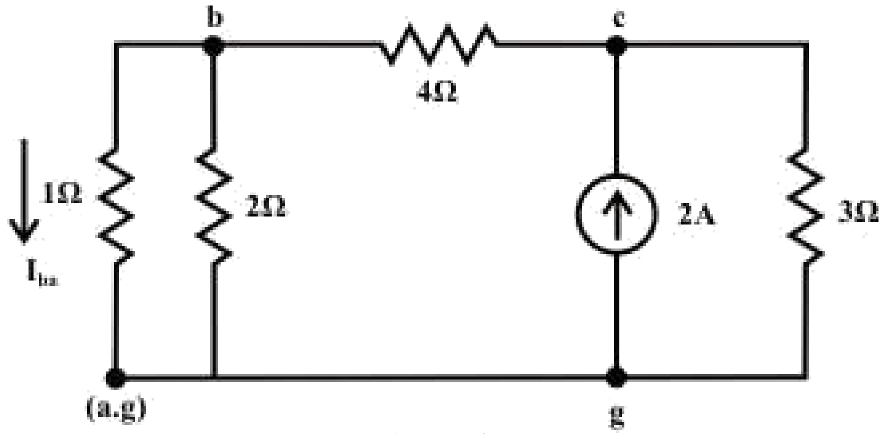


Fig. 7.2(d)

Current in the following branches:

$3\Omega\ resistor = \dfrac{(14/3)\cdot 2}{(14/3)+3} = 1.217\ A$; $\quad 4\ \Omega\ resistor = 2 - 1.217 = 0.783A$

$1\Omega\ resistor = \dfrac{2}{3}\cdot 0.783 = 0.522\ A\ (\ b\ to\ a)$

Voltage across 3Ω resistor (c & g terminals) $V_{cg} = 1.217 \cdot 3 = 3.651 volts$

The total current flowing through 1Ω resistor (due to the both sources) from a to b = 0.913 (due to voltage source only; current flowing from ' a ' to ' b ') – 0.522 (due to current source only; current flowing from ' b 'to ' a ') $= 0.391\ A$.

Total voltage across the current source $V_{cg} = 2.61 volt$ (due to voltage source ; ' c ' is higher potential than ' g ') + 3.651 $volt$ (due to current source only; ' c ' is higher potential than ' g ') = 6.26 $volt$.

Example L.7.2 For the circuit shown in fig.7.3(a), the value of V_{s1} *and* I_s are fixed. When $V_{s\,2} = 0$, the current $I = 4\ A$. Find the value of I when $V_{s\,2} = 32V$.

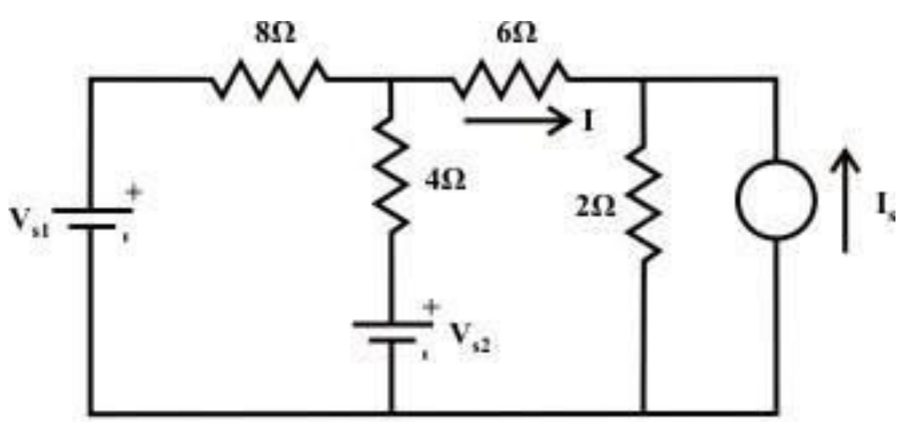


Fig. 7.3(a)

Solution: Let us assume that the current flowing 6 Ω resistors due to the voltage and current sources are given by (assume circuit linearity)

$$I = \alpha V_{s1} + \beta V_{s2} + \eta I_s = I'_{(\text{due to } V_{s1})} + I''_{(\text{due to } V_{s2})} + I'''_{(\text{due to } I_s)} \tag{7.1}$$

where the parameters α, β, *and* η represent the positive constant numbers. The parameters α *and* β are the total conductance of the circuit when each voltage source acting alone in the circuit and the remaining sources are replaced by their internal resistances. On the other hand, the parameter η represents the total resistance of the circuit when the current source acting alone in the circuit and the remaining voltage sources are replaced by their internal resistances. The expression (7.1) for current I is basically written from the concept of superposition theorem.

From the first condition of the problem statement one can write an expression as (when the voltage source V_{s1} and the current source I_s acting jointly in the circuit and the other voltage source V_{s2} is not present in the circuit.)

$$4 = I = \alpha V_{s1} + \eta I_s = I'_{(\text{due to } V_{s1})} + I'''_{(\text{due to } I_s)} \text{ (Note both the sources are fixed)} \tag{7.2}$$

Let us assume the current following through the 6 Ω resistor when all the sources acting in the circuit with $V_{s2} = 32V$ is given by the expression (7.1). Now, one can determine the current following through 6 Ω resistor when the voltage source $V_{s2} = 32V$ acting alone in the circuit and the other sources are replaced by their internal resistances. For the circuit shown in fig.7.3 (b), the current delivered by the voltage source to the 6 Ω resistor is given by

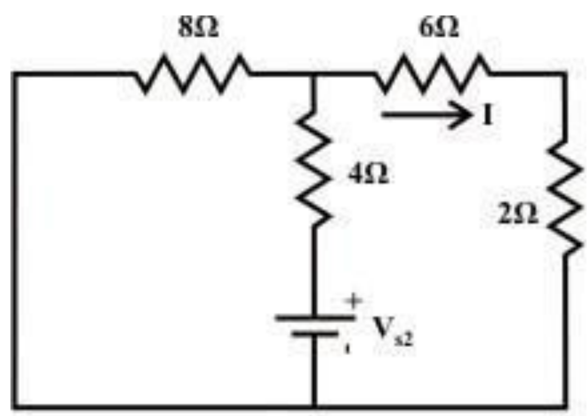


Fig. 7.3(b)

$$I_1 = \frac{V_{S2}}{R_{eq}} = \frac{32}{(8 \| 8) + 4} = 4\,A \tag{7.3}$$

The current following through the 6 Ω due to the voltage source $V_{S2} = 32V$ only is 2 A (flowing from left to right; i,e. in the direction as indicated in the figure 7.3(b)). Using equation (7.1), the total current I flowing the 6 Ω resistor can be obtained as

$$I = \alpha V_{s1} + \beta V_{s2} + \eta I_s = I'_{(due\ to\ V_{s1})} + I''_{(due\ to\ V_{s2})} + I'''_{(due\ to\ I_s)} = I'_{(due\ to\ V_{s1})} + I'''_{(due\ to\ I_s)} + I''_{(due\ to\ V_{s2})}$$

$$= 4A + 2A = 6A \text{ (note: } I'_{due\ to\ V_{s1}} + I'''_{due\ to\ I_s}) = 4\,A \text{ (see eq. 7.2)}$$

Example L.7.3: Calculate the current I_{ab} flowing through the resistor 3Ω as shown in fig.7.4(a), using the superposition theorem.

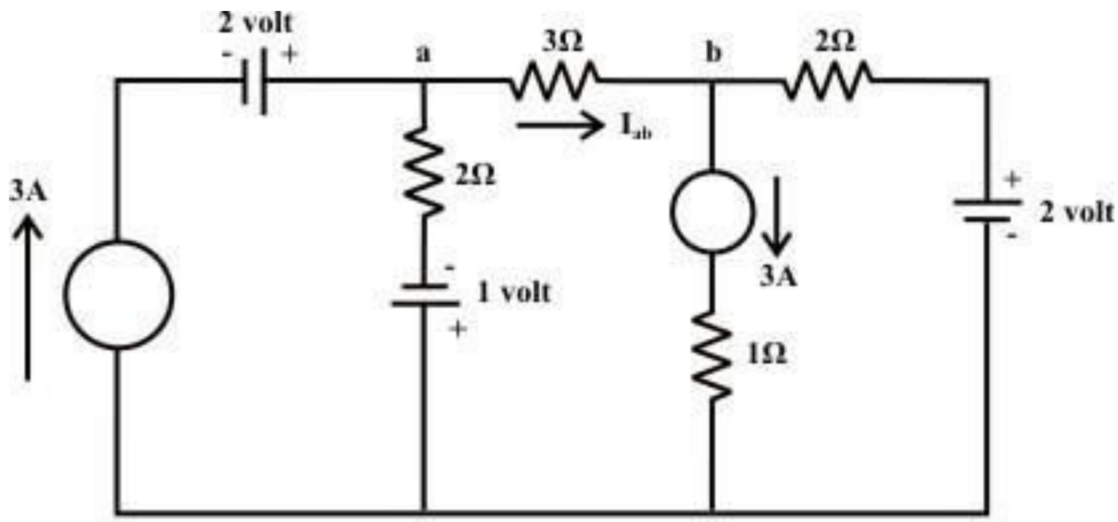


Fig. 7.4 (a)

Solution: Assume that the current source 3 A (left to the 1*volt* source) is acting alone in the circuit and the internal resistances replace the other sources. The current flowing through 3Ω resistor can be obtained from fig.7.4(b)

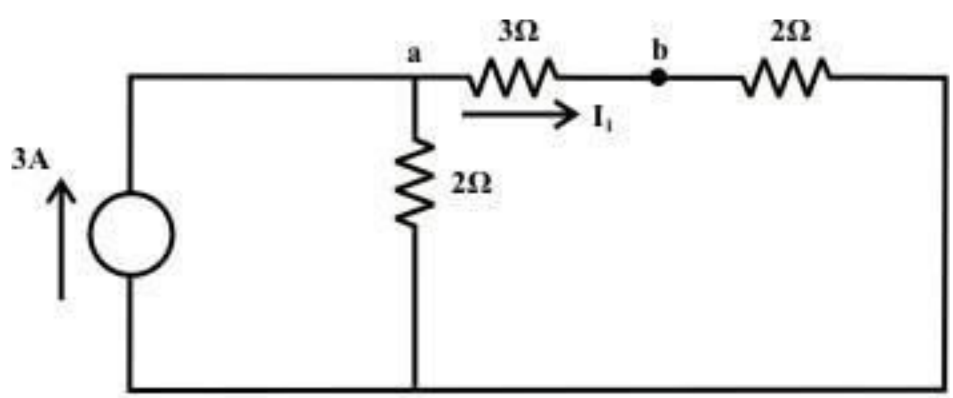


Fig. 7.4 (b)

and it is given by

$$I_{1(due\ to\ 3\,A\ current\ source)} = 3 \cdot \frac{2}{7} = \frac{6}{7} A\ (a\ to\ b) \tag{7.4}$$

Current flowing through 3Ω resistor **due to** $2V$ **source (only)** can be obtained from fig.7.4(c)

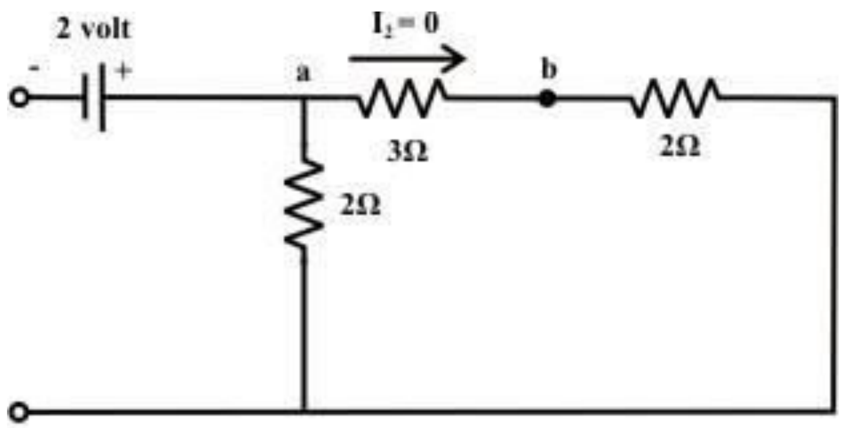


Fig. 7.4 (c)

and it is seen from no current is flowing.

$$I_{2(\text{ due to } 2V \text{ voltage source})} = 0A\,(a \text{ to } b) \tag{7.5}$$

Current through 3 Ω resistor **due to** $1V$ **voltage source only** (see fig.7.3(d)) is given by

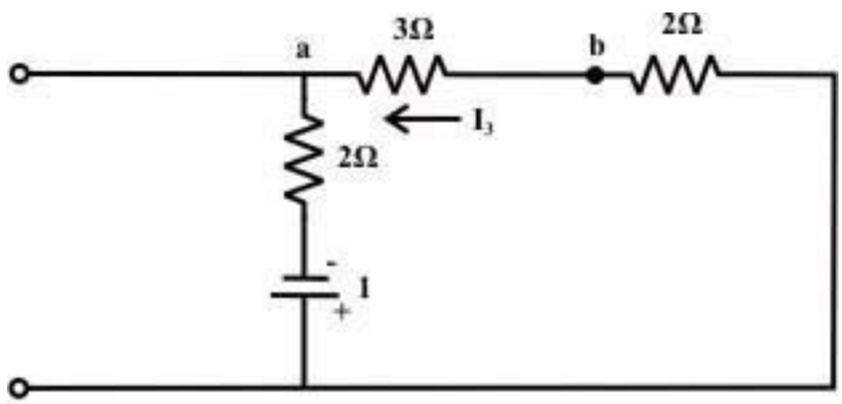


Fig. 7.4 (d)

$$I_{3(\text{ due to } 1V \text{ voltage source})} = \frac{1}{7}A\ (\,b \text{ to } a) \tag{7.6}$$

Current through 3 Ω resistor **due to** 3 A **current source only** (see fig.7.3(e)) is obtained by

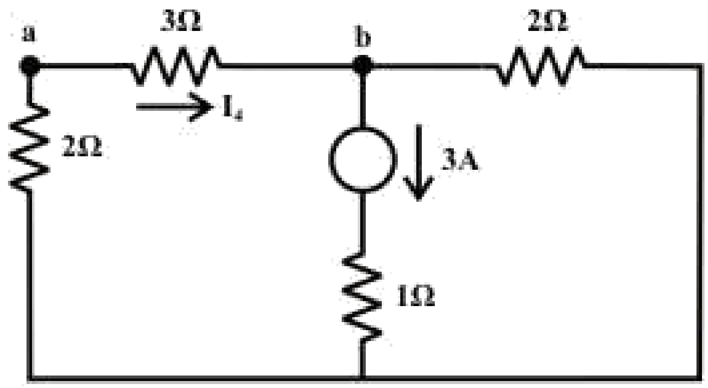


Fig. 7.4 (e)

$$I_{4(\text{due to } 3A \text{ current source})} = 3 \cdot \frac{2}{7} = \frac{6}{7} A \;(a \text{ to } b) \qquad (7.7)$$

Current through 3 Ω resistor **due to** 2*V* **voltage source only** (see fig.7.3(f)) is given by

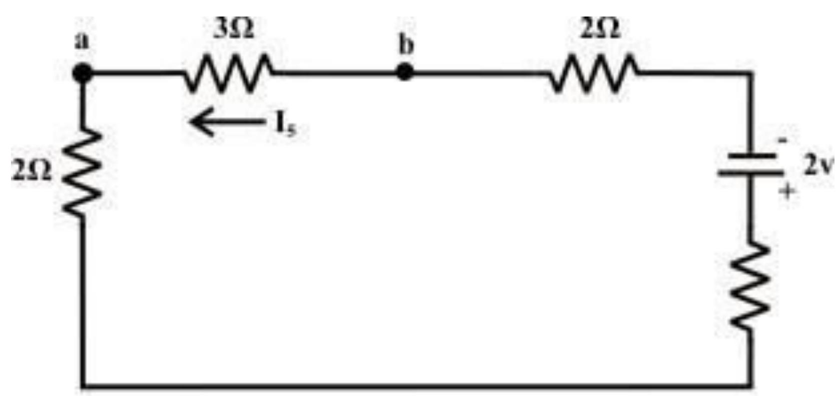


Fig. 7.4 (f)

$$I_{5(\text{due to } 2V \text{ voltage source})} = \frac{2}{7} A \;(b \text{ to } a) \qquad (7.8)$$

Resultant current I_{ab} flowing through 3Ω resistor due to the combination of all sources is obtained by the following expression (the algebraic sum of all currents obtained in eqs. (7.4)-(7.8) with proper direction of currents)

$$I_{ab} = I_{1(\text{due to } 3A \text{ current source})} + I_{2(\text{due to } 2V \text{ voltage source})} + I_{3(\text{due to } 1V \text{ voltage source})} + I_{4(\text{due to } 3A \text{ current source})} + I_{5(\text{due to } 2V \text{ voltage source})}$$

$$= \frac{6}{7} + 0 - \frac{1}{7} + \frac{6}{7} - \frac{2}{7} = \frac{9}{7} = 1.285 \;(a \text{ to } b)$$

3 Limitations of superposition Theorem

- Superposition theorem doesn't work for power calculation. Because **power calculations** involve either the product of voltage and current, the square of current or the square of the voltage, they are not linear operations. This statement can be explained with a simple example as given below.

Example: Consider the circuit diagram as shown in fig.7.5.

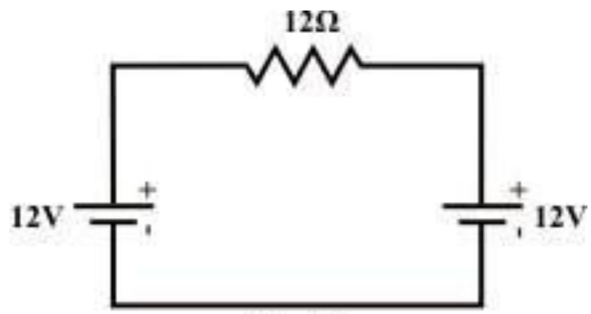


Fig. 7.5

Using superposition theorem one can find the resultant current flowing through 12 Ω resistor is zero and consequently power consumed by the resistor is also zero. For power consumed in an any resistive element of a network can not be computed using superposition theorem. Note that the power consumed by each individual source is given by

$P_{W1(\text{due to } 12\,V\,\text{source (left)})} = 12\,watts$; $P_{W2(\text{due to } 12V\,\text{source (right)})} = 12\ watts$

The total power consumed by 12 Ω = 24 *watts* (applying superposition theorem). This result is wrong conceptually. In fact, we may use the superposition theorem to find a current in any branch or a voltage across any branch, from which power is then can be calculated.

- Superposition theorem can not be applied for non linear circuit (Diodes or Transistors).
- This method has weaknesses:- In order to calculate load current I_L or the load voltage V_L for the several choices of load resistance R_L of the resistive network , one needs to solve for every source voltage and current, perhaps several times. With the simple circuit, this is fairly easy but in a large circuit this method becomes an painful experience.

4 Test Your Understanding

[Marks: 40]

T.7.1 When using the superposition theorem, to find the current produced independently by one voltage source, the other voltage source(s) must be ----------- and the current source(s) must be --------------. [2]

T.7.2 For a linear circuit with independent sources p_1 , p_2 , p_3 ……… p_n and if y_i is the response of the circuit to source p_i , with all other independent sources set to zero), then resultant response $y =$. [1]

T.7.3 Use superposition theorem to find the value of the voltage v_a in fig.7.6. [8]

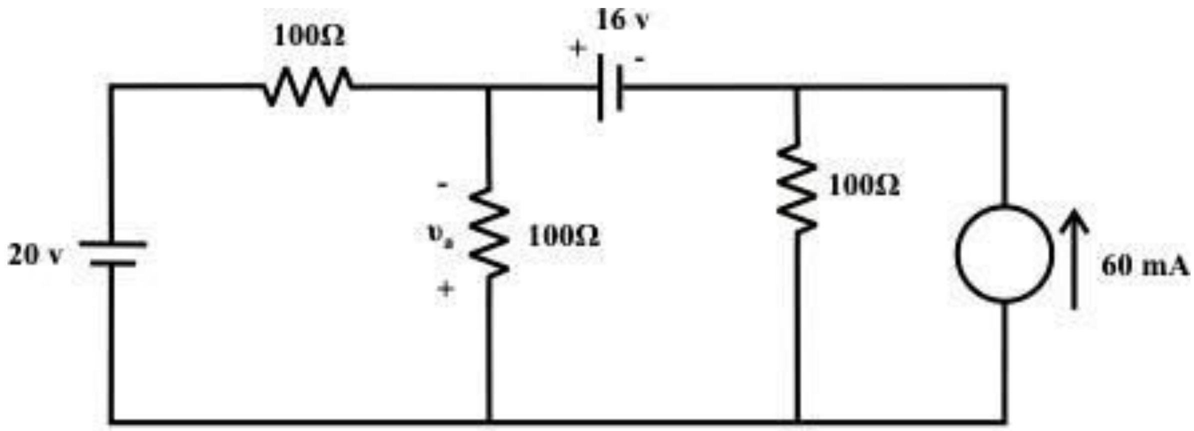


Fig. 7.6

(Ans. 14 *volts*)

T.7.4 For the circuit shown in fig.7.7, calculate the value of source current I_x that yields $I = 0$ if V_A *and* V_C are kept fixed at 7 V *and* 28V . [7]

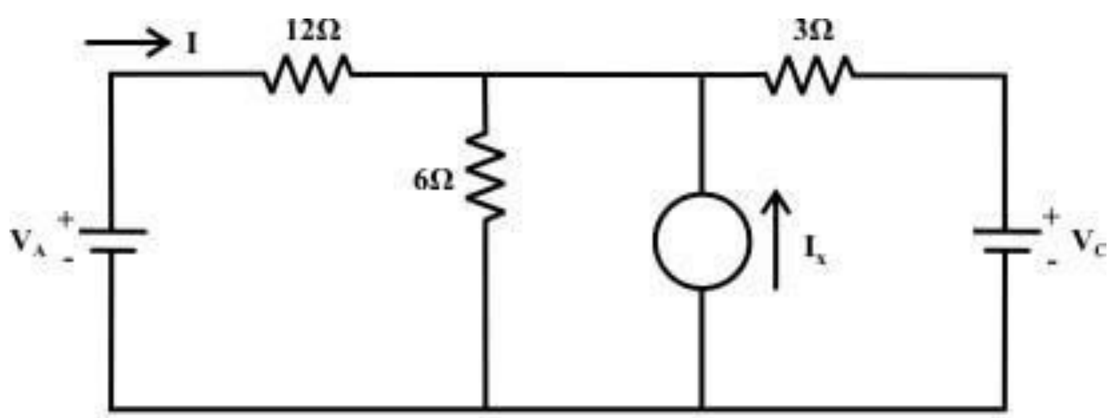


Fig. 7.7

(Ans. $I_x = -5.833\ A$)

T.7.5 For the circuit shown below (see fig.7.8), it follows from linearly that we can write $V_{ab} = \alpha I_x + \beta V_A + \eta V_B$, *where* α, β, *and* η *are* constants. Find the values of (i) (*i*)α (*ii*) β *and* (*iii*) η . [7]

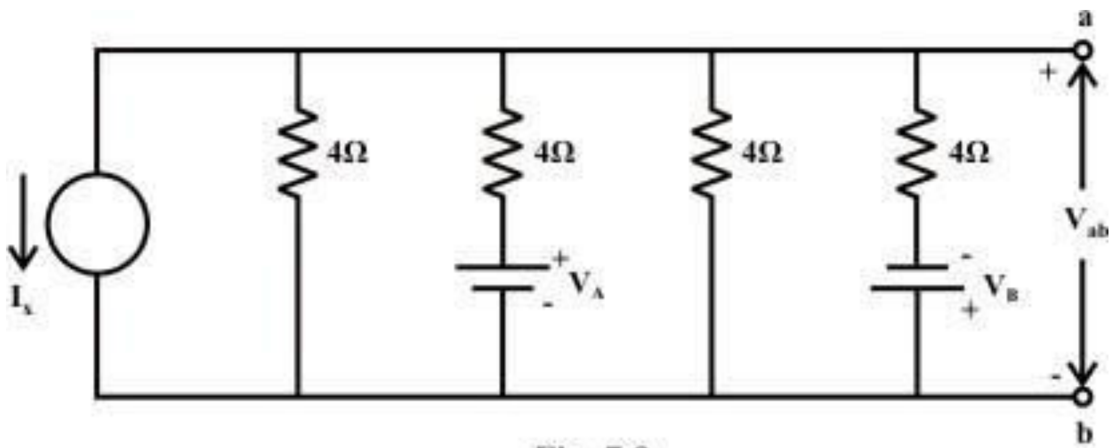


Fig. 7.8

(Ans. $\alpha = -1$; $\beta = 0.063$; *and* $\eta = -0.063$)

T.7.6 Using superposition theorem, find the current i through $5\ \Omega$ resistor as shown in fig.7.9. [8]

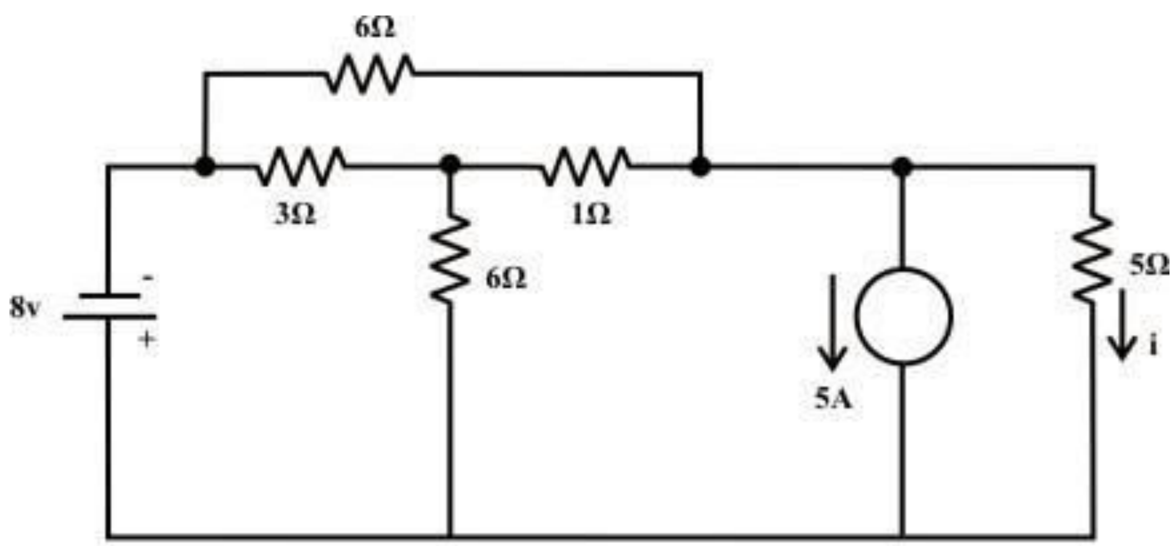


Fig. 7.9

(Ans. $-0.538\ A$)

T.7.7 Consider the circuit of fig.7.10

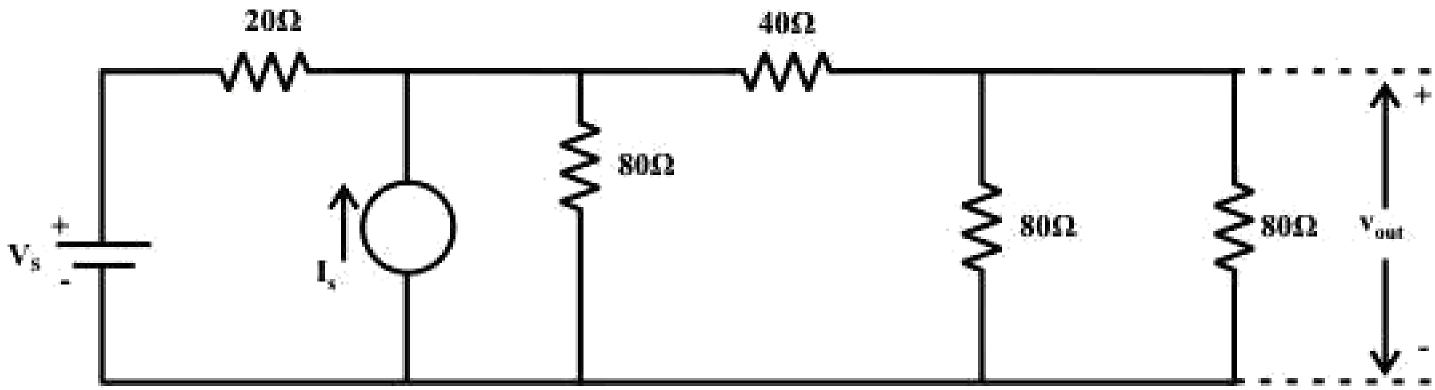


Fig. 7.10

(a) Find the linear relationship between V_{out} and input sources V_s *and* I_s

(b) If $V_s = 10\ V\ and\ I_s = 1, find\ V_{out}$

(c) What is the effect of doubling all resistance values on the coefficients of the linear relationship found in part (a)? [7]

(Ans. (a) $V_{out} = 0.3333V_s + 6.666\ I_s$; (b)$V_{out} = 9.999V$ (c) $V_{out} = 0.3333V_s + 13.332I_s$)

UNIT-1

DC Circuit

Loop Analysis of resistive circuit in the context of dc voltages and currents.

Objectives

- Meaning of circuit analysis; distinguish between the terms mesh and loop.
- To provide more general and powerful circuit analysis tool based on Kirchhoff's voltage law (KVL) only.

1 Introduction

The Series-parallel reduction technique that we learned in lesson-3 for analyzing DC circuits simplifies every step logically from the preceding step and leads on logically to the next step. Unfortunately, if the circuit is complicated, this method (the simplify and reconstruct) becomes mathematically laborious, time consuming and likely to produce mistake in calculations. In fact, to elevate these difficulties, some methods are available which do not require much thought at all and we need only to follow a well-defined faithful procedure. One most popular technique will be discussed in this lesson is known as 'mesh or loop' analysis method that based on the fundamental principles of circuits laws, namely, Ohm's law and Kirchhoff's voltage law. Some simple circuit problems will be analyzed by hand calculation to understand the procedure that involve in mesh or loop current analysis.

1.1 Meaning of circuit analysis

The method by which one can determine a variable (either a voltage or a current) of a circuit is called analysis. Basic difference between 'mesh' and 'loop' is discussed in lesson-3 with an example. A 'mesh' is any closed path in a given circuit that does not have any element (or branch) inside it. A mesh has the properties that (i) every node in the closed path is exactly formed with two branches (ii) no other branches are enclosed by the closed path. Meshes can be thought of a resembling window partitions. On the other hand, 'loop' is also a closed path but inside the closed path there may be one or more than one branches or elements.

2.Solution of Electric Circuit Based on Mesh (Loop) Current Method

Let us consider a simple dc network as shown in Figure 4.1 to find the currents through different branches using Mesh (Loop) current method

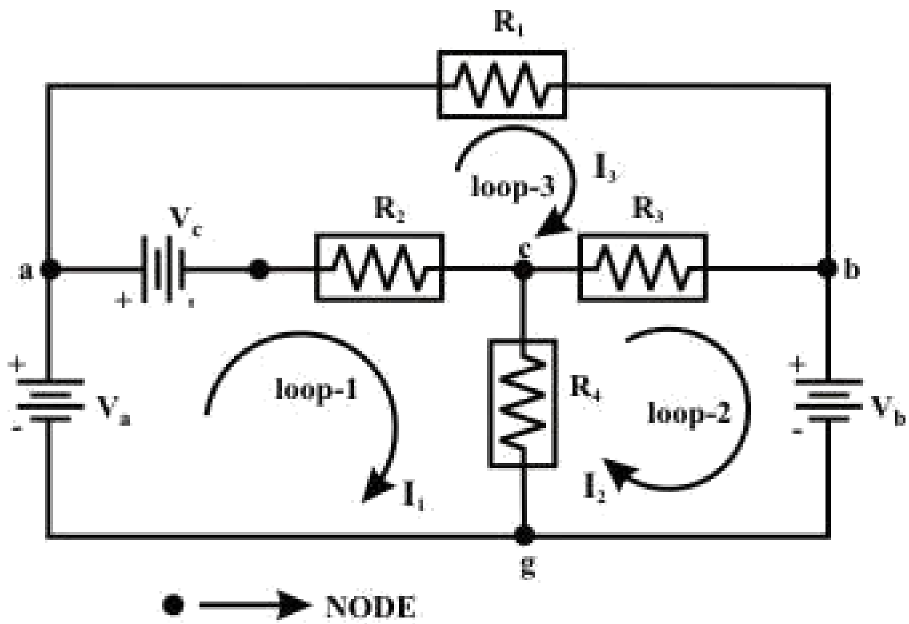


Figure 4.1

Applying KVL around mesh (loop)-1:(note in mesh-1, I_1 is known as local current and other mesh currents I_2 & I_3 are known as foreign currents.)

$$V_a - V_c - (I_1 - I_3) R_2 - (I_1 - I_2) R_4 = 0$$

$$V_a - V_c = (R_2 + R_4) I_1 - R_4 I_2 - R_2 I_3 = R_{11} I_1 - R_{12} I_2 - R_{13} I_3 \qquad (4.1)$$

Applying KVL around mesh (loop)-2:(similarly in mesh-2, I_2 is local current and I_1 & I_3 are known as foreign currents)

$$-V_b - (I_2 - I_3) R_3 - (I_2 - I_1) R_4 = 0$$

$$-V_b = -R_4 I_1 + (R_3 + R_4) I_2 - R_3 I_3 = -R_{21} I_1 + R_{22} I_2 - R_{23} I_3 \qquad (4.2)$$

Applying KVL around mesh (loop)-3:

$$V_c - I_3 R_1 - (I_3 - I_2) R_3 - (I_3 - I_1) R_2 = 0$$

$$V_c = -R_2 I_1 - R_3 I_2 + (R_1 + R_2 + R_3) I_3 = -R_{31} I_1 - R_{32} I_2 + R_{33} I_3 \qquad (4.3)$$

** In general, we can write for i^{th} mesh (for $i = 1, 2, ... N$)

$$\sum V_{ii} = -R_{i1} I_1 - R_{i2} I_2 \, ... + R_{ii} I_i - R_{i,i+1} I_{i+1} - .... R_{iN} I_N$$

$\sum V_{ii} \rightarrow$ simply und the mesh.

Note: Generally, $R_{ij} = R_{ji}$ (true only for linear bilateral circuits)

I_i → the unknown mesh currents for the network.

Summarize:

Step-I: Draw the circuit on a flat surface with no conductor crossovers.

Step-2: Label the mesh currents (I_i) carefully in a clockwise direction.

Step-3: Write the mesh equations by inspecting the circuit (No. of independent mesh (loop) equations=no. of branches (b) - no. of principle nodes (n) + 1).

Note:

To analysis, a resistive network containing voltage and current sources using 'mesh' equations method the following steps are essential to note:

- If possible, convert current source to voltage source.
- Otherwise, define **the voltage** across the current source and write the mesh equations as if these source voltages were known. Augment the set of equations with one equation for each current source expressing a known mesh current or difference between two mesh currents.
- Mesh analysis is valid only for circuits that can be drawn in a two-dimensional plane in such a way that no element crosses over another.

Example-L-4.1: Find the **current** through 'ab-branch' (I_{ab}) and **voltage (**V_{cg} **)** across the current source using Mesh-current method.

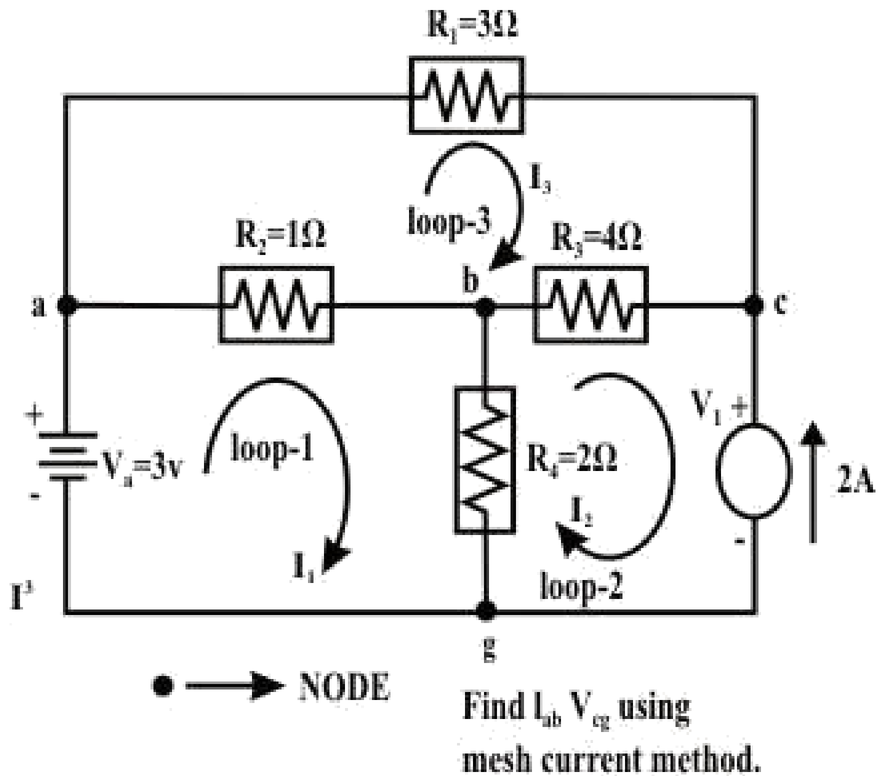


Figure 4.2

Solution: Assume voltage across the current source is v_1 ('c' is higher potential than 'g' (ground potential and assumed as zero potential) and note I_2 = *-2A* (since assigned current direction (I_2) is opposite to the source current)

Loop - 1: (Appling KVL)

$$V_a - (I_1 - I_3) R_2 - (I_1 - I_2) R_4 = 0 \quad 3 = 3I_1 - 2I_2 - I_3$$

$$3I_1 - I_3 = -1 \tag{4.4}$$

Loop - 2: (Appling KVL)

Let us assume the voltage across the current source is v_1 and its top end is assigned with a positive sign.

$$-v_1 - (I_2 - I_1) R_4 - (I_2 - I_3) R_3 = 0 - v_1 = -2I_1 + 6I_2 - 4I_3$$

$$2I_1 + 12 + 4I_3 = v_1 \qquad \text{(note: } I_2 = -2A\text{)} \tag{4.5}$$

Loop - 3: (Appling KVL)

$$-I_3 R_1 - (I_3 - I_2) R_3 - (I_3 - I_1) R_2 = 0 \quad -I_1 - 4I_2 + 8I_3 = 0$$

$$I_1 - 8I_3 = 8 \qquad \text{(Note, } I_2 = -2A\text{)} \tag{4.6}$$

Solving equations (4.4) and (4.6), we get $I_1 = -\frac{48}{69} = -0.6956\,A$ and

$I_3 = -\frac{25}{23} = -1.0869\,A$, $I_{ab} = I_1 - I_3 = 0.39A$, $I_{bc} = I_2 - I_3 = -0.913A$ and

$I_{bg} = I_1 - I_2 = 1.304\,A$

- ve sign of current means that the current flows in reverse direction (in our case, the current flows through 4Ω resistor from 'c' to 'b' point). From equation (4.5), one can get $v_1 ==$ *6.27 volt.*

Another way: $-v_1 + v_{bg} + v_{bc} = 0$ $v_1 = v_{cg} = 6.27$ *volt.*

Example-L-4.2 For the circuit shown Figure 4.3 (a) find V_x using the mesh current method.

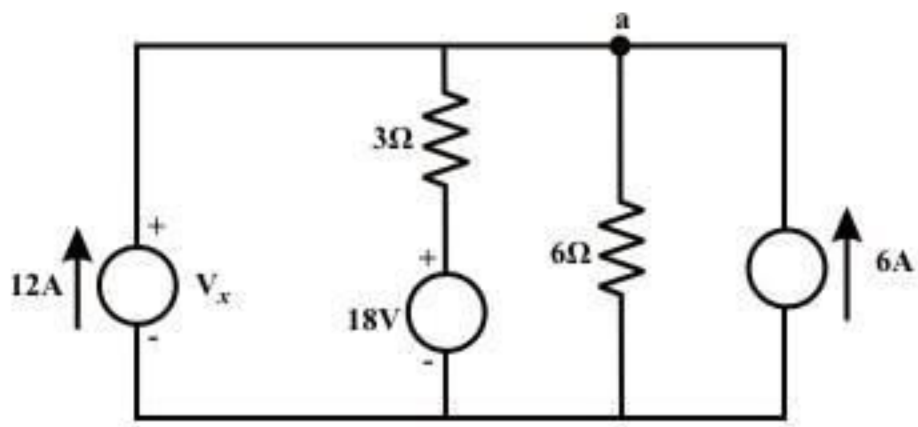


Fig. 4.3(a)

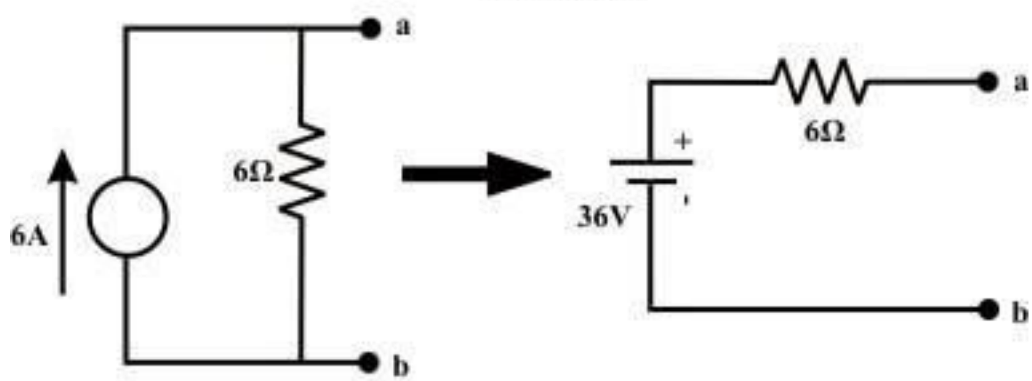


Fig. 4.3(b)

Solution: One can easily convert the extreme right current source (6 A) into a voltage source. Note that the current source magnitude is 6 A and its internal resistance is 6 Ω . The given circuit is redrawn and shown in Figure 4.3 (c)

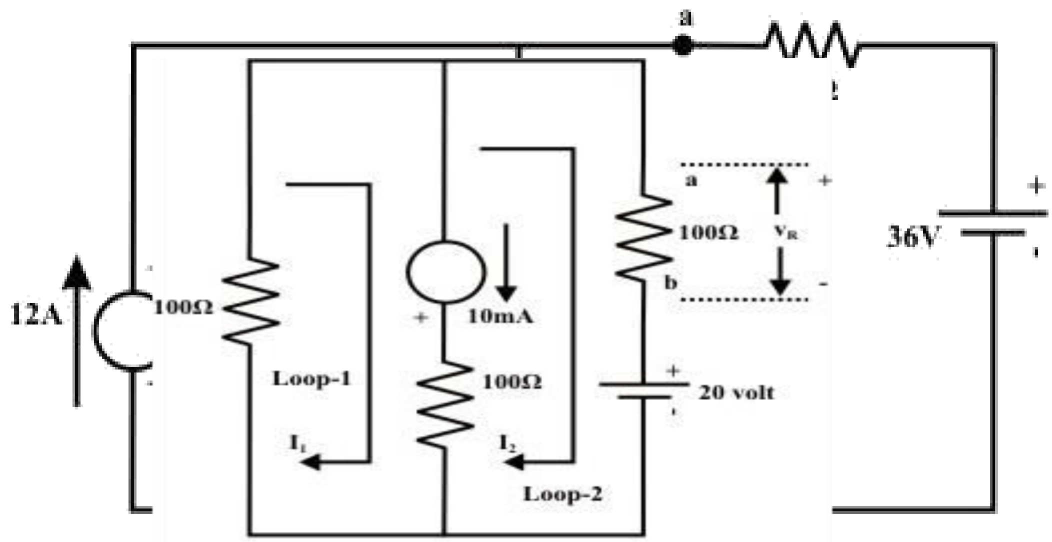


Figure 4.4

Loop-1: (Write KVL, note $I_1 = 12\,A$)

$V_x - (I_1 - I_2)\cdot 3 - 18 = 0 \quad V_x + 3\,I_2 = 54$ (4.7)

Loop-2: (write KVL)

$18 - (I_2 - I_1)\cdot 3 - I_2 \cdot 6 - 36 = 0 \qquad 9\,I_2 = 18 \quad I_2 = 2\,A$

Using the value of $I_2 = 2\,A$ in equation (4.7), we get $V_x = 48\,volt$.

Example-L-4.3 Find v_R for the circuit shown in figure 4.4 using 'mesh current method. Calculate the power absorbed or delivered by the sources and all the elements.

Solution: Assume the voltage across the current source is ' v ' and the bottom end of current source is marked as positive sign.

For loop No. 1: (KVL equation)

$v - (I_1 - I_2)\cdot 100 - I_1 \cdot 100 = 0 \quad v - 200\,I_1 + 100\,I_2 = 0$ (4.8)

It may be noted that from the figure that the current flowing through the 100 Ω resistor (in the middle branch) is $10\,mA$. More specifically, one can write the following expression

$I_1 - I_2 = 10\cdot 10^{-3}$ (4.9)

For loop No. 2: (KVL equation)

$-20 - (I_2 - I_1)\cdot 100 - v - I_2 \cdot 100 = 0 \quad v + 200\,I_2 - 100\,I_1 = -20$ (4.10)

Solving equations (4.8)–(4.10), one can obtained the loop currents as $I_1 = -0.095 = -95\,mA$ (-ve sign indicates that the assigned loop current direction is not correct or in other words loop current (I_1) direction is anticlockwise.) and

$I_2 = -0.105 = -105\ mA$ (note, loop current (I_2) direction is anticlockwise). Now the voltage across the $100\,\Omega$ resistor (extreme right branch) is given by $v_R = I_2 \cdot 100 = -0.105 \cdot 100 = -10.5\ volt.$ .This indicates that the resistor terminal (b) adjacent to the voltage source is more positive than the other end of the resistor terminal (a). From equation (4.8) $v = -8.5\ volt$ and this implies that the 'top' end of the current source is more positive than the bottom 'end'.

Power delivered by the voltage source = $20 \cdot 0.105 = 2.1W$ (note that the current is leaving the positive terminal of the voltage source). On the other hand, the power received or absorbed by the current source = $8.5 \cdot 0.01 = 0.085W$ (since current entering to the positive terminal (top terminal) of the current source). Power absorbed by the all resistance is given

$= (0.105)^2 \cdot 100 + (0.095)^2 \cdot 100 + (10 \cdot 10^{-3})^2 \cdot 100 = 2.015W$.

Further one can note that the power delivered ($P_d = 2.1W$) = power absorbed ($P_{ab} =$ $0.085 + 2.015 = 2.1W$) = $2.1W$

---

3 Test Your Understanding

[Marks:50]

T.4.1 To write the Kirchhoff's voltage law equation for a loop, we proceed clockwise around the loop, considering voltage rises into the loop equation as ------- terms and voltage drops as -------- terms. [2]

T.4.2 When writing the Kirchhoff's voltage law equation for a loop, how do we handle the situation when an ideal current source is present around the loop? [2]

T.4.3 When a loop current emerges with a positive value from mathematical solution of the system of equations, what does it mean? What does it mean when a loop current emerges with a negative value? [2]

T.4.4 In mesh current method, the current flowing through a resistor can be computed with the knowledge of ------ loop current and ---------- loop current. [2]

T.4.5 Find the current through $6\ \Omega$ resistor for the circuit Figure 4.5 using 'mesh current' method and hence calculate the voltage across the current source. [10]

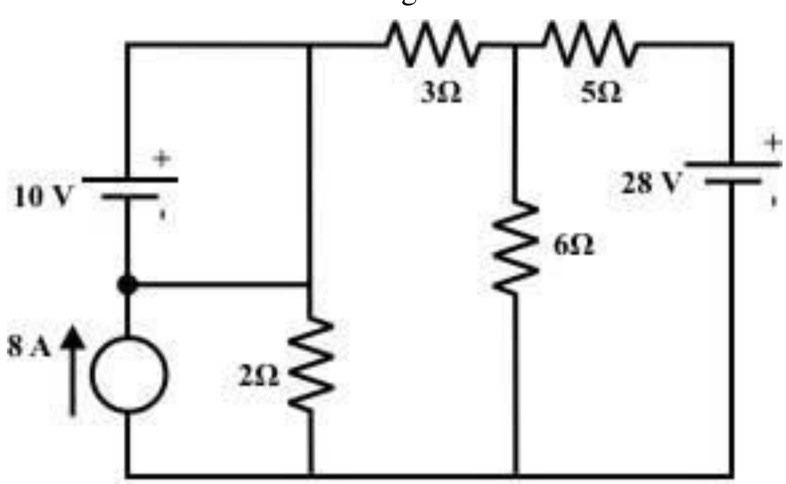


Figure 4.5

(Answer: 3.18 A ; 13.22 V)

T.4.6 For the circuit shown in Figure 4.6, find the current through I_{AB} , I_{AC} , I_{CD} and I_{EF} using 'mesh current' method. [12]

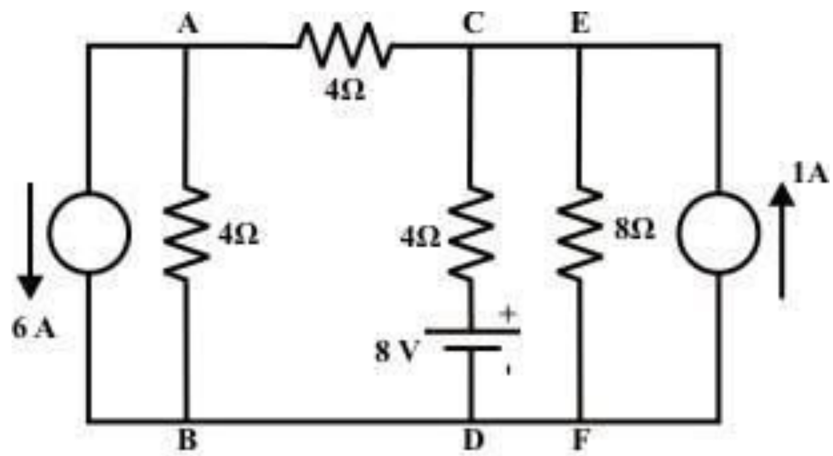


Figure 4.6

(Answer: $I_{AB} = -3\ A$; $I_{AC} = -3\ A$; $I_{CD} = -2\ A$ and $I_{EF} = 0\ A$.)

T.4.7 Find the current flowing through the $R_L = 1k\Omega$ resistor for the circuit shown in Figure 4.7 using 'mesh current' method. What is the power delivered or absorbed by the independent current source? [10]

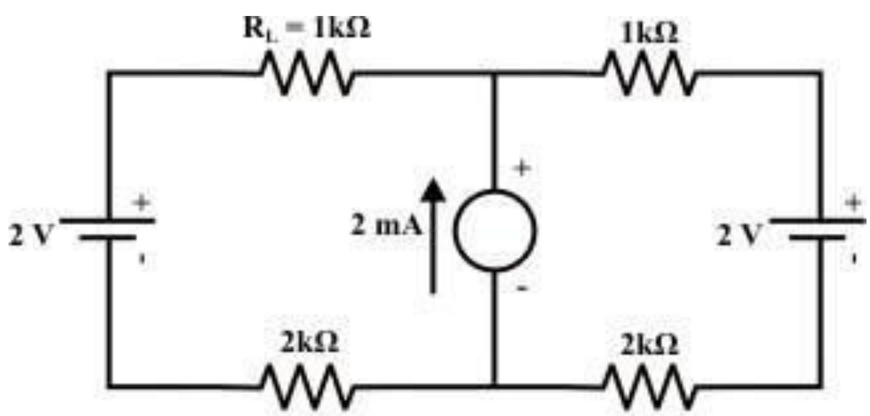


Figure 4.7

(Answer: 1 mA;10 mW)

T.4.8 Using 'mesh current' method, find the current flowing through 2 Ω resistor for the circuit shown in Figure 4.8 and hence compute the power consumed by the same 2 Ω resistor. [10]

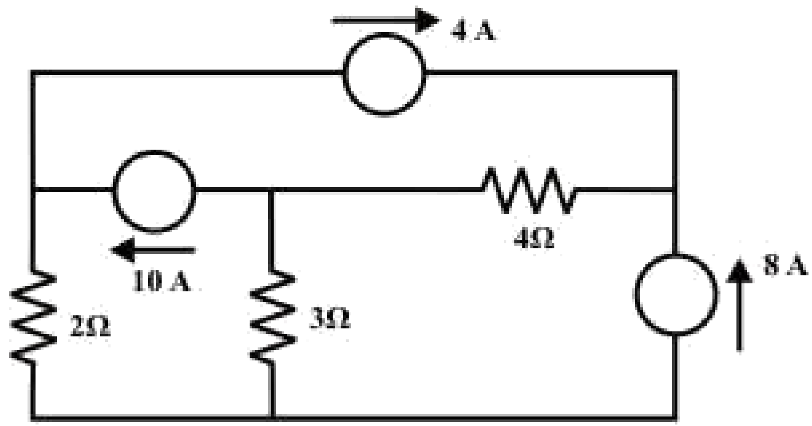


Figure 4.8

(Answer: 6 A; 72W)

UNIT-2

Three-phase AC Circuits

Three-phaseBalanced Supply

In the module, containing six lessons (12-17), the study of circuits, consisting of the linear elements – resistance, inductance and capacitance, fed from single-phase ac supply, has been presented. In this module, which may also be termed as an extension of the previous one, containing three lessons (18-20), the solution of currents in the balanced circuits, fed from three-phase ac supply, along with the measurement of power, will be described.

In this (first) lesson of this module, the generation of three-phase balanced voltages is taken up first. Then, the two types of connections (star and delta), normally used for the above supply, followed by line and phase quantities (voltages and currents) for the connections, in both supply and load sides (both being balanced), are described.

Keywords: Three-phase balanced voltage, star- and delta-connections, balanced load.

After going through this lesson, the students will be able to answer the following questions:

1. How to generate three-phase balanced voltages?
2. What are the two types of connections (star and delta) normally used for three-phase balanced supply?
3. What are meant by the terms – line and phase quantities (voltages and currents), for the two types of connections in both supply and load sides (both being balanced)

Generation of Three-phase Balanced Voltages

In the first lesson (No. 12) of the previous module, the generation of single-phase voltage, using a multi-turn coil placed inside a magnet, was described. It may be noted that, the scheme shown was a schematic one, whereas in a machine, the windings are distributed in number of slots. Same would be the case with a normal three -phase generator. Three windings, with equal no. of turns in each one, are used, so as to obtain equal voltage in magnitude in all three phases. Also to obtain a balanced three-phase voltage, the windings are to be placed at an electrical angle of 120° with each other, such that the voltages in each phase are also at an angle of 120° with each other, which will be described in the next section. The schematic diagram with multi-turn coils, as was shown earlier in Fig. 12.1 for a single-phase one, placed at angle of 120° with each other, in a 2-pole configuration, is shown in Fig. 18.1a. The waveforms in each of the three windings

(R, Y & B), are also shown in Fig. 18.1b. The windings are in the stator, with the poles shown in the rotor, which is rotating at a synchronous speed of N_s (r/min, or rpm), to obtain a frequency of $f = \left((p\ N_s)/120 \right)$ (Hz), p being no. of poles [$p = 2$] (see lesson no. 12).

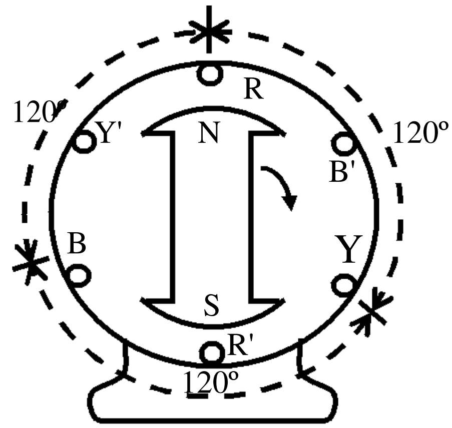


Fig. 18.1 (a) Schematic diagram of three windings of stator for the generation of three phased balanced voltage (2-pole rotor).

Three-phase Voltages for Star Connection

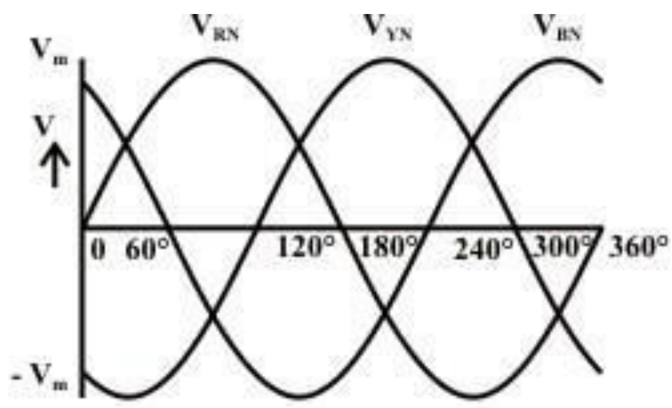


Fig. 18.1 (b)Three-phase balanced voltage waveforms with the source star-connected (the phase sequence, R-Y-B)

The connection diagram of a star (Y)-connected three-phase system is shown in Fig. 18.2a, along with phasor representation of the voltages (Fig. 18.2b). These are in continuation of the figures 18.1a-b. Three windings for three phases are R (+) & R'(−),Y (+) & Y'(−), and B (+) & Y'(−). Taking the winding of one phase, say phase R as an example, then R with sign (+) is taken as start, and R' with sign (−) is taken as finish. Same is the case with two other phases. For making star (Y)-connection, R', Y' & B' are connected together, and the point is taken as neutral, N. Three phase voltages are:

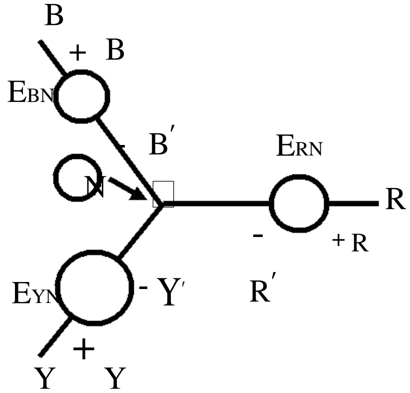
$e_{RN} = E_m \sin\theta$; $e_{YN} = E_m \sin(\theta - 120°)$; $= E_m \sin$

e_{BN} $(\theta - 240°) = E_m \sin(\theta + 120°)$

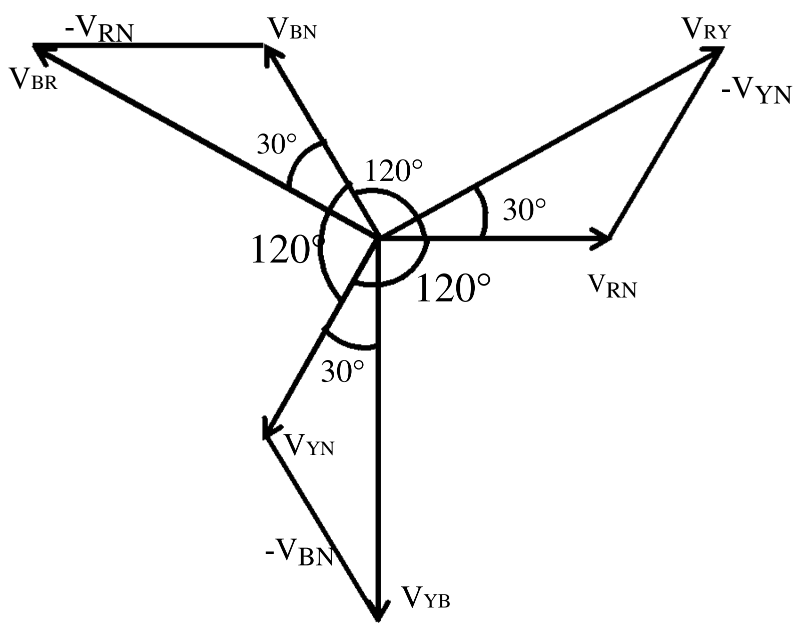
It may be noted that, if the voltage in phase R (e_{RN}) is taken as reference as stated earlier, then the voltage in phase Y(e_{YN}) lags e_{RN} by 120° , and the voltage in phase B(e_{BN}) lags e_{YN} by 120° , or leads e_{RN} by 120° . The phasors are given as:

E_{RN} $0° = E\,(1.0 + j0.0)$: E_{YN} $-120° = E\,(-0.5 - j0.866)$;

E_{BN} $+120° = E\,(-0.5 + j0.866)$.



(a)



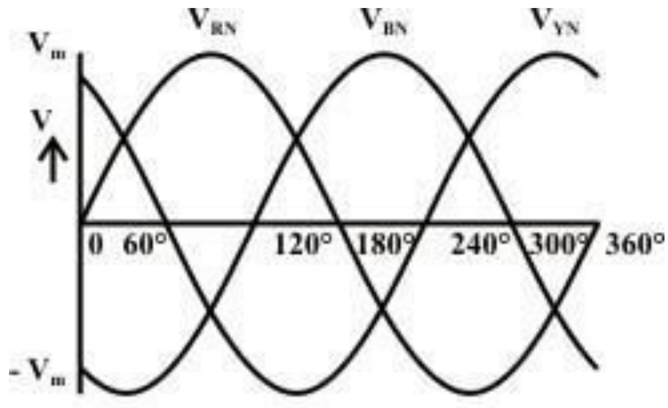
(b)

Fig. 18.2 (a) Three-phase balanced voltages, with the source star-connected (the phase sequence, R-Y-B)
(b) Phasor diagram of the line and phase voltages

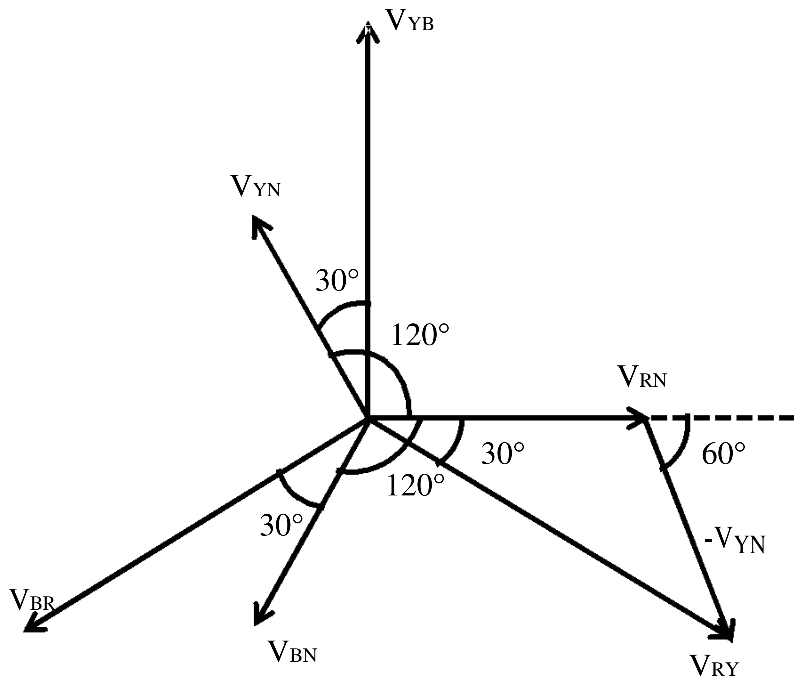
The phase voltages are all equal in magnitude, but only differ in phase. This is also shown in Fig. 18.2b. The relationship between E and E_m is $E = E_m / \sqrt{2}$. The phase sequence is R- Y-B. It can be observed from Fig. 18.1b that the voltage in phase Y attains the maximum value, after $\theta = \omega t = 120°$ from the time or angle, after the voltage in phase R attains the maximum value, and then the voltage in phase B attains the maximum value. The angle of lag or lead from the reference phase, R is stated earlier.

Reversal of phase sequence from R-Y-B to R-B-Y

If the phase sequence is reversed from R-Y-B to R-B-Y, the waveforms and the corresponding phasor diagram are shown in figures 18.3 (a -b) respectively. It can be observed from Fig. 18.3a that the voltage in phase B attains the maximum value, after $\theta = 120°$ from the time (or angle), after the voltage in phase R attains the maximum value, and then the voltage in phase Y attains the maximum value. The angle of lag or lead from the reference phase, R is stated earlier. The same sequence is observed in the phasor diagram (Fig. 18.3b), when the phase sequence is reversed to R-B-Y.



(a)



(b)

Fig. 18.3 (a) Three-phase balanced voltage waveforms with the source star-connected (the phase sequence, R-B-Y)
(b) Phasor diagram of the line and phase voltages

Relation between the Phase and Line Voltages for Star Connection

Three line voltages (Fig. 18.4) are obtained by the following procedure. The line voltage, E_{RY} is

$$E_{RY} = E_{RN} - E_{YN} = E\angle 0° - E\angle -120° = E\,[(1 + j0) - (-0.5 - j0.866)]$$
$$= E\,(1.5 + j0.866) = \sqrt{3}\,E\angle 30°$$

The magnitude of the line voltage, E_{RY} is $\sqrt{3}$ times the magnitude of the phase voltage E_{RN}, and E_{RY} leads E_{RN} by 30°. Same is the case with other two line voltages as shown in brief (the steps can easily be derived by the procedure given earlier).

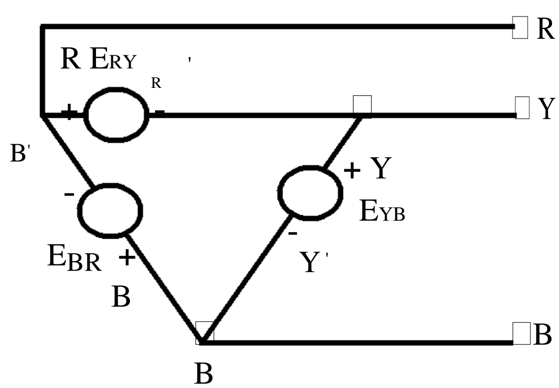
$$E_{YB} = E_{YN} - E_{BN} = E\angle -120° - E\angle +120° = \sqrt{3}E\angle -90°$$

$$E_{BR} = E_{BN} - E_{RN} = E\angle +120° - E\angle 0° = \sqrt{3}E\angle +150°$$

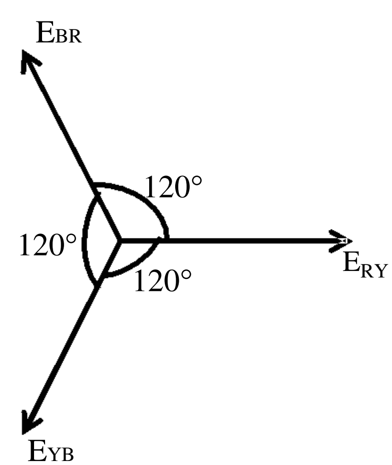
So, the three line voltages are balanced, with their magnitudes being equal, and the phase angle being displaced from each other in sequence by 120° . Also, the line voltage, say E_{RY} , <u>leads</u> the corresponding phase voltage, E_{RN} by 30°

Relation between the Phase and Line Voltages for Delta Connection

The connection diagram of a delta ( )-connected three-phase system is shown in Fig. 18.4a, along with phasor representation of the voltages (Fig. 18.4b). For making delta ( )-connection, the start of one winding is connected to the finish of the next one in sequence, for instance, starting from phase R, R’ is to connected to Y, and then Y’ to B, and so on (Fig. 18.4a). The line and phase voltages are the same in this case, and are given as



(a)



(b)

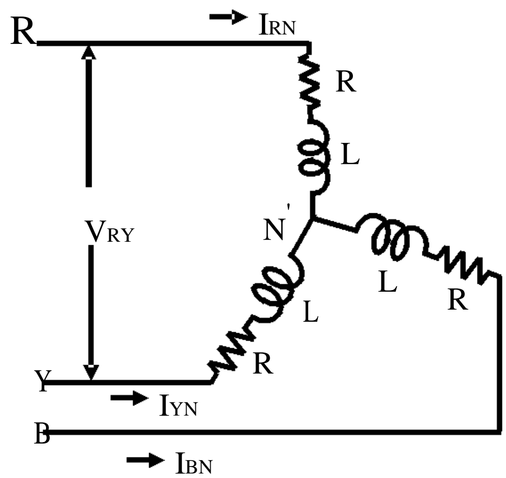
Fig. 18.4 (a) Three-phase balanced voltages, with the source delta-connected (the phase sequence, R-Y-B)

(b) Phasor diagram of the line and phase voltages

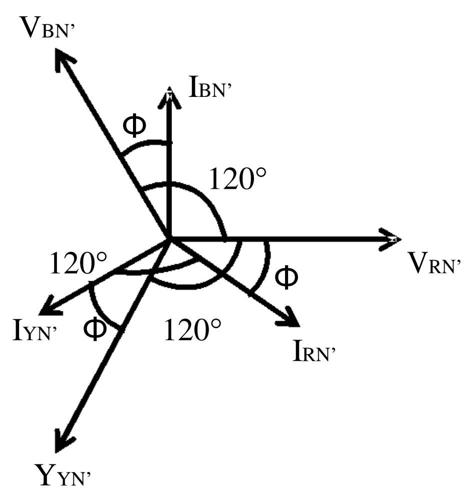
$E_{RY} = E \quad 0°$; $\qquad E_{YB} = E \quad -120°$; $\qquad E_{BR} = E \quad +120°$

If the phasor sum of the above three phase (or line) voltages are taken, the result is zero (0). The phase or line voltages form a balanced one, with their magnitudes being equal, and the phase being displaced from each other in sequence by$120°$.

Currents for Circuit with Balanced Load (Star-connected)



(a)



(b)

Fig. 18.5 (a) Circuit diagram for a three-phase balanced star-connected load
(b) Phasor diagram of the phase voltages, and the line & phase currents

A three-phase star (Y)-connected balanced load (Fig. 18.5a) is fed from a balanced three-phase supply, which may be a three-wire one. A balanced load means that, the magnitude of the impedance per phase, is same, i.e.,$|Z_p| = |Z_{RN'}| = |Z_{YN'}| = |Z_{BN'}|$, and their angle is also same, as $\varphi_p = \varphi_{RN'} = \varphi_{YN'} = \varphi_{BN'}$. In other words, if the impedance per phase is given as, $Z_p\ \varphi_p = R_p + j\,X_p$, then $R_p = R_{RN'} = R_{YN'} = R_{BN'}$, and also

$X_p = X_{RN'} = X_{YN'} = X_{BN'}$. The magnitude and phase angle of the impedance per phase are: $Z_p = \sqrt{R_p^2 + X_p^2}$, and $\varphi_p = \tan^{-1}\left(X_p / R_p\right)$. For balanced load, the magnitudes of the phase voltages, $|V_p| = |V_{RN'}| = |V_{YN'}| = |V_{BN'}|$ are same, as those of the source voltages per phase $|V_{RN}| = |V_{YN}| = |V_{BN}|$, if it is connected in star, as given earlier. So, this means that, the point N' , star point on the load side is same as the star point, N of the load side. The phase currents (Fig. 18.5b) are obtained as,

$$I_{RN'} \angle -\varphi_p = \frac{V_{RN} \angle 0^\circ}{Z_{RN'} \angle \varphi_p} = \frac{V_{RN}}{Z_{RN'}} \angle -\varphi_p$$

$$I_{YN'} \angle -(120^\circ + \varphi_p) = \frac{V_{YN} \angle -120^\circ}{Z_{YN'} \angle \varphi_p} = \frac{V_{YN}}{Z_{YN'}} \angle -(120^\circ + \varphi_p)$$

$$I_{BN'} \angle (120^\circ - \varphi_p) = \frac{V_{BN} \angle +120^\circ}{Z_{BN'} \angle \varphi_p} = \frac{V_{BN}}{Z_{BN'}} \angle (120^\circ - \varphi_p)$$

In this case, the phase voltage, V_{RN} is taken as reference. This shows that the phase currents are equal in magnitude, i.e., ($|I_p| = |I_{RN'}| = |I_{YN'}| = |I_{BN'}|$), as the magnitudes of the voltage and load impedance, per phase, are same, with their phase angles displaced from each other in sequence by 120° . The magnitude of the phase currents, is expressed as $|I_p| = \left(|V_p| / |Z_p|\right)$. These phase currents are also line currents ($|I_L| = |I_R| = |I_Y| = |I_B|$), in this case.

Total Power Consumed in the Circuit (Star-connected)

In the lesson No. 14 of the previous module, the power consumed in a circuit fed from a single-phase supply was presented. Using the same expression for the above star-connected balanced circuit, fed from three-phase supply (Fig. 18.4a-b), the power consumed per phase is given by

$$W_p = V_p \; I_p \; \cos \varphi_p = V_p \; I_p \; \cos\left(V_p , I_p\right)$$

It has been shown earlier that the magnitude of the phase voltage is given by $|V_p| = |V_L| / \sqrt{3}$, where the magnitude of the line voltage is $|V_L|$. The magnitudes of the phase and line current are same, i.e., $|I_p| = |I_L|$. Substituting the two expressions, the total power consumed is obtained as

$$W = 3\left(V_l / \sqrt{3}\right) I_L \; \cos \varphi_p = \sqrt{3}\; V_L \; I_L \; \cos \varphi_p$$

Please note that the phase angle, φ_p is the angle between the phase voltage V_p , and the phase current , I_p .

Before taking up an example, the formulas for conversion from delta-connected circuit to its star equivalent and vice versa (conversion from star to delta connection) using impedances, and also ideal inductances/capacitances, are presented here, starting with circuits with resistances, as derived in lesson #6 on dc circuits.

Delta(Δ)-Star(*Y*) conversion and Star-Delta conversion

Before taking up the examples, the formula for Delta( )-Star(Y) conversion and also Star-Delta conversion, using impedances as needed, instead of resistance as elements, which is given in lesson #6 in the module on DC circuit, are presented. The formulas for delta-star conversion, using resistances (Fig. 18.6), are,

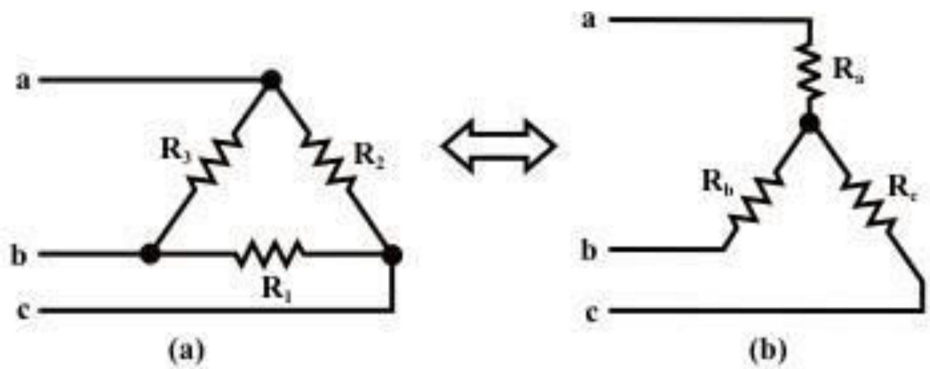


Fig. 18.6: Resistances connected (a) in delta, and (b) in star configurations

$$R_a = \frac{R_2\, R_3}{R_1 + R_2 + R_3} \qquad R_b = \frac{R_3\, R_1}{R_1 + R_2 + R_3} \qquad R_c = \frac{R_1\, R_2}{R_1 + R_2 + R_3}$$

The formulas for delta-star conversion, using resistance, are,

$$R_1 = R_b + R_c + \frac{R_b\, R_c}{R_a} = \frac{R_a\, R_b + R_b\, R_c + R_c\, R_a}{R_a}$$

$$R_2 = R_c + R_a + \frac{R_c\, R_a}{R_b} = \frac{R_a\, R_b + R_b\, R_c + R_c\, R_a}{R_b}$$

$$R_3 = R_a + R_b + \frac{R_a\, R_b}{R_c} = \frac{R_a\, R_b + R_b\, R_c + R_c\, R_a}{R_c}$$

The derivation of these formulas is given in lesson #6. If three equal resistances ($R_1 = R_2 = R_3 = R$) connected in delta, are converted into its equivalent star, the resistances obtained are equal, its value being $R_a = R_b = R_c = (R / 3) = R'$, which is derived using formulas given earlier. Similarly, if three equal resistances connected in star, are converted into its equivalent delta, the resultant resistances, using formulas, are equal ($R_1 = R_2 = R_3 = 3\, R' = 3\, (R / 3) = R$).

The formula for the above conversions using impedances, instead of resistances, are same, replacing resistances by impedances, as the formula for series and parallel combination using impedances, instead of resistances, remain same as shown in the previous module on ac single phase circuits.

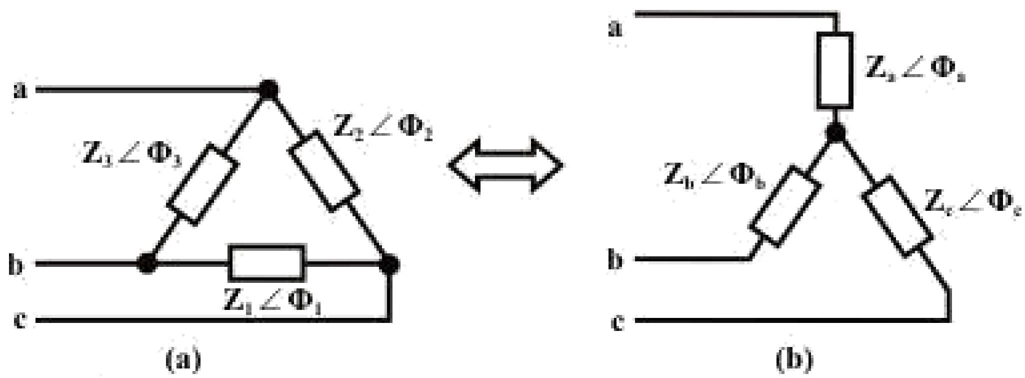


Fig. 18.7: Impedances connected (a) in delta, and (b) in star configurations

The formulas for delta-star conversion, using impedances (Fig. 18.7), are,

$$Z_a = \frac{Z_2\, Z_3}{Z_1 + Z_2 + Z_3} \qquad Z_b = \frac{Z_3\, Z_1}{Z_1 + Z_2 + Z_3} \qquad Z_c = \frac{Z_1\, Z_2}{Z_1 + Z_2 + Z_3}$$

The formulas for delta-star conversion, using impedance, are,

$$Z_1 = Z_b + Z_c + \frac{Z_b\, Z_c}{Z_a} = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_a}$$

$$Z_2 = Z_c + Z_a + \frac{Z_c\, Z_a}{Z_b} = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_b}$$

$$Z_3 = Z_a + Z_b + \frac{Z_a\, Z_b}{Z_c} = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_c}$$

Please note that all the impedances used in the formula given here are complex quantities, like $Z_1\ \varphi_1$, , $Z_a\ \varphi_a$, , having both magnitude and angle as given. The formulas can be derived by the same procedure as given in lesson #6.

An example is taken up, when three equal impedances connected in delta are to be converted into its equivalent star. The impedances are equal, both in magnitude and angle, such that $|Z_1| = |Z_2| = |Z_3| = |Z|$, and $\varphi_1 = \varphi_2 = \varphi_3 = \varphi$. The impedances connected in delta are of the form $Z\ \varphi = R \pm j\,X$. Using the formula given here, the impedances of the star equivalent are also equal, having the magnitude as $|Z_a| = |Z_b| = |Z_b| = (|Z|/3) = |Z'|$ and angle as $\varphi_a = \varphi_b = \varphi_c = \varphi$.

The angles of the equivalent impedance connected in star are equal to the angles of the impedances connected in delta. The impedances connected in delta are also equal, both in magnitude and angle, and are of the form $Z'\ \varphi = (Z/3)\ \varphi = (R/3) \pm j\,(X/3)$.

Similarly, if three equal impedances connected in star are converted into its equivalent delta, the magnitude and angle of the impedances using the formulas given here, are $|Z_1| = |Z_2| = |Z_3| = (3\ |Z'|) = |Z|$ and $\varphi_1 = \varphi_2 = \varphi_2 = \varphi$ respectively. This shows that three impedances are equal, both in magnitude and angle, with its value being $Z\ \varphi = (3\,Z')\ \varphi = [3\,(R/3)] \pm j[3\,(X/3)] = R \pm j\,X$

which can also be obtained simply from the result given earlier.

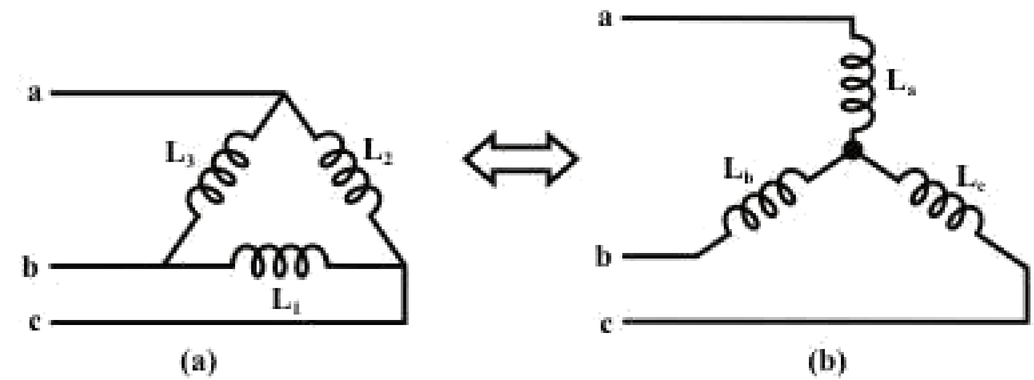


Fig. 18.8: Inductances (ideal) connected (a) in delta, and (b) in star configurations

Now, let us use the above formula for the circuits (Fig. 18.8), using inductances only. The symbols used for the inductances are same (L_1 , , L_a ,). The impedances of the inductances connected in delta, are computed as $Z_1\ \varphi_1 = 0.0 + j\omega L_1 = X_1\ 90°$, the angles in three cases are $90°$. The magnitudes of the impedances are proportional to the respective inductances as $|Z_1| = X_1\ L_1$. Converting the combination into its equivalent star, the inductances using the formulas given here, are

$$L_a = \frac{L_2 L_3}{L_1 + L_2 + L_3} \qquad L_b = \frac{L_3 L_1}{L_1 + L_2 + L_3} \qquad L_c = \frac{L_1 L_2}{L_1 + L_2 + L_3}$$

These relations can also be derived. Further, these are of the same form, as has been earlier obtained for resistances. It may be observed here that the formulas for series and parallel combination using inductances, instead of resistances, remain same, as shown in the previous module on ac single phase circuits, and also can be derived from first principles, such as relationship of induced emf in terms of inductance, as compared with Ohm's law for resistance. The inductances are all ideal, i.e. lossless, having no resistive component. The formulas for star-delta conversion using inductances (conversion of star-connected inductances into its equivalent delta) are,

$$L_1 = L_b + L_c + \frac{L_b L_c}{L_a} = \frac{L_a L_b + L_b L_c + L_c L_a}{L_a}$$

$$L_2 = L_c + L_a + \frac{L_c L_a}{L_b} = \frac{L_a L_b + L_b L_c + L_c L_a}{L_b}$$

$$L_3 = L_a + L_b + \frac{L_a L_b}{L_c} = \frac{L_a L_b + L_b L_c + L_c L_a}{L_c}$$

These are of the same form as derived for circuits with resistances.

If three equal inductances ($L_1 = L_2 = L_3 = L$) connected in delta, are converted into its equivalent star, the inductances obtained are equal, its value being $L_a = L_b = L_c = (L / 3) = L'$, which is derived using formulas given earlier. Similarly, if three equal inductances connected in star, are converted into its equivalent delta, the resultant inductances, using formulas, are equal ($L_1 = L_2 = L_3 = 3\ L' = 3\ (L / 3) = L$).

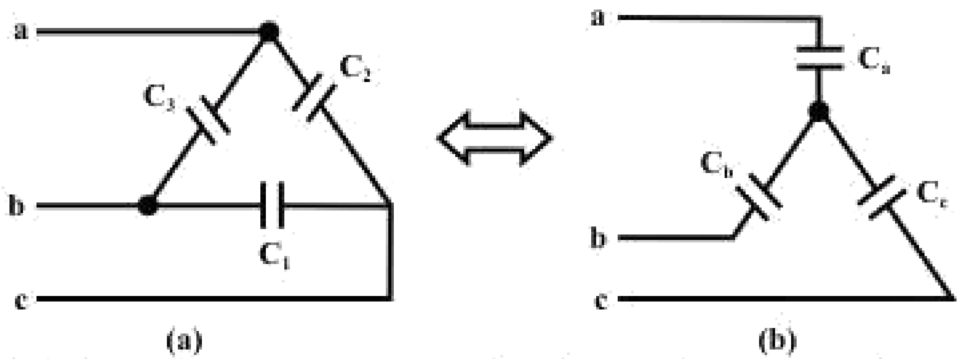


Fig. 18.9: Capacitances connected (a) in delta, and (b) in star configurations

The formulas for the circuits (Fig. 18.9) using capacitances are derived here. The symbols used for the capacitances are same (C_1 , , C_a ,). The impedances of the inductances connected in delta, are computed as $Z_1 \ \varphi_1 = 0.0 - jX_C = X_1 \ -90°$, the angles in three cases are ($-90°$). The magnitudes of the impedances are inversely proportional to the respective capacitances as, $Z_1 = |X_C| = |X_1| = (1/\omega C)\ (1/C_1)$.
Converting the combination into its equivalent star, the resultant capacitances using the formulas given here, are

$$1/C_a = \frac{(1/C_2)\,(1/C_3)}{1/C_1 + 1/C_2 + 1/C_3}$$

or $$C_a = \frac{C_1 C_2 + C_2 C_3 + C_3 C_1}{C_1} = C_2 + C_3 + \frac{C_2 C_3}{C_1}$$

Similarly,

$$C_b = C_3 + C_1 + \frac{C_3 C_1}{C_2} = \frac{C_1 C_2 + C_2 C_3 + C_3 C_1}{C_2}$$

$$C_c = C_1 + C_2 + \frac{C_1 C_2}{C_3} = \frac{C_1 C_2 + C_2 C_3 + C_3 C_1}{C_3}$$

The capacitances in this case are all ideal, without any loss, specially at power frequency, which is true in nearly all cases, except otherwise stated. The formulas for star-delta conversion using capacitances (conversion of star-connected capacitances into its equivalent delta) are,

$$1/C_1 = \frac{(1/C_a)\,(1/C_b) + (1/C_b)\,(1/C_c) + (1/C_c)\,(1/C_a)}{1/C_a}$$

or $$C_1 = \frac{C_b C_c}{C_a + C_b + C_c}$$

Similarly,

$$C_2 = \frac{C_c C_a}{C_a + C_b + C_c} \qquad C_3 = \frac{C_a C_b}{C_a + C_b + C_c}$$

If three equal capacitances ($C_1 = C_2 = C_3 = C$) connected in delta, are converted into its equivalent star, the capacitances obtained are equal, its value being

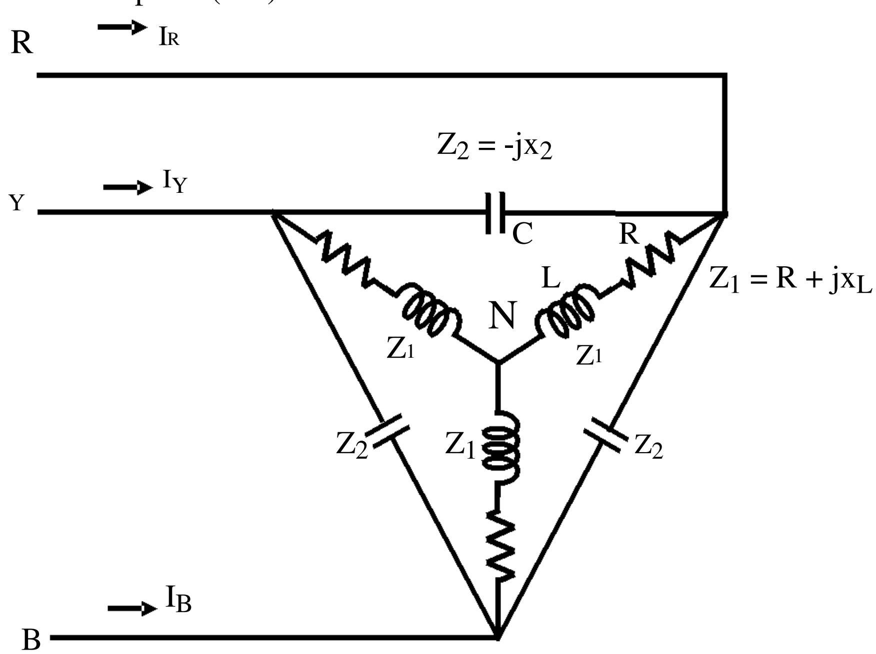
$C_a = C_b = C_c = (3\ C) = C'$, which is derived using formulas given earlier. Similarly, if three equal capacitances connected in star, are converted into its equivalent delta, the resultant capacitances, using formulas, are equal ($C_1 = C_2 = C_3 = C' / 3 = (3\ C) / 3 = C$).

The formulas for conversion of three equal inductances/capacitances connected in delta into its equivalent star and vice versa (star-delta conversion) can also be obtained from the formulas using impedances as shown earlier, only by replacing inductance with impedance, and for capacitance by replacing it reciprocal of impedance (in both cases using magnitude of impedance only, as the angles are equal (90° for inductance and − 90° for capacitance). Another point to note is left for observation by the reader. Please have a close look at the formulas needed for delta-star conversion and vice versa (star-delta conversion) for capacitances, including those with equal values of capacitances, and then compare them with the formulas needed for such conversion using resistances/inductances (may be impedances also). The rules for conversion of capacitances in series/parallel into its equivalent one can be compared to the rules for conversion of resistances/inductances in series/parallel into its equivalent one.

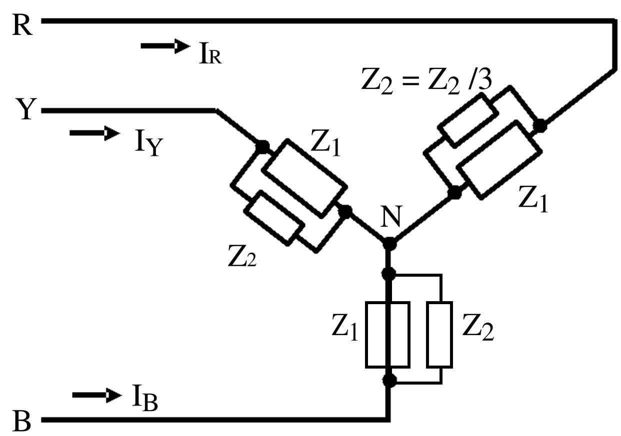
The reader is referred to the comments given after the example 18.1.

Example 18.1

The star-connected load consists of a resistance of 15 Ω, in series with a coil having resistance of 5 Ω, and inductance of 0.2 H, per phase. It is connected in parallel with the delta-connected load having capacitance of 90 μF per phase (Fig. 18.10a). Both the loads being balanced, and fed from a three-phase, 400 V, 50 Hz, balanced supply, with the phase sequence as R-Y-B. Find the line current, power factor, total power & reactive VA, and also total volt-amperes (VA).



(a)



(b)

Fig. 18.10 (a) Circuit diagram (Example 18.1)
(b) Equivalent balanced star-connected circuit

Solution

$f = 50\ Hz$ $\qquad\qquad \omega = 2\pi f = 2 \cdot \pi \cdot 50 = 314.16\ rad\,/\,s$

For the balanced star-connected load, $\quad R = 15\ \Omega$

For the inductance coil, $r = 5\ \Omega$ $\qquad X_L = \omega L = 314.16 \cdot 0.2 = 62.83\ \Omega$

with the above values taken per phase.

The impedance per phase is,

$Z_1 = R + (r + j\,X_L) = 15 + (5 + j\,62.83) = (20 + j\,62.83) = 65.94\ \ 72.34°\ \Omega$

For the balanced delta-connected load, $\quad C = 90\ \ \mu F$

Converting the above load into its equivalent star, $C_1 = C\,/\,3 = 90\,/\,3 = 30$

$\mu F\ X_{C1} = 1/\omega\,C_1 = 1/(314.16 \cdot 30 \cdot 10^{-6}) = 106.1\ \Omega$

The impedance per phase is $Z_2' = -\,j106.1 = 106.1\ \ -90°$

In the equivalent circuit for the load (Fig. 18.10b), the two impedances, Z_1 & Z_2' are in parallel. So, the total admittance per phase is,

$$Y_p = Y_1 + Y_2' = \frac{1}{Z_1} + \frac{1}{Z_2'} = \frac{1}{65.94\ \ +72.34°} + \frac{1}{106.1\ \ -90°}$$

$= 0.01517\ \ -72.34° + 0.009425\ \ +90° = [(4.6 - j14.46) + j\,9.425]\cdot 10^{-3}$

$= (4.6 - j\,5.03)\cdot 10^{-3}\ = 0.006816\ \ -47.56°\ \Omega^{-1}$

The total impedance per phase is,

$Z_p = 1/\,Y_p = 1/(0.006816\ \ -47.56°) = 146.71\ \ +47.56° = (99.0 + j108.27)\ \Omega$

The phasor diagram is shown in Fig. 18.10c.

Taking the phase voltage, V_{RN} as reference,

$|V_{RN}| = |V_p| = |V_L|\,/\sqrt{3} = 400\,/\sqrt{3} = 231.0\ V$

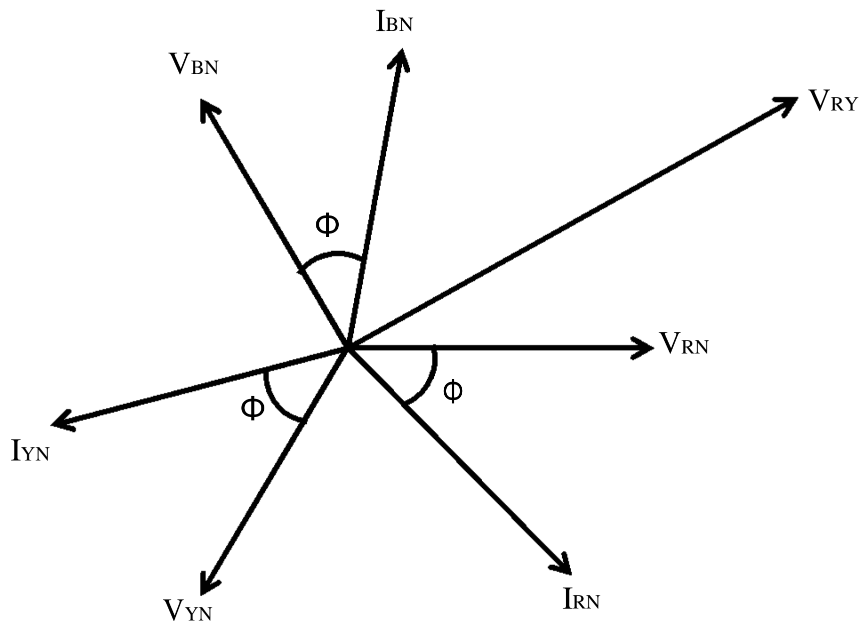


Fig. 18.10 (c) Phasor diagram

The phase voltages are,

$\vec{V}_{RN} = 231.0 \;\; 0°$; $\quad \vec{V}_{YN} = 231.0 \;\; -120°$; $\quad V_{BN} = 231.0 \;\; +120°$

So, the phase current, I_{RN} is,

$$\vec{I}_{RN} = \frac{\vec{V}_{RN}}{Z_p} = \frac{231.0 \;\; 0°}{146.71 \;\; +47.56°} = 1.575 \;\; -47.56° = (1.0625 - j1.162)\, A$$

The two other phase currents are,

$\vec{I}_{YN} = 1.575 \;\; -167.56°$; $\quad I_{BN} = 1.575 \;\; +72.44°$

As the total circuit (Fig. 18.5b) is taken as star-connected, the line and phase currents are same, i.e., $|I_L| = |I_p| = 1.575\, A$

Also, the phase angle of the total impedance is positive.

So, the power factor is $\cos \varphi_p = \cos 47.56° = 0.675\ lag$

The total volt-amperes is $S = 3\, V_p \;\, I_p = 3 \cdot 231 \cdot 1.575 = 1.0915\ kVA$

The total VA is also obtained as $S = \sqrt{3}\, V_L\, I_L = \sqrt{3} \cdot 400 \cdot 1.575 = 1.0915\ kVA$

The total power is $P = 3\, V_p\, I_p \cos \varphi_p = 3 \cdot 231 \cdot 1.575 \cdot 0.675 = 737\ W$

The total reactive volt-amperes is,

$Q = 3\, V_p \;\, I_p \;\sin \varphi_p = 3 \cdot 231 \cdot 1.575 \cdot \sin 47.56° = 805\ VAR$

This example can be solved by converting the star-connected part into its equivalent delta, as shown in Example 19.1 (next lesson). A simple example (20.1) of a balanced star-connected load is also given in the last lesson (#20) of this module.

After starting with the generation of three- phase balanced voltage system, the phase and line voltages, both being balanced, first for star-connection, and then for delta-connection (both on source side), are discussed. The currents (both phase and line) for balanced star-connected load, along with total power consumed, are also described in this lesson. An example is given in detail. In the next lesson, the currents (both phase and line) for balanced delta-connected load will be presented.

Problems

18.1 A balance load of (16+j12)Ω per phase, connected in star, is fed from a three-phase, 230V supply. Find the line current, power factor, total power, reactive VA and total VA.

18.2 Find the three voltages $V_{an,}$ $V_{bn,}$ & $V_{cn,}$ in the circuit shown in Fig. 18.11. The circuit components are: R = 10 Ω, jX_L = j17.3 Ω.

List of Figures

UNIT-2

Three-phase AC Circuits

Three-phase Delta-Connected Balanced Load

In the previous (first) lesson of this module, the two types of connections (star and delta), normally used for the three-phase balanced supply in source side, along with the line and phase voltages, are described. Then, for balanced star-connected load, the phase and line currents, along with the expression for total power, are obtained. In this lesson, the phase and line currents for balanced delta-connected load, along with the expression for total power, will be presented.

Keywords: line and phase currents, star- and delta-connections, balanced load.

After going through this lesson, the students will be able to answer the following questions:

1. How to calculate the currents (line and phase), for the delta-connected balanced load fed from a three-phase balanced system?

2. Also how to find the total power fed to the above balanced load, for the two types of load connections – star and delta?

Currents for Circuits with Balanced Load (Delta-connected)

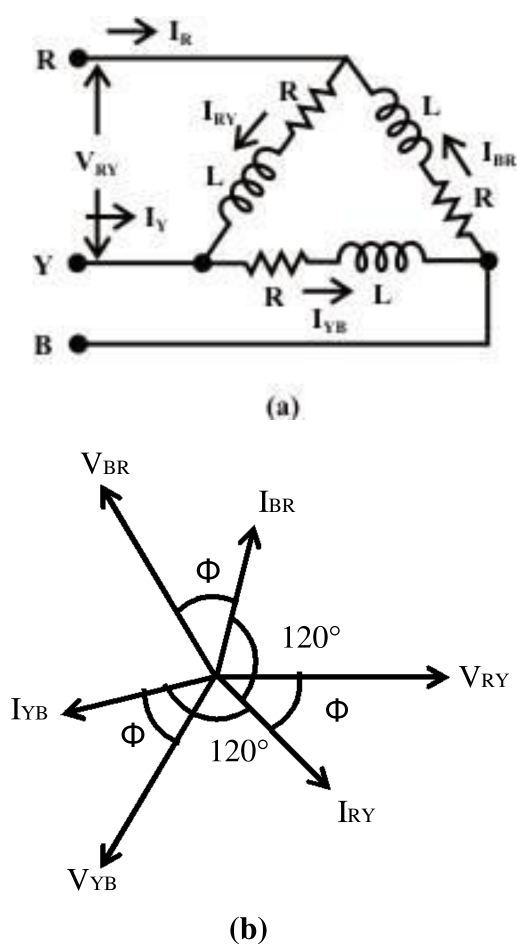


Fig. 19.1 **(a) Balanced delta-connected load fed from a three-phase balanced supply**
(b) Phasor diagram

A three-phase delta ( )-connected balanced load (Fig. 19.1a) is fed from a balanced three-phase supply. A balanced load means that, the magnitude of the impedance per phase, is same, i.e., $|Z_p| = |Z_{RY}| = |Z_{YB}| = |Z_{BR}|$, and their angle is also same, as $\varphi_p = \varphi_{RY} = \varphi_{YB} = \varphi_{BR}$. In other words, if the impedance per phase is given as, $Z_p \angle \varphi_p = R_p + j\,X_p$, then $R_p = R_{RY} = R_{YB} = R_{BR}$, and also $X_p = X_{RY} = X_{Yb} = X_{BR}$. The magnitude and phase angle of the impedance per phase are: $Z_p = \sqrt{R_p^2 + X_p^2}$, and $\varphi_p = \tan^{-1}\left(X_p / R_p\right)$. In this case, the magnitudes of the phase voltages $|V_p|$ are same, as those of the line voltages $|V_L| = |V_{RY}| = |V_{YB}| = |V_{BR}|$. The phase currents (Fig. 19.1b) are obtained as,

$$I_{RY} \angle -\varphi_p = \frac{V_{RY} \angle 0^\circ}{Z_{RY} \angle \varphi_p} = \frac{V_{RY}}{Z_{RY}} \angle -\varphi_p$$

$$I_{YB} \angle -(120^\circ + \varphi_p) = \frac{V_{YN} \angle -120^\circ}{Z_{YB} \angle \varphi_p} = \frac{V_{YB}}{Z_{YB}} \angle -(120^\circ + \varphi_p)$$

$$I_{BR} \angle (120^\circ - \varphi_p) = \frac{V_{BR} \angle +120^\circ}{Z_{BR} \angle \varphi_p} = \frac{V_{BR}}{Z_{BR}} \angle (120^\circ - \varphi_p)$$

In this case, the phase voltage, V_{RY} is taken as reference. This shows that the phase currents are equal in magnitude, i.e., ($|I_p| = |I_{RY}| = |I_{YB}| = |I_{BR}|$), as the magnitudes of the voltage and load impedance, per phase, are same, with their phase angles displaced from each other in sequence by 120°. The magnitude of the phase currents, is expressed as $|I_p| = \left(|V_p| / |Z_p|\right)$.

The line currents (Fig. 19.1b) are given as

$$I_R \angle -\theta_R = I_{RY} - I_{BR} = I_p \angle(-\varphi_p) - I_p \angle(120^\circ - \varphi_p) = \sqrt{3}\, I_p \angle -(30^\circ + \varphi_p)$$
$$= I_L \angle -(30^\circ + \varphi_p)$$

$$I_Y \angle -\theta_Y = I_{YB} - I_{RY} = I_p \angle -(120^\circ + \varphi_p) - I_p \angle(-\varphi_p) = \sqrt{3}\, I_p \angle -(150^\circ + \varphi_p)$$
$$= I_L \angle -(150^\circ + \varphi_p)$$

$$I_B \angle -\theta_B = I_{BR} - I_{YB} = I_p \angle(120^\circ - \varphi_p) - I_p \angle -(120^\circ + \varphi_p) = \sqrt{3}\, I_p \angle(90^\circ - \varphi_p)$$
$$= I_L \angle(90^\circ - \varphi_p)$$

The line currents are balanced, as their magnitudes are same and $\sqrt{3}$ times the magnitudes of the phase currents ($I_L = \sqrt{3} \cdot |I_p|$), with the phase angles displaced from each other in sequence by 120°. Also to note that the line current, say I_R, lags the corresponding phase current, I_{RY} by 30°.

If the phase current, I_{RY} is taken as reference, the phase currents are

$I_{RY} \angle 0^\circ = I_p\,(1.0 + j0.0)$: $\qquad I_{YB} \angle -120^\circ = I_p\,(-0.5 - j0.866)$;

$I_{BR} \angle +120^\circ = I_p\,(-0.5 + j0.866)$.

The line currents are obtained as

$I_R = I_{RY}\angle 0° - I_{BR}\angle +120° = I_p\{(1.0 + j\,0.0) - (-0.5 + j\,0.866)\} = I_p(1.5 - j\,0.866)$

$= \sqrt{3}\,I_p \angle -30° = I_L \angle -30°$

$I_Y = I_{YB}\angle -120° - I_{RY}\angle 0° = I_p\{(-0.5 - j\,0.866) - (1.0 + j\,0.0)\} = -I_p(1.5 + j\,0.866)$

$= \sqrt{3}\,I_p \angle -150° = I_L \angle -150°$

$I_B = I_{BR}\angle +120° - I_{YB}\angle -120° = I_p\{(-0.5 + j\,0.866) - (-0.5 - j\,0.866)\}$

$= I_p(j1.732) = \sqrt{3}\,I_p \angle +90° = I_L \angle +90°$

Total Power Consumed in the Circuit (Delta-connected)

In the last lesson (No. 18), the equation for the power consumed in a star-connected balanced circuit fed from a three-phase supply, was presented. The power consumed per phase, for the delta-connected balanced circuit, is given by

$$W_p = V_p \cdot I_p \cdot \cos\varphi_p = V_p \cdot I_p \cdot \cos\left(V_p, I_p\right)$$

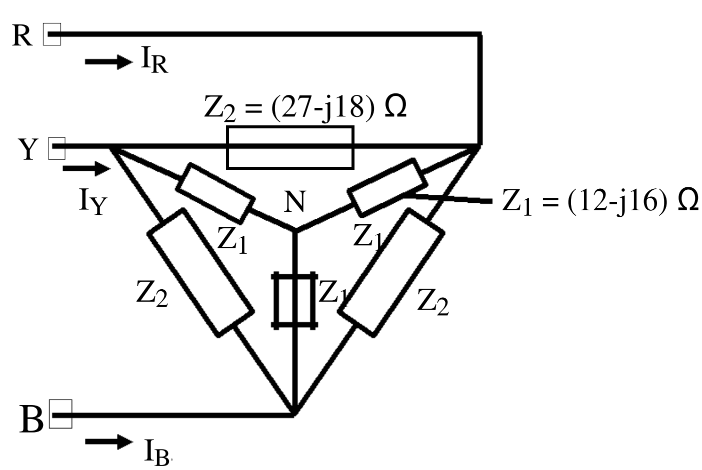
It has been shown earlier that the magnitudes of the phase and line voltages are same, i.e., $|V_p| = |V_L|$. The magnitude of the phase current is $(1/\sqrt{3})$ times the magnitude of the line current, i.e., $|I_p| = \left(|I_L|/\sqrt{3}\right)$. Substituting the two expressions, the total power consumed is obtained as

$$W = 3 \cdot V_L \cdot \left(I_L/\sqrt{3}\right) \cdot \cos\varphi_p = \sqrt{3}\, V_L \cdot I_L \cdot \cos\varphi_p$$

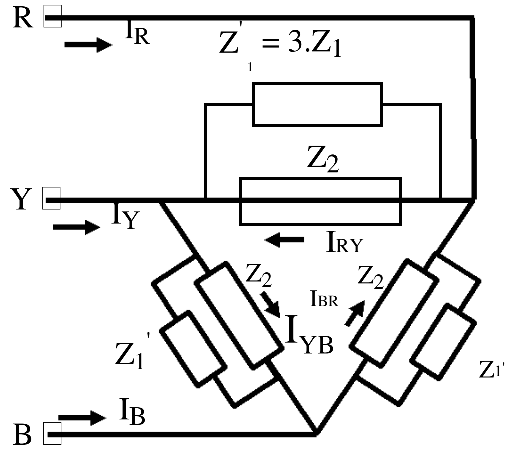
It may be observed that the phase angle, φ_p is the angle between the phase voltage V_p, and the phase current, I_p. Also that the expression for the total power in a three-phase balanced circuit is the same, whatever be the type of connection – star or delta.

Example 19.1

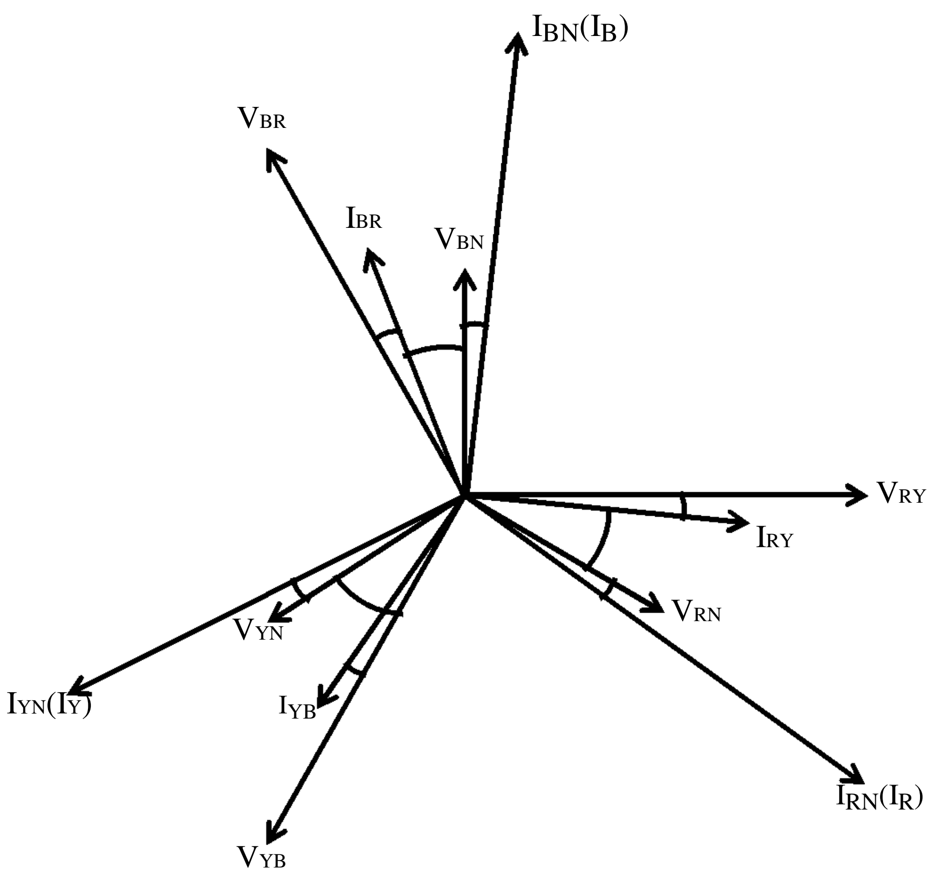
The star-connected load having impedance of $(12 - j16)\ \Omega$ per phase is connected in parallel with the delta-connected load having impedance of $(27 + j18)\ \Omega$ per phase (Fig. 19.2a), with both the loads being balanced, and fed from a three-phase, 230 V, balanced supply, with the phase sequence as R-Y-B. Find the line current, power factor, total power & reactive VA, and also total volt-amperes (VA).



(a)



(b)



(c)

Fig. 19.2 (a) Circuit diagram (Example 19.1)
(b) Equivalent circuit (delta-connected)
(c) Phasor diagram

Solution

For the balanced star-connected load, the impedance per phase is,

$Z_1 = (12 - j16) = 20.0 \angle -53.13^\circ\ \Omega$

The above load is converted into its equivalent delta. The impedance per phase is,

$Z_1' = 3 \cdot Z_1 = 3 \cdot (12 - j16) = (36 - j\,48) = 60.0 \angle -53.13^\circ\ \Omega$

For the balanced delta-connected load, the impedance per phase is,

$Z_2 = (27 + j18) = 32.45 \angle +33.69^\circ\ \Omega$

In the equivalent circuit for the load (Fig. 19.2b), the two impedances, Z_1' & Z_2 are in parallel. So, the total admittance per phase is,

$$Y_p = Y_1' + Y_2 = \frac{1}{Z_1'} + \frac{1}{Z_2} = \frac{1}{60.0 \angle -53.13^\circ} + \frac{1}{32.45 \angle +33.69^\circ}$$

$= 0.0167 \angle +53.13^\circ + 0.03082 \angle -33.69^\circ$

$= [(0.01 + j\,0.01333) + (0.02564 - j\,0.017094)] = (0.03564 - j\,0.003761)$

$= 0.03584 \angle -6.024^\circ\ \Omega^{-1}$

The total impedance per phase is,

$Z_p = 1/\,Y_p = 1/(0.03584 \angle -6.024^\circ) = 27.902 \angle +6.024^\circ = (27.748 + j\,2.928)\ \Omega$

The phasor diagram is shown in Fig. 19.2c.

Taking the line voltage, V_{RY} as reference, $V_{RY} = 230 \angle 0^\circ\ V$

The other two line voltages are,

$V_{YB} = 230 \angle -120^\circ$; $\quad V_{BR} = 230 \angle +120^\circ$

For the equivalent delta-connected load, the line and phase voltages are same.

So, the phase current, I_{RY} is,

$$I_{RY} = \frac{V_{RY}}{Z_p} = \frac{230.0 \angle 0^\circ}{27.902 \angle +6.024^\circ} = 8.243 \angle -6.024^\circ = (8.198 - j\,0.8651)\ A$$

The two other phase currents are,

$I_{YB} = 8.243 \angle -126.024^\circ$; $\quad I_{BR} = 8.243 \angle +113.976^\circ$

The magnitude of the line current is $\sqrt{3}$ times the magnitude of the phase current.

So, the line current is $|I_L| = \sqrt{3} \cdot |I_p| = \sqrt{3} \cdot 8.243 = 14.277\ A$

The line current, I_R lags the corresponding phase current, I_{RY} by 30°.

So, the line current, I_R is $I_R = 14.277 \angle -36.024^\circ\ A$

The other two line currents are,

$I_Y = 14.277 \angle -156.024^\circ$; $\quad I_B = 14.277 \angle +83.976^\circ$

Also, the phase angle of the total impedance is positive.

So, the power factor is $\cos \varphi_p = \cos 6.024^\circ = 0.9945$ *lag*

The total volt-amperes is $S = 3 \cdot V_p \cdot I_p = 3 \cdot 230 \cdot 8.243 = 5.688\ kVA$

The total VA is also obtained as $S = \sqrt{3} \cdot V_L \cdot I_L = \sqrt{3} \cdot 230 \cdot 14.277 = 5.688\ kVA$

The total power is $P = 3 \cdot V_p \cdot I_p \cdot \cos \varphi_p = 3 \cdot 230 \cdot 8.243 \cdot 0.9945 = 5.657\ k\,W$

The total reactive volt-amperes is,

$Q = 3 \cdot V_p \cdot I_p \cdot \sin \varphi_p = 3 \cdot 230 \cdot 8.243 \cdot \sin 6.024^\circ = 597.5\ VAR$

An alternative method, by converting the delta-connected part into its equivalent star is given, as shown earlier in Ex. 18.1.

For the balanced star-connected load, the impedance per phase is,

$Z_1 = (12 - j16) = 20.0 \angle -53.13° \ \Omega$

For the balanced delta-connected load, the impedance per phase is,

$Z_2 = (27 + j18) = 32.45 \angle +33.69° \ \Omega$

Converting the above load into its equivalent star, the impedance per phase is,

$Z_2' = Z_2 / 3 = (27 + j18) / 3 = (9 + j\,6) = 10.817 \angle +33.69° \ \Omega$

In the equivalent circuit for the load, the two impedances, Z_1 & Z_2' are in parallel. So, the total admittance per phase is,

$$Y_p = Y_1 + Y_2' = \frac{1}{Z_1} + \frac{1}{Z_2'} = \frac{1}{20.0 \angle -53.13°} + \frac{1}{10.817 \angle +33.69°}$$

$$= 0.05 \angle +53.13° + 0.09245 \angle -33.69° = [(0.03 + j\,0.04) + (0.0769 - j\,0.05128)]$$

$$= (0.1069 - j\,0.01128) = 0.1075 \angle -6.0235° \ \Omega^{-1}$$

The total impedance per phase is,

$Z_p = 1/Y_p = 1/(0.1075 \angle -6.0235°) = 9.3023 \angle +6.0235° = (9251 + j\,0.976) \ \Omega$

The phasor diagram is shown in Fig. 18.5c. The magnitude of the phase voltage is,

$|V_{RN}| = |V_p| = |V_L| / \sqrt{3} = 230 / \sqrt{3} = 132.8 \ V$

The line voltage, V_{RY} is taken as reference as given earlier. The corresponding phase voltage, V_{RN} lags V_{RY} by 30° . So, the phase voltage, V_{RN} is $V_{RN} = 132.8 \angle -30°$

The phase current, I_{RN} is,

$$I_{RN} = \frac{V_{RN}}{Z_p} = \frac{132.8 \angle -30°}{9.3023 \angle +6.0235°} = 14.276 \angle -36.0235° \ A$$

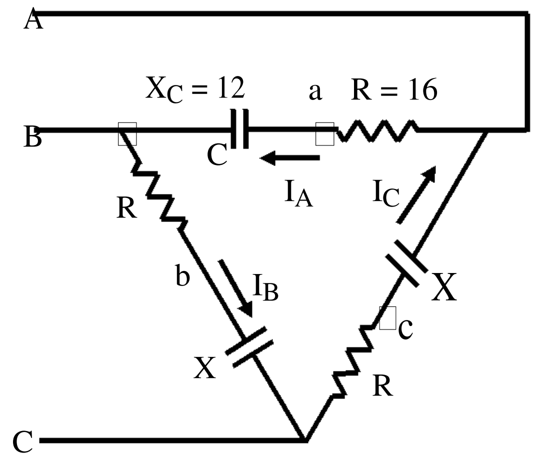
As the total load is taken as star-connected, the line and phase currents are same, in this case. The phase angle of the total impedance is positive, with is value as $\varphi = 6.0235°$. The power factor is $\cos 6.0235° = 0.9945$ *lag*

The total volt-amperes is $S = 3 \cdot V_p \cdot I_p = 3 \cdot 132.8 \cdot 14.276 = 5.688 \ kVA$

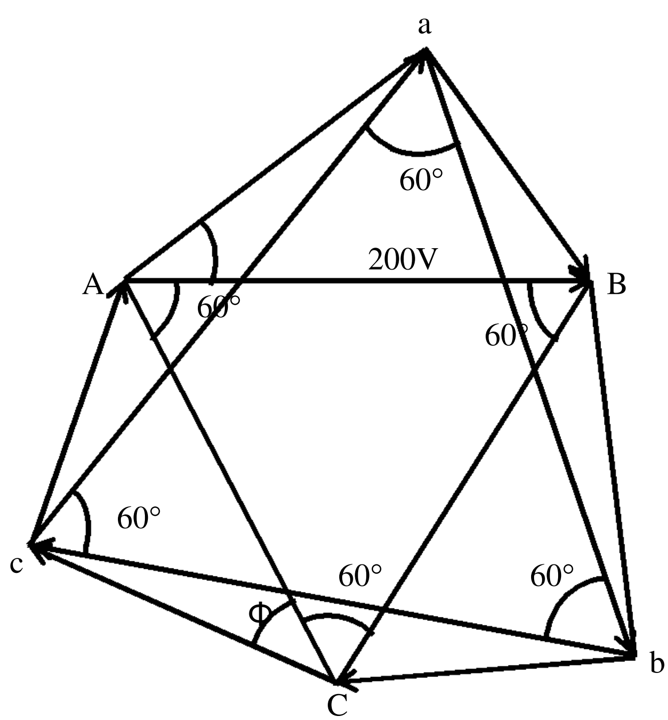
The remaining steps are not given, as they are same as shown earlier.

Example 19.2

A balanced delta-connected load with impedance per phase of $(16 - j12)\ \Omega$ shown in Fig. 19.3a, is fed from a three-phase, 200 V balanced supply with phase sequence as A-B-C. Find the voltages, V_{ab} , V_{bc} & V_{ca} , and show that they (voltages) are balanced.



(a)



(b

Fig. 19.3 (a) Circuit diagram (Example 19.2)
(b) Phasor diagram

Solution

$R_p = 16\ \Omega$; $X_{Cp} = 12\ \Omega$

$Z_{AB} = Z_p = R_p - j\,X_{Cp} = 16 - j12 = 20 \angle -36.87°\ \Omega$

For delta-connected load, $|V_L| = |V_p| = 200\ V$

Taking the line or phase voltage V_{AB} as reference, the line or phase voltages are,

$V_{AB} = 200 \angle 0°$; $V_{BC} = 200 \angle -120°$; $V_{CA} = 200 \angle +120°$

The phasor diagram is shown in Fig. 19.3b. The phase current, I_{AB} is,

$I_{AB} = V_{AB} / Z_p = (200 \angle 0°) / (20 \angle -36.87°) = 10.0 \angle +36.87° = (8.0 + j\,6.0)\ A$

The other two phase currents are,

$I_{BC} = 10.0 \angle -83.13° = (1.196 - j\,9.928)\ A$

$I_{CA} = 10.0 \angle +156.87° = (-9.196 + j\,3.928)\ A$

The voltage, V_{ab} is,

$V_{ab} = V_{aB} + V_{Bb} = (-j\,X_{Cp}) \cdot I_{AB} + R_p \cdot I_{BC}$

$= (12 \cdot 10) \angle(36.87° - 90°) + (16 \cdot 10) \angle -83.13° = 120 \angle -53.13° + 160 \angle -83.13°$

$= (72.0 - j\,96.0) + (19.14 - j158.85) = (91.14 - j\,254.85) = 270.66 \angle -70.32°$

V Alternatively,

$V_{ab} = (-j12) \cdot (8.0 + j\,6.0) + 16 \cdot (1.196 - j\,9.928) = (91.14 - j\,254.85)$

$= 270.66 \angle -70.32°\ V$

Similarly, the voltage, V_{bc} is,

$V_{bc} = V_{bC} + V_{Cc} = (-j\,X_{Cp}) \cdot I_{BC} + R_p \cdot I_{CA}$

$= (12 \cdot 10) \angle -(83.13° + 90°) + (16 \cdot 10) \angle 156.87° = 120 \angle -173.13° + 160 \angle 156.87°$

$= -(119.14 + j14.35) + (-147.14 + j\,62.85) = (-266.28 + j\,48.5)$

$= 270.66 \angle +169.68\ V$

In the same way, the voltage, V_{ca} is obtained as $V_{ca} = 270.66 \angle +49.68°\ V$

The steps are not shown here.

The three voltages, as computed, are equal in magnitude, and also at phase difference of 120° with each other in sequence. So, the three voltages can be termed as balanced ones.

A simple example (20.3) of a balanced delta-connected load is given in the following lesson

The phase and line currents for a delta-connected balanced load, fed from a three-phase supply, along with the total power consumed, are discussed in this lesson. Also some worked out problems (examples) are presented. In the next lesson, the measurement of power in three-phase circuits, both balanced and unbalanced, will be described.

Problems

19.1 A balanced load of (9-j6) Ω per phase, connected in delta, is fed from a three phase, 100V supply. Find the line current, power factor, total power, reactive VA and total VA.

19.2 Three star-connected impedances, Z_1 = (8-jb) Ω per phase, are connected in parallel with three delta-connected impedances, Z_2 = (30+j15) Ω per phase, across a three-phase 230V supply. Find the line current, total power factor, total power, reactive VA, and total VA.

List of Figures

UNIT-2

Single-phase AC Circuits

RESONANCE IN SERIES CIRCUITS

In the last lesson, the following points were described:

1. How to compute the total impedance in parallel and series-parallel circuits?
2. How to solve for the current(s) in parallel and series-parallel circuits, fed from single phase ac supply, and then draw complete phasor diagram?
3. How to find the power consumed in the circuits and also the different components, and the power factor (lag/lead)?

In this lesson, the phenomenon of the resonance in series and parallel circuits, fed from single phase variable frequency supply, is presented. Firstly, the conditions necessary for resonance in the above circuits are derived. Then, the terms, such as bandwidth and half power frequency, are described in detail. Some examples of the resonance conditions in series and parallel circuits are presented in detail, along with the respective phasor diagrams.

Keywords: Resonance, bandwidth, half power frequency, series and parallel circuits,

After going through this lesson, the students will be able to answer the following questions;

1. How to derive the conditions for resonance in the series and parallel circuits, fed from a single phase variable frequency supply?
2. How to compute the bandwidth and half power frequency, including power and power factor under resonance condition, of the above circuits?
3. How to draw the complete phasor diagram under the resonance condition of the above circuits, showing the currents and voltage drops in the different components?

Resonance in Series circuit

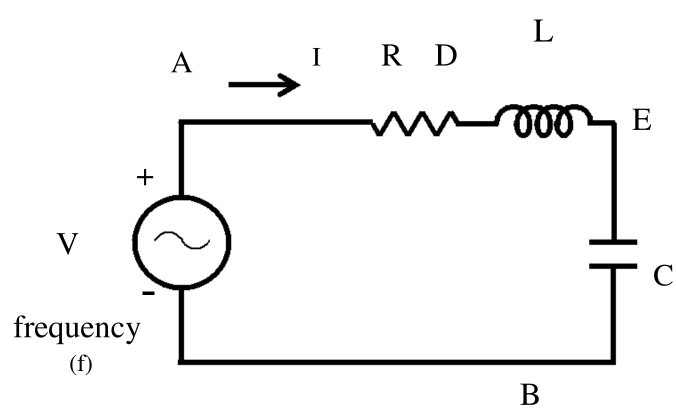


Fig. 17.1 (a) Circuit diagram.

The circuit, with resistance R, inductance L, and a capacitor, C in series (Fig. 17.1a) is connected to a single phase variable frequency (f) supply.

The total impedance of the circuit is

$$Z\angle\varphi = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

where,

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\ ; \qquad \varphi = \tan^{-1}\frac{(\omega L - 1/\omega C)}{R}\ ;\ \omega = 2\pi f$$

The current is

$$I\angle-\varphi = \frac{V\angle 0^\circ}{Z\angle\varphi} = (V/Z)\angle-\varphi$$

where $I = \dfrac{V}{\left[R^2 + (\omega L - (1/\omega C))^2\right]^{\frac{1}{2}}}$

The current in the circuit is maximum, if $\omega L = \dfrac{1}{\omega C}$.

The frequency under the above condition is

$$f_o = \frac{\omega_o}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

This condition under the magnitude of the current is maximum, or the magnitude of the impedance is minimum, is called resonance. The frequency under this condition with the constant values of inductance L, and capacitance C, is called resonant frequency. If the capacitance is variable, and the frequency, f is kept constant, the value of the capacitance needed to produce this condition is

$$C = \frac{1}{\omega^2 L} = \frac{1}{(2\pi f)^2 L}$$

The magnitude of the impedance under the above condition is $|Z| = R$, with the reactance $X = 0$, as the inductive reactance $X_l = \omega L$ is equal to capacitive reactance $X_C = 1/\omega C$. The phase angle is $\varphi = 0^\circ$, and the power factor is unity ($\cos\varphi = 1$), which means that the current is in phase with the input (supply) voltage.. So, the magnitude of the current ($|V/R|$) in the circuit is only limited by resistance, R. The phasor diagram is shown in Fig. 17.1b.

The magnitude of the voltage drop in the inductance L/capacitance C (both are equal, as the reactance are equal) is $I\cdot\omega_o L = I\cdot(1/\omega_o C)$.

The magnification of the voltage drop as a ratio of the input (supply) voltage is

$$Q = \frac{\omega_o L}{R} = \frac{2\pi f_o L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

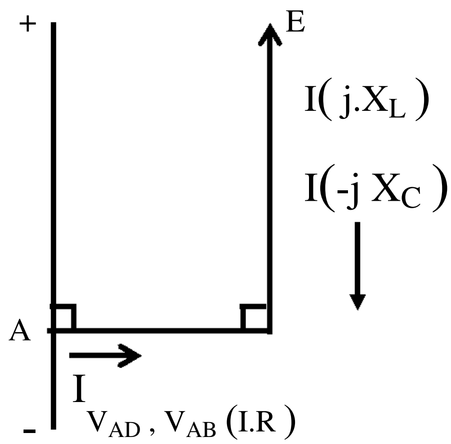


Fig. 17.1 (b) Phasor Diagram

It is termed as Quality (Q) factor of the coil.

The impedance of the circuit with the constant values of inductance L, and capacitance C is minimum at resonant frequency (f_o), and increases as the frequency is changed, i.e. increased or decreased, from the above frequency. The current is maximum at $f = f_o$, and decreases as frequency is changed ($f > f_o$, or $f < f_o$), i.e. $f \neq f_o$. The variation of current in the circuit having a known value of capacitance with a variable frequency supply is shown in Fig. 17.2.

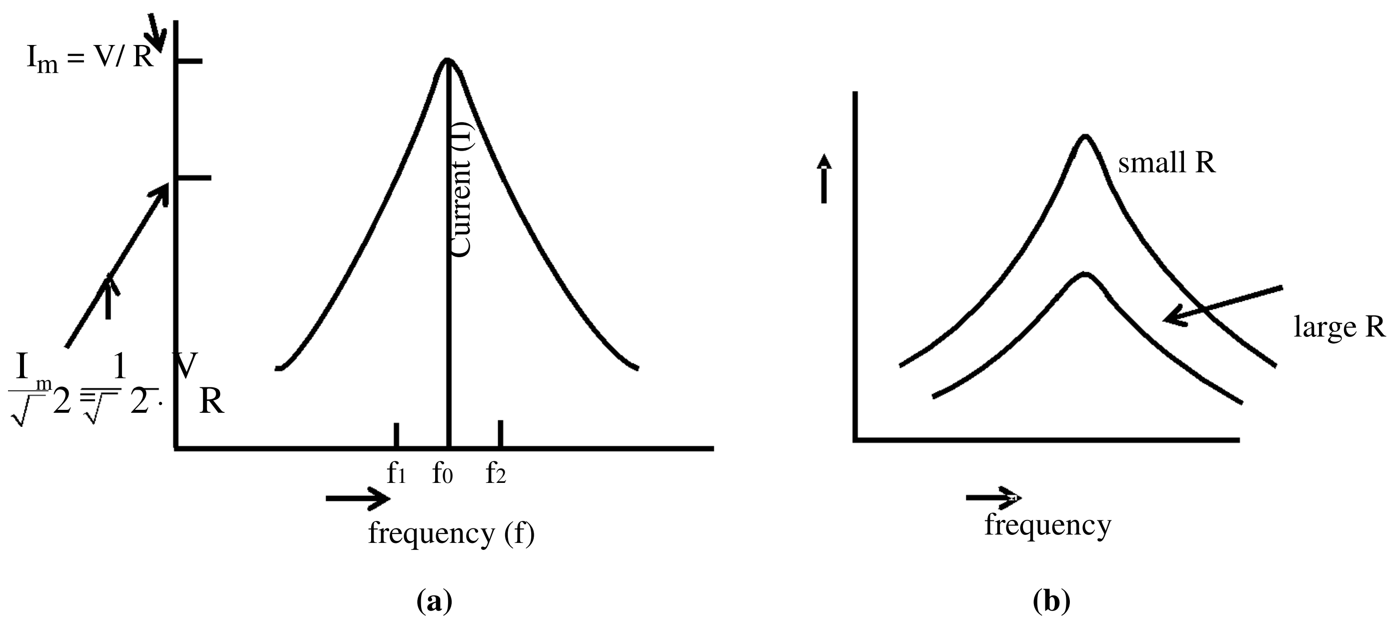


Fig. 17.2 Variation of current under variable frequency supply

The maximum value of the current is (V / R). If the magnitude of the current is reduced to (1/ $\sqrt{2}$) of its maximum value, the power consumed in R will be half of that with the maximum current, as power is $I^2 R$. So, these points are termed as half power points. If the two frequencies are taken as f_1 and f_2, where $f_1 = f_0 - f/2$ and $f_2 = f_0 + f/2$, the band width being given by $f = f_2 - f_1$.

The magnitude of the impedance with the two frequencies is

$$Z = \left[R^2 + \left(2\pi(f_0 \pm f/2)\, L - \frac{1}{2\pi(f_0 \pm f/2)\, C} \right)^2 \right]^{\frac{1}{2}}$$

As ($2\pi f_0 L = 1/2\pi f_0 C$) and the ratio ($f/2f_0$) is small, the magnitude of the reactance of the circuit at these frequencies is $X = X_{L0}\,(f/f_0)$. As the current is $(1/\sqrt{2})$ of its maximum value, the magnitude of the impedance is ($\sqrt{2}$) of its minimum value (R) at resonant frequency.

So, $Z = \sqrt{2}\cdot R = \left[R^2 + (X_{L0}(f/f_0))^2\right]^{\frac{1}{2}}$

From the above, it can be obtained that $(f/f_0)X_{L0} = R$

or $f = f_2 - f_1 = \dfrac{R f_0}{X_{L0}} = \dfrac{R f_0}{2\pi f_0 L} = \dfrac{R}{2\pi L}$

The band width is given by $f = f_2 - f_1 = R/(2\pi L)$

It can be observed that, to improve the quality factor (Q) of a coil, it must be designed to have its resistance, R as low as possible. This also results in reduction of band width and losses (for same value of current). But if the resistance, R cannot be decreased, then Q will decrease, and also both band width and losses will increase.

Example 17.1

A constant voltage of frequency, 1 MHz is applied to a lossy inductor (r in series with L), in series with a variable capacitor, C (Fig. 17.3). The current drawn is maximum, when C = 400 pF; while current is reduced to $(1/\sqrt{2})$ of the above value, when C = 450 pF. Find the values of r and L. Calculate also the quality factor of the coil, and the bandwidth.

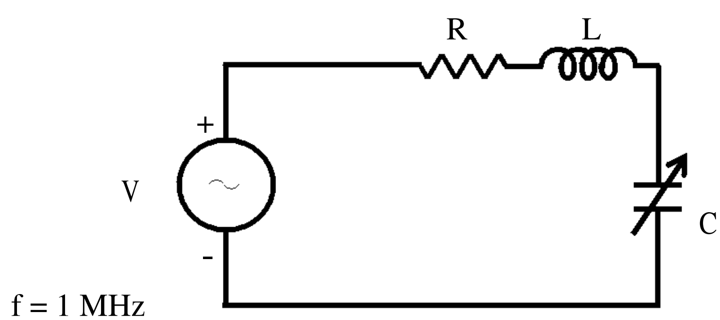


Fig. 17.3 Circuit diagram

Solution

$f = 1\text{ MHz} = 10^6\text{ Hz}$ $\qquad \omega = 2\pi f \qquad C = 400\, pF = 400\cdot 10^{-12}\, F$

$I_{max} = V/r$ as $X_L = X_C$ $\qquad X_c = \dfrac{1}{2\pi f C} = \dfrac{1}{2\pi\cdot 10^6\cdot 400\cdot 10^{-12}} = 398\ \Omega$

$X_L = X_C = 2\pi f L = 398\ \Omega \qquad L = \dfrac{398.0}{2\pi \cdot 10^6} = 63.34\ \mu H$

$C_1 = 450\ pF \qquad X_{C1} == \dfrac{1}{2\pi \cdot 10^6 \cdot 450 \cdot 10^{-12}} = 353.7\ \Omega$

$Z\angle\varphi = r + j\left(X_L - X_{C1}\right) = r + j\,(398.0 - 353.7) = (r + j\,44.3)\ \Omega$

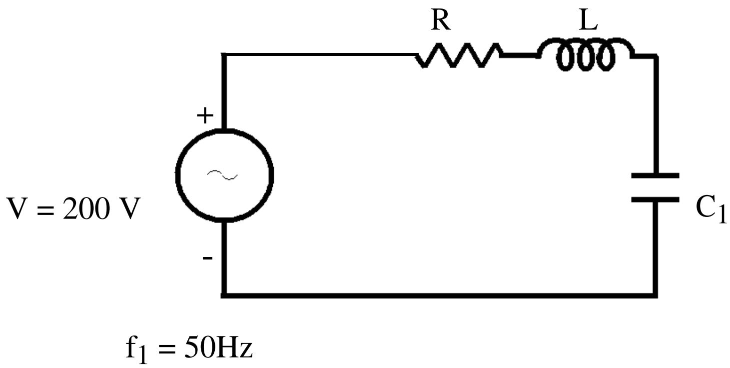
$I = \dfrac{I_{max}}{\sqrt{2}} = \dfrac{V}{\sqrt{2} \cdot r} = \dfrac{V}{Z} = \dfrac{V}{\sqrt{r^2 + (44.3)^2}}$

From above, $\sqrt{2} \cdot r = \sqrt{r^2 + (44.3)^2}$ or $2r^2 = r^2 + (44.3)^2$
or $r = 44.3\ \Omega$

The quality factor of the coil is $Q = \dfrac{X_L}{r} = \dfrac{398.0}{44.3} = 8.984$

The band with is

$f = f_2 - f_1 = \dfrac{r}{2\pi L} = \dfrac{44.3}{2\pi \cdot 63.34 \cdot 10^{-6}} = \dfrac{44.3}{398 \cdot 10^{-6}} = 0.1113 \cdot 10^6 = 0.1113\ MHz$
$= 111.3 \cdot 10^3 = 111.3\ kHz$



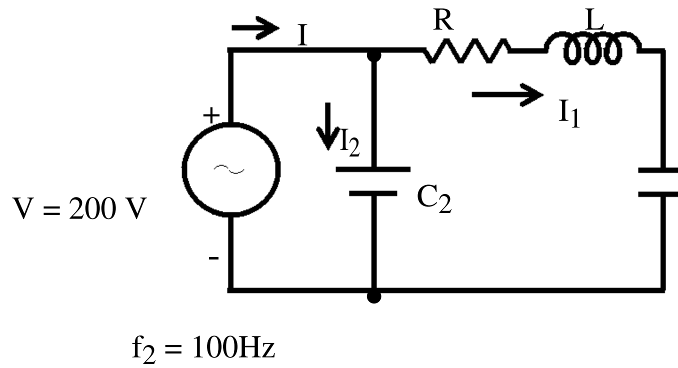


Fig. 17.6 (a) Circuit diagram **Fig. 17.6 (b) Circuit diagram**

Solution

$f_1 = 50$ Hz V = 200 V R = 15 Ω L = 0.75 H

From the condition of resonance at 50 Hz in the series circuit,

$$X_{L1} = \omega_1 L = 2\pi f_1 L = X_{C1} = \frac{1}{\omega_1 C_1} = \frac{1}{2\pi f_1 C_1}$$

$$\text{So, } C_1 = \frac{1}{(2\pi f_1)^2 L} = \frac{1}{(2\pi\cdot 50)^2\cdot 0.75} = 13.5\cdot 10^{-6} = 13.5\ \mu F$$

The maximum current drawn from the supply is, $I_{max} = V/R = 200/15 = 13.33\ A$

$f_2 = 100$ Hz $\omega_2 = 2\pi f_2 = 2\pi\cdot 100 = 628.3\ rad/s$

$$X_{L2} = 2\pi f_2 L = 2\pi\cdot 100\cdot 0.75 = 471.24\ \Omega$$

$$X_{C2} = \frac{1}{2\pi f_2 C_1} = \frac{1}{2\pi\cdot 100\cdot 13.5\cdot 10^{-6}} = 117.8.\ \Omega$$

$$Z_1\angle\varphi_1 = R + j\,(X_{L2} - X_{C2}) = 15 + j\,(471.24 - 117.8) = 15 + j\,353.44$$

$$= 353.75\angle 87.57^\circ\ \Omega$$

$$Y_1\angle -\varphi_1 = \frac{1}{Z_1\angle\varphi_1} = \frac{1}{15 + j\,353.44} = \frac{1}{353.75\angle 87.57^\circ} = 2.827\cdot 10^{-3}\angle -87.57^\circ$$

$$= (0.12 - j\,2.824)\cdot 10^{-3}\ \Omega^{-1}$$

$$Y_2 = 1/Z_2 = j\,(\omega_2 C_2)$$

As the combination is resistive in nature, the total admittance is

$$Y\angle 0^\circ = Y + j\,0 = Y_1 + Y_2 = (0.12 - j\,2.824)\cdot 10^{-3} + j\omega_2 C_2$$

From the above expression, $\omega_2 C_2 = 628.3\cdot C_2 = 2.824\cdot 10^{-3}$

$$\text{or, } C_2 = \frac{2.824\cdot 10^{-3}}{628.3} = 4.5\cdot 10^{-6} = 4.5\ \mu F$$

The total admittance is $Y = 0.12 \cdot 10^{-3}\ \Omega^{-1}$
The total impedance is $Z = 1/Y = 1/\left(0.12 \cdot 10^{-3}\right) =$
$8.33 \cdot 10^{3}\ \Omega = 8.33\ k\Omega$ The total current drawn from
the supply is

$I = V \cdot Y = V/Z = 200 \cdot 0.12 \cdot 10^{-3} = 0.024\ A = 24 \cdot 10^{-3} = 24\ mA$ The phasor diagram for the circuit (Fig. 17.6b) is shown in Fig. 17.6c.

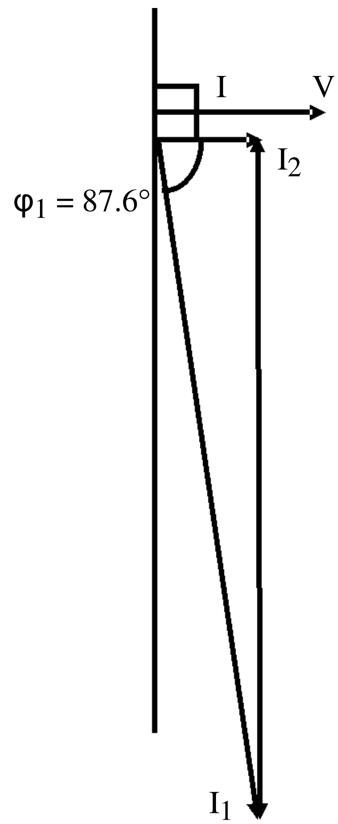


Fig. 17.6 (c) Phasor diagram

The condition for resonance in both series and parallel circuits fed from single phase ac supply is described. It is shown that the current drawn from the supply is at unity power factor (upf) in both cases. The value of the capacitor needed for resonant condition with a constant frequency supply, and the resonant frequency with constant value of capacitance, have been derived. Also taken up is the case of a lossy inductance coil in parallel with a capacitor under variable frequency supply, where the total current will be at upf. The quality factor of the coil and the bandwidth of the series circuit with known value of capacitance have been determined. This is the final lesson in this module of single phase ac circuits. In the next module, the circuits fed from three phase ac supply will be described.

Problems

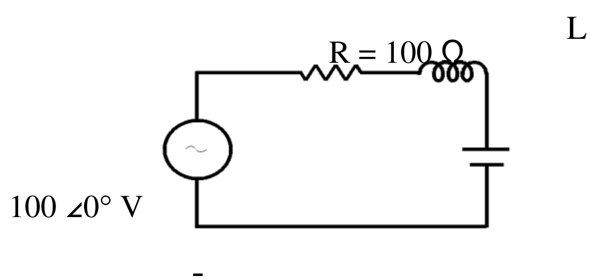
17.1 A coil having a resistance of 20 Ω and inductance of 20 mH, in series with a capacitor is fed from a constant voltage variable frequency supply. The maximum current is 10 A at 100 Hz. Find the two cut-off frequencies, when the current is 0.71 A.

17.2 With the ac voltage source in the circuit shown in Fig. 17.7 operating a frequency of f, it was found that $I = 1.0 \angle 0° $ A. When the source frequency was doubled (2f), the current became $I = 0.707 \angle -45°$ A. Find:

a) The frequency f, and
b) The inductance L, and also the reactances, X_L and X_C at 2f

17.3 For the circuit shown in Fig. 17.8,

a) Find the resonant frequency f_0, if R = 250 Ω, and also calculate Q_0 (quality factor), BW (band width) in Hz, and lower and upper cut-off frequencies (f_1 and f_2) of the circuit.
b) Suppose it was desired to increase the selectivity, so that BW was 65 Hz. What value of R would accomplish this?



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UNIT-2

Single-phase AC Circuits

Solution of Current in R-L-C Series Circuits

In the last lesson, two points were described:

1. How to represent a sinusoidal (ac) quantity, i.e. voltage/current by a phasor?
2. How to perform elementary mathematical operations, like addition/ subtraction and multiplication/division, of two or more phasors, represented as complex quantity?

Some examples are also described there. In this lesson, the solution of the steady state currents in simple circuits, consisting of resistance R, inductance L and/or capacitance C connected in series, fed from single phase ac supply, is presented. Initially, only one of the elements R / L / C, is connected, and the current, both in magnitude and phase, is computed. Then, the computation of total reactance and impedance, and the current, in the circuit consisting of two components, R & L / C only in series, is discussed. The process of drawing complete phasor diagram with current(s) and voltage drops in the different components is described. Lastly, the computation of total power and also power consumed in the components, along with the concept of power factor, is explained.

Keywords: Series circuits, reactance, impedance, phase angle, power, power factor.

After going through this lesson, the students will be able to answer the following questions;

1. How to compute the total reactance and impedance of the R-L-C series circuit, fed from single phase ac supply of known frequency?
2. How to compute the current and also voltage drops in the components, both in magnitude and phase, of the circuit?
3. How to draw the complete phasor diagram, showing the current and voltage drops?
4. How to compute the total power and also power consumed in the components, along with power factor?

Solution of Steady State Current in Circuits Fed from Single-phase AC Supply

Elementary Circuits

1. Purely resistive circuit (R only)

The instantaneous value of the current though the circuit (Fig. 14.1a) is given by,

$$i = \frac{v}{R} = \frac{V_m}{R} \sin \omega t = I_m \sin \omega t$$

where,

I_m and V_m are the maximum values of current and voltage respectively.

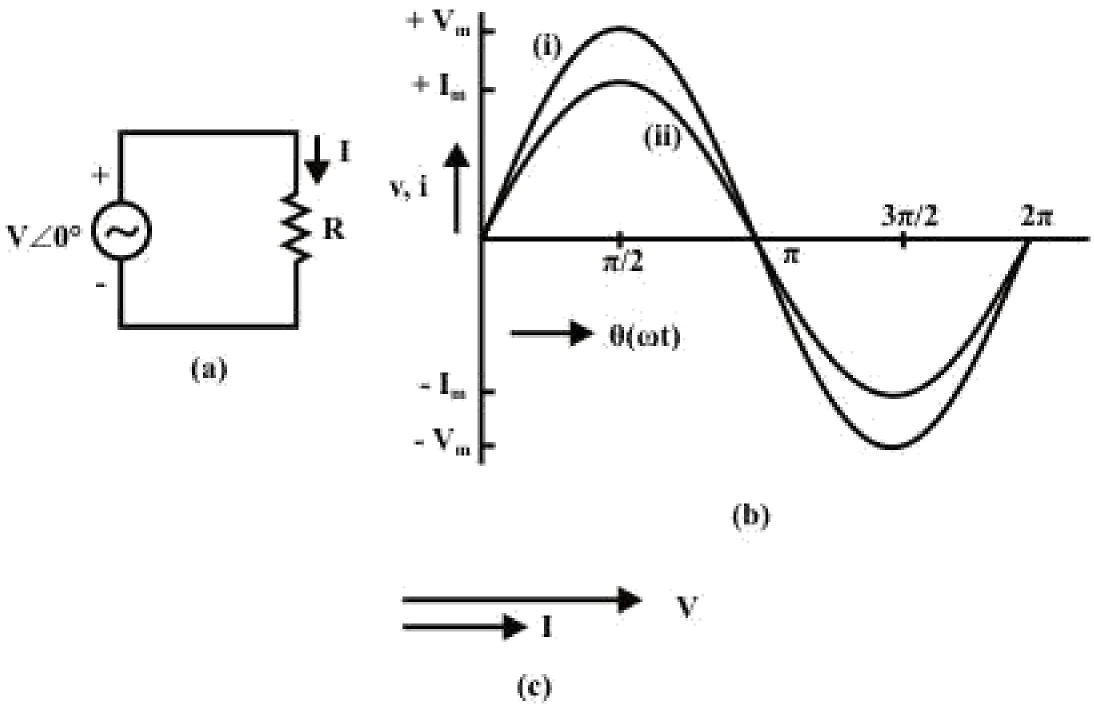


Fig. 14.1: Circuit with Resistance (R)
(a) Circuit diagram
(b) Waveforms: (i) Voltage (ii) Current
(c) Phasor diagram

The rms value of current is given by

$I = \frac{I_m}{\sqrt{2}} =$ [redacted] R R

In phasor notation,

$\bar{V} = V\ 0° = V\,(1 + j\,0) = V + j\,0$

$\bar{I} = I\ 0° = I\,(1 + j\,0) = I + j\,0$

The impedance or resistance of the circuit is obtained as,

$$\frac{\bar{V}}{\bar{I}} = \frac{V\ 0°}{I\ 0°} = Z\ 0° = R + j\,0$$

Please note that the voltage and the current are in phase ($\varphi = 0°$), which can be observed from phasor diagram (Fig. 14.1b) with two (voltage and current) phasors, and also from the two waveforms (Fig. 14.1c).

In ac circuit, the term, Impedance is defined as voltage/current, as is the resistance in dc circuit, following Ohm's law. The impedance, Z is a complex quantity. It consists of real part as resistance R, and imaginary part as reactance X, which is zero, as there is no inductance/capacitance. All the components are taken as constant, having linear V-I characteristics. In the three cases being considered, including this one, the power

consumed and also power factor in the circuits, are not taken up now, but will be described later in this lesson.

2. Purely inductive circuit (L only)

For the circuit (Fig. 14.2a), the current i, is obtained by the procedure described here.

As $v = L\dfrac{di}{dt} = V_m \sin \omega t = \sqrt{2}\, V \sin \omega t$,

$$di = \frac{\sqrt{2}\, V}{L} \sin (\omega t)\, dt$$

Integrating,

$$i = -\frac{\sqrt{2}\, V}{\omega L} \cos \omega t = \frac{\sqrt{2}\, V}{\omega L} \sin (\omega t - 90°) = I_m \sin (\omega t - 90°) = \sqrt{2}\, I \sin (\omega t - 90°)$$

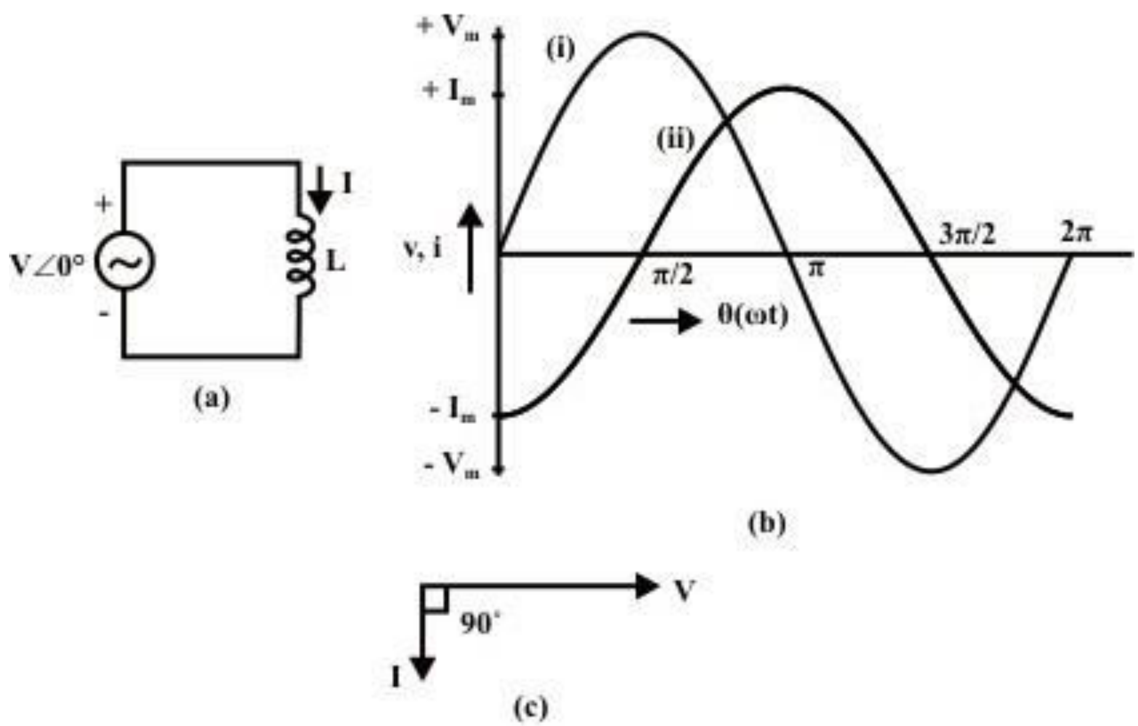


Fig. 14.2: Circuit with Inductance (L)
(a) Circuit diagram
(b) Waveforms: (i) Voltage (ii) Current
(c) Phasor diagram

It may be mentioned here that the current i, is the steady state solution, neglecting the constant of integration. The rms value, I is

$$\bar{I} = \frac{\bar{V}}{\omega L} = I \angle -90°$$

$$\bar{V} = V \angle 0° = V + j\,0 \; ; \qquad \bar{I} = I \angle -90° = 0 - j\,I$$

The impedance of the circuit is

$$Z\angle\varphi = \frac{\bar{V}}{\bar{I}} = \frac{V\angle 0^\circ}{I\angle -90^\circ} = \frac{V}{-jI} = j\omega L = 0 + jX_L = X_L\angle 90^\circ = \omega L\angle 90^\circ$$

where, the inductive reactance is $X_L = \omega L = 2\pi f L$.

Note that the current lags the voltage by $\varphi = +90^\circ$. This can be observed both from phasor diagram (Fig. 14.2b), and waveforms (Fig. 14.2c). As the circuit has no resistance, but only inductive reactance $X_L = \omega L$ (positive, as per convention), the impedance Z is only in the y-axis (imaginary).

3. **Purely capacitive circuit (C only)**

The current i, in the circuit (Fig. 14.3a), is,

$$i = C\frac{dv}{dt}$$

Substituting $v = \sqrt{2}\,V\sin\omega t = V_m \sin\omega t$, i is

$$i = C\frac{d}{dt}\left(\sqrt{2}\,V\sin\omega t\right) = \sqrt{2}\,\omega C V\cos\omega t = \sqrt{2}\,\omega C V\sin(\omega t + 90^\circ) = \sqrt{2}\,I\sin(\omega t + 90^\circ)$$

$$= I_m \sin(\omega t + 90^\circ)$$

The rms value, I is

$$\bar{I} = \omega C \bar{V} = \frac{\bar{V}}{1/(\omega C)} = I\angle 90^\circ$$

$$\bar{V} = V\angle 0^\circ = V + j0\ ; \qquad \bar{I} = I\angle 90^\circ = 0 + jI$$

The impedance of the circuit is

$$Z\angle\varphi = \frac{\bar{V}}{\bar{I}} = \frac{V\angle 0^\circ}{I\angle 90^\circ} = \frac{V}{jI} = \frac{1}{j\omega C} = -\frac{j}{\omega C} = 0 - jX_C = X_C\angle -90^\circ = \frac{1}{\omega C}\angle 90^\circ$$

where, the capacitive reactance is $X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$.

Note that the current leads the voltage by $\varphi = 90^\circ$ (this value is negative, i.e. $\varphi = -90^\circ$), as per convention being followed here. This can be observed both from phasor diagram (Fig. 14.3b), and waveforms (Fig. 14.3c). As the circuit has no resistance, but only capacitive reactance, $X_c = 1/(\omega C)$ (negative, as per convention), the impedance Z is only in the y-axis (imaginary).

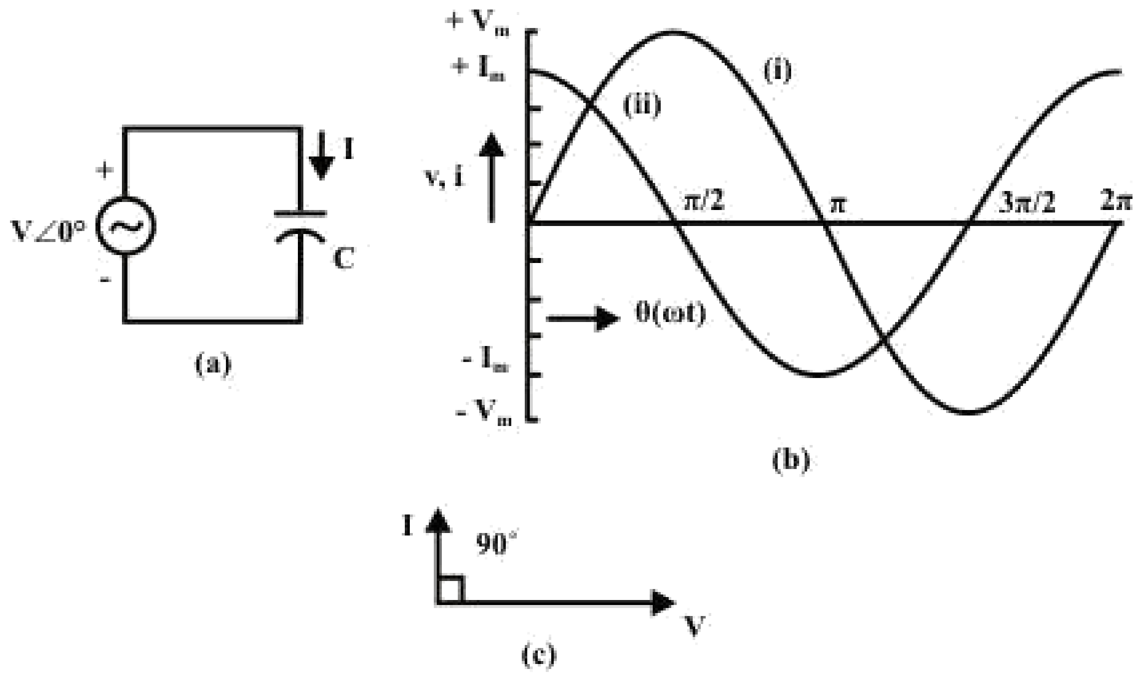


Fig. 14.3: Circuit with Capacitance (C)
(a) Circuit diagram
(b) Waveforms: (i) Voltage (ii) Current
(c) Phasor diagram

Series Circuits

1. Inductive circuit (R and L in series)

The voltage balance equation for the R-L series circuit (Fig. 14.4a) is,

$$v = R\,i + L\frac{di}{dt}$$

where, $v = \sqrt{2}\,V \sin \omega t = V_m \sin \omega t = \sqrt{2}\,V \sin\theta$, θ being ωt.

The current, i (in steady state) can be found as

$$i = \sqrt{2}\,I \sin(\omega t - \varphi) = I_m \sin(\omega t - \varphi) = \sqrt{2}\,I \sin(\theta - \varphi)$$

The current, $i(t)$ in steady state is sinusoidal in nature (neglecting transients of the form shown in the earlier module on dc transients). This can also be observed, if one sees the expression of the current, $i = I_m \sin(\omega t)$ for purely resistive case (with R only), and $i = I_m \sin(\omega t - 90°)$ for purely inductive case (with L only).

Alternatively, if the expression for i is substituted in the voltage equation, the equation as given here is obtained.

$2\ V\sin\omega t = R\quad 2\ I\sin(\omega t - \varphi) + \omega L\ 2\ I\cos(\omega t - \varphi)$

If, first, the trigonometric forms in the RHS side is expanded in terms of sin ωt and cos ωt, and then equating the terms of sin ωt and cos ωt from two (LHS & RHS) sides, the two equations as given here are obtained.

$V = (R\cos\varphi + \omega L\sin\varphi)\ I$, and

$0 = (-R\sin\varphi + \omega L\cos\varphi)$

From these equations, the magnitude and phase angle of the current, I are derived.

From the second one, $\tan\varphi = (\omega L / R)$

So, phase angle, $\varphi = \tan^{-1}(\omega L / R)$

Two relations, $\cos\varphi = (R / Z)$, and $\sin\varphi = (\omega L / Z)$, are derived, with the term (impedance), $Z = \sqrt{R^2 + (\omega L)^2}$

If these two expressions are substituted in the first one, it can be shown that the magnitude of the current is $I = V / Z$, with both V and Z in magnitude only.

The steps required to find the rms value of the current I, using complex form of impedance, are given here.

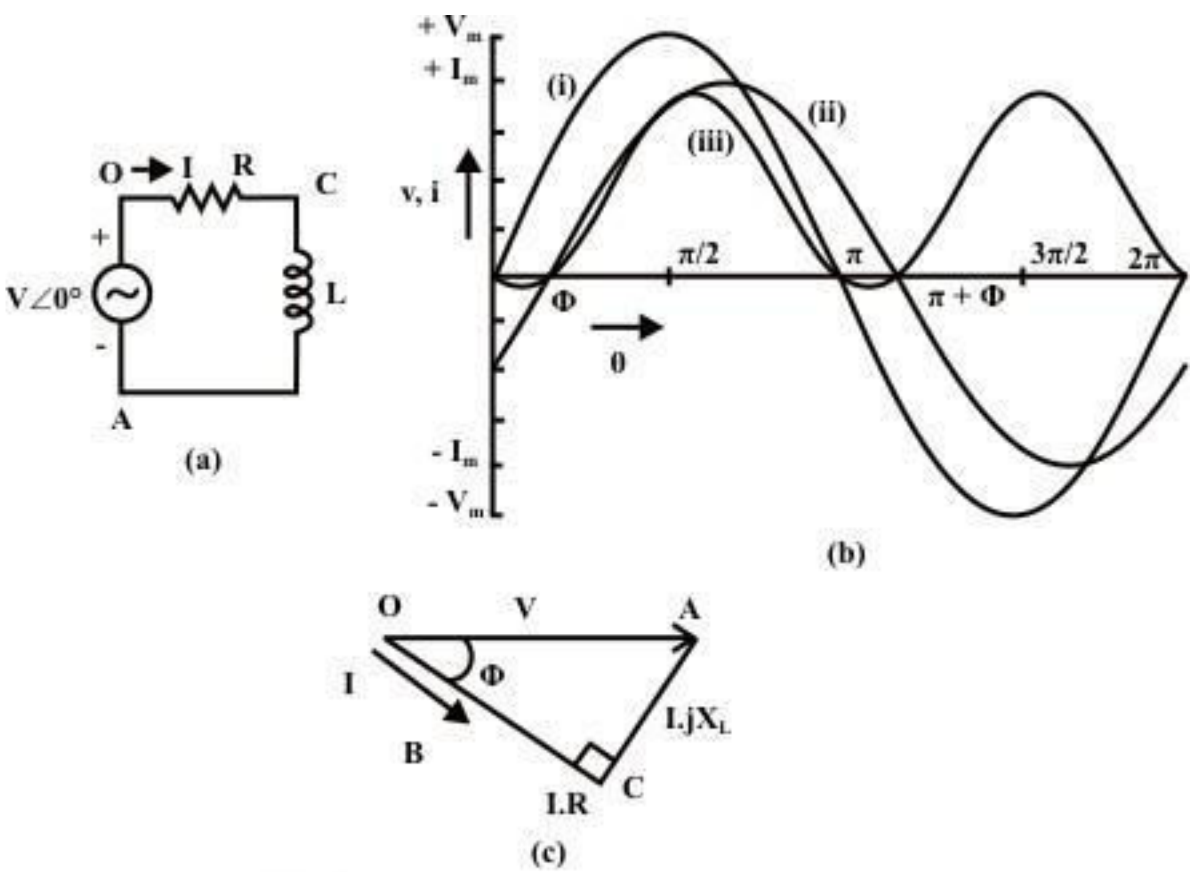


Fig. 14.4: Circuit with Resistance (R) and Inductance (L) in series.
(a) Circuit diagram
(b) Waveforms: (i) Voltage (ii) Current (iii) Power
(c) Phasor diagram

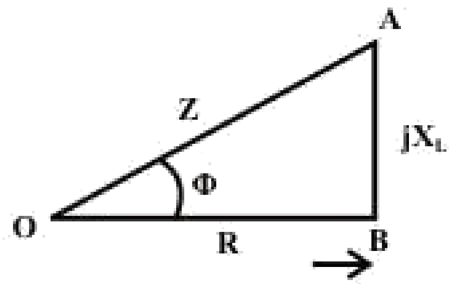


Fig. 14.5: The complex form of the impedance (R-L series circuit)

The impedance (Fig. 14.5) of the inductive (R-L) circuit is,

$$Z\angle\varphi = R + jX_L = R + j\,\omega L$$

where,

$$Z = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + (\omega L)^2} \quad \text{and} \quad \varphi = \tan^{-1}\frac{X_L}{R} = \tan^{-1}\frac{\omega L}{R}$$

$$\bar{I} = \frac{\bar{V}}{\bar{Z}} = \frac{V + j0}{Z\angle\varphi} = \frac{V + j0}{R + jX_L} = \frac{V + j0}{R + j\omega L}$$

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + X_L^2}} = \frac{V}{\sqrt{R^2 + (\omega L)^2}}$$

Note that the current lags the voltage by the angle φ, value as given above. In this case, the voltage phasor has been taken as reference phase, with the current phasor lagging the voltage phasor by the angle, φ. But normally, in the case of the series circuit, the current phasor is taken as reference phase, with the voltage phasor leading the current phasor by φ. This can be observed both from phasor diagram (Fig. 14.4b), and waveforms (Fig. 14.4c). The inductive reactance X_L is positive. In the phasor diagram, as one move from voltage phasor to current phasor, one has to go in the clockwise direction, which means that phase angle, φ is taken as positive, though both phasors are assumed to move in anticlockwise direction as shown in the previous lesson.

The complete phasor diagram is shown in Fig. 14.4b, with the voltage drops across the two components and input (supply) voltage (OA), and also current (OB). The voltage phasor is taken as reference. It may be observed that

$$V_{OC}\,(= I\,R) + V_{CA}\,[= I\,(j\,X_L)] = V_{OA}\,(= I\,Z)\,,$$

using the Kirchoff's second law relating to the voltage in a closed loop. The phasor diagram can also be drawn with the current phasor as reference, as will be shown in the next lesson.

Power consumed and Power factor

From the waveform of instantaneous power ($W = v\,i$) also shown in Fig. 14.4c for the above circuit, the average power is,

$$W = \frac{1}{\pi}\int_0^{\pi} v\,i\,d\theta = \frac{1}{\pi}\int_0^{\pi} \sqrt{2}\,V \sin\theta \,\sqrt{2}\,I \sin(\theta-\varphi)\,d\theta = \frac{1}{\pi}\int_0^{\pi} V\,I\left[\cos\varphi - \cos(2\theta-\varphi)\right] d\theta$$

$$= \frac{1}{\pi} V\,I \cos\varphi\,\theta\Big|_0^{\pi} - \frac{V\,I}{2}\sin(2\theta-\varphi)\Big|_0^{\pi}$$

$$= \frac{1}{\pi} V\,I\cos\varphi\,(\pi-0) - \frac{V\,I}{2}\left[\sin(2\pi-\varphi)+\sin\varphi\right] = V\,I\cos\varphi$$

Note that power is only consumed in resistance, R only, but not in the inductance, L. So, $W = I^2 R$.

$$\text{Power factor} = \frac{\textit{average power}}{\textit{apparent power}} = \frac{V\,I\cos\varphi}{V\,I} = \cos\varphi = \frac{R}{Z} = \frac{R}{\sqrt{R^2+(\omega L)^2}}$$

The power factor in this circuit is less than 1 (one), as $0° \le \varphi \le 90°$, φ being positive as given above.

For the resistive (R) circuit, the power factor is 1 (one), as $\varphi = 0°$, and the average power is $V\,I$.

For the circuits with only inductance, L or capacitance, C as described earlier, the power factor is 0 (zero), as $\varphi = \pm 90°$. For inductance, the phase angle, or the angle of the impedance, $\varphi = +90°$ (lagging), and for capacitance, $\varphi = -90°$ (leading). It may be noted that in both cases, the average power is zero (0), which means that no power is consumed in the elements, L and C.

The complex power, Volt-Amperes (VA) and reactive power will be discussed after the next section.

2. Capacitive circuit (R and C in series)

This part is discussed in brief. The voltage balance equation for the R-C series circuit (Fig. 14.6a) is,

$$v = R\,i + \frac{1}{C}\int i\,dt = \sqrt{2}\,V\sin\omega t$$

The current is

$$i = \sqrt{2}\,I\sin(\omega t + \varphi)$$

The reasons for the above choice of the current, i, and the steps needed for the derivation of the above expression, have been described in detail, in the case of the earlier example of inductive (R-L) circuit. The same set of steps has to be followed to derive the current, *i* in this case.

Alternatively, the steps required to find the rms value of the current I, using complex form of impedance, are given here.

The impedance of the capacitive (R-C) circuit is,

$$Z - \varphi = R - j X_C = R - j \frac{1}{\omega C}$$

$$Z = \sqrt{R^2 + X_c^2} = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} \text{ and}$$

$$\varphi = \tan^{-1}\left(-\frac{X_C}{R}\right) = \tan^{-1}\left(-\frac{1}{\omega C R}\right) = -\tan^{-1}\left(\frac{1}{\omega C R}\right)$$

$$I\angle\varphi = \frac{V\angle 0°}{Z\angle -\varphi} = \frac{V + j\,0}{R - jX_C} = \frac{V + j\,0}{R - j\,(1/\omega C)}$$

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + X_C^2}} = \frac{V}{\sqrt{R^2 + (1/\omega C)^2}}$$

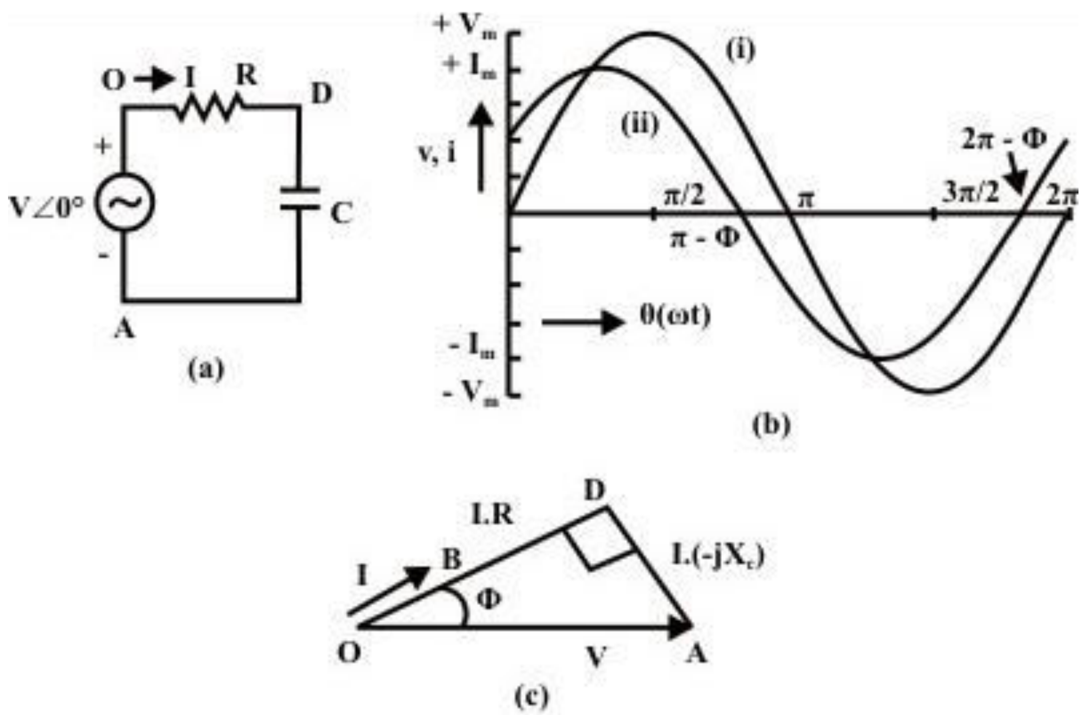


Fig. 14.6: Circuit with Resistance (R) and Capacitance (C) in series.
(a) Circuit diagram
(b) Waveforms: (i) Voltage (ii) Current
(c) Phasor diagram

Note that the current leads the voltage by the angle φ, value as given above. In this case, the voltage phasor has been taken as reference phase, with the current phasor leading the voltage phasor by the angle, φ. But normally, in the case of the series circuit, the current phasor is taken as reference phase, with the voltage phasor lagging the current phasor by φ. This can be observed both from phasor diagram (Fig. 14.6b), and waveforms (Fig. 14.6c). The capacitive reactance X_C is negative. In the phasor diagram, as one move from voltage phasor to current phasor, one has to go in the anticlockwise direction, which means that phase angle, φ is taken as negative. This is in contrast to the case as described earlier. The complete phasor diagram is shown in Fig. 14.6b, with the voltage drops across the two components and input (supply) voltage, and also current. The voltage phasor is taken as reference.

The power factor in this circuit is less than 1 (one), with φ being same as given above. The expression for the average power is $P = V\,I\cos\varphi$, which can be obtained by the method shown above. The power is only consumed in the resistance, R, but not in the capacitance, C. One example is included after the next section.

Complex Power, Volt-Amperes (VA) and Reactive Power

The complex power is the product of the voltage and complex conjugate of the current, both in phasor form. For the inductive circuit, described earlier, the voltage ($V\angle 0°$) is taken as reference and the current ($I\angle -\varphi = I\cos\varphi - j\,I\sin\varphi$) is lagging the voltage by an angle, φ. The complex power is

$$\bar{S} = V\bar{I}^{*} = V\angle 0°\; I\angle\varphi = (V\,I)\angle\varphi = V\,I\cos\varphi + j\,V\,I\sin\varphi = P + j\,Q$$

The Volt-Amperes (S), a scalar quantity, is the product of the magnitudes the voltage and the current. So, $S = V\,I = \sqrt{P^2 + Q^2}$. It is expressed in VA.

The active power (W) is

$P = \text{Re}\,(\bar{S}) = \text{Re}\,(V\bar{I}^{*}) = V\,I\cos\varphi$, as derived earlier.

The reactive power (VAr) is given by $Q = \text{Im}\,(\bar{S}) = \text{Im}\,(V\bar{I}^{*}) = V\,I\sin\varphi$.

As the phase angle, φ is taken as positive in inductive circuits, the reactive power is positive. The real part, ($I\cos\varphi$) is in phase with the voltage V, whereas the imaginary part, $I\sin\varphi$ is in quadrature ($-90°$) with the voltage V. But in capacitive circuits, the current ($I\angle\varphi$) leads the voltage by an angle φ, which is taken as negative. So, it can be stated that the reactive power is negative here, which can easily be derived

Example 14.1

A voltage of 120 V at 50 Hz is applied to a resistance, R in series with a capacitance, C (Fig. 14.7a). The current drawn is 2 A, and the power loss in the resistance is 100 W. Calculate the resistance and the capacitance.

Solution

$V = 120$ V $I = 2$ A $P = 100$ W $f = 50$ Hz

$R = P / I^2 = 100 / 2^2 = 25\ \Omega$

$Z = \sqrt{R^2 + X_C^2} = V / I = 120 / 2 = 60\ \Omega$

$X_c = 1/(2\pi f\, C) = \sqrt{Z^2 - R^2} = \sqrt{(60)^2 - (25)^2} = 54.54\ \Omega$

$$C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi\ 50.0 \cdot 54.54} = 58.36\ 10^{-6} = 58.36\ \mu F$$

The power factor is, $\cos\varphi = R / Z = 25 / 60 = 0.417\ (lead)$

The phase angle is $\varphi = \cos^{-1}(0.417) = 65.38°$

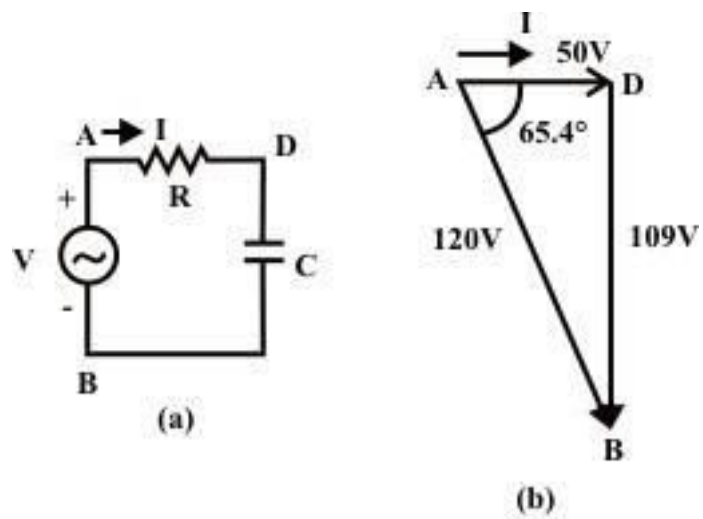


Fig. 14.7: (a) Circuit diagram
(b) Phasor diagram

The phasor diagram, with the current as reference, is shown in Fig. 14.7b. The examples, with lossy inductance coil (r in series with L), will be described in the next lesson. The series circuit with all elements, R. L & C, along with parallel circuits, will be taken up in the next lesson.

Problems

14.1 Calculate the power factor in the following cases for the circuit with the elements, as given, fed from a single phase ac supply.

(i) With resistance, R only, but no L and C

(a) 1.0 (Φ=0°) (b) 0.0 lagging (Φ=+90°)

(c) 0.0 leading (Φ=-90°) (d) None of the above

(ii) with only pure/lossless inductance, L, but no R and C

(a) 1.0 (Φ=0°) (b) 0.0 lagging (Φ=+90°)

(c) 0.0 leading (Φ=-90°) (d) None of the above

(iii) with only pure capacitance, C, but no R and L.

(a) 1.0 (Φ=0°) (b) 0.0 lagging (Φ=+90°)

(c) 0.0 leading (Φ=-90°) (d) None of the above

14.2 Calculate the current and power factor (lagging / leading) in the following cases for the circuits having impedances as given, fed from an ac supply of 200 V. Also draw the phasor diagram in all cases.

(i) $Z = (15+j20)\ \Omega$

(ii) $Z = (14-j14)\ \Omega$

(iii) $Z = R + j\,(X_L - X_C)$, where $R = 10\ \Omega$, $X_L = 20\ \Omega$, and $X_C = 10\ \Omega$.

14.3 A 200 V, 50 Hz supply is connected to a resistance (R) of 20 Ω in series with an iron cored choke coil (r in series with L). The readings of the voltmeters across the resistance and across the coil are 120 V and 150 V respectively. Find the loss in the coil. Also find the total power factor. Draw the phasor diagram.

14.4 A circuit, with a resistance, R and a lossless inductance in series, is connected across an ac supply (V) of known frequency (f). A capacitance, C is now connected in series with R-L, with V and f being constant. Justify the following statement with reasons.

The current in the circuit normally increases with the introduction of C.

Under what condition, the current may also decrease. Explain the condition with reasons.

UNIT-2
Single-phase AC Circuits

Representation of Sinusoidal Signal by a Phasor and Solution of Current in R-L-C Series Circuits

In the last lesson, two points were described:

1. How a sinusoidal voltage waveform (ac) is generated?
2. How the average and rms values of the periodic voltage or current waveforms, are computed?

Some examples are also described there. In this lesson, the representation of sinusoidal (ac) voltage/current signals by a phasor is first explained. The polar/Cartesian (rectangular) form of phasor, as complex quantity, is described. Lastly, the algebra, involving the phasors (voltage/current), is presented. Different mathematical operations – addition/subtraction and multiplication/division, on two or more phasors, are discussed.

Keywords: Phasor, Sinusoidal signals, phasor algebra

After going through this lesson, the students will be able to answer the following questions;

1. What is meant by the term, 'phasor' in respect of a sinusoidal signal?
2. How to represent the sinusoidal voltage or current waveform by phasor?
3. How to write a phasor quantity (complex) in polar/Cartesian (rectangular) form?
4. How to perform the operations, like addition/subtraction and multiplication/division on two or more phasors, to obtain a phasor?

This lesson forms the background of the following lessons in the complete module of single ac circuits, starting with the next lesson on the solution of the current in the steady state, in R-L-C series circuits.

Symbols

i or i(t)	Instantaneous value of the current (sinusoidal form)
I	Current (rms value)
I_m	Maximum value of the current

$\bar{I}$ Phasor representation of the current

φ Phase angle, say of the current phasor, with respect to the reference phasor Same symbols are used for voltage or any other phasor.

Representation of Sinusoidal Signal by a Phasor

A sinusoidal quantity, i.e. current, $i\ (t) = I_m \sin \omega t$, is taken up as an example. In Fig. 13.1a, the length, OP, along the x-axis, represents the maximum value of the current I_m , on a certain scale. It is being rotated in the anti-clockwise direction at an angular speed, ω, and takes up a position, OA after a time t (or angle, $\theta = \omega t$, with the x-axis). The vertical projection of OA is plotted in the right hand side of the above figure with respect to the angle θ. It will generate a sine wave (Fig. 13.1b), as OA is at an angle, θ with the x-axis, as stated earlier. The vertical projection of OA along y-axis is OC = AB =

$i\ (\theta) = I_m \sin\theta$, which is the instantaneous value of the current at any time t or angle θ. The angle θ is in rad., i.e. $\theta = \omega t$. The angular speed, ω is in rad/s, i.e. $\omega = 2\pi f$, where f is the frequency in Hz or cycles/sec. Thus,

$$i = I_m \sin\theta = I_m \sin\omega t = I_m \sin 2\pi ft$$

So, OP represents the phasor with respect to the above current, i.

The line, OP can be taken as the rms value, $I = I_m / \sqrt{2}$, instead of maximum value, I_m . Then the vertical projection of OA, in magnitude equal to OP, does not represent exactly the instantaneous value of I, but represents it with the scale factor of $1/\sqrt{2} = 0.707$. The reason for this choice of phasor as given above, will be given in another lesson later in this module.

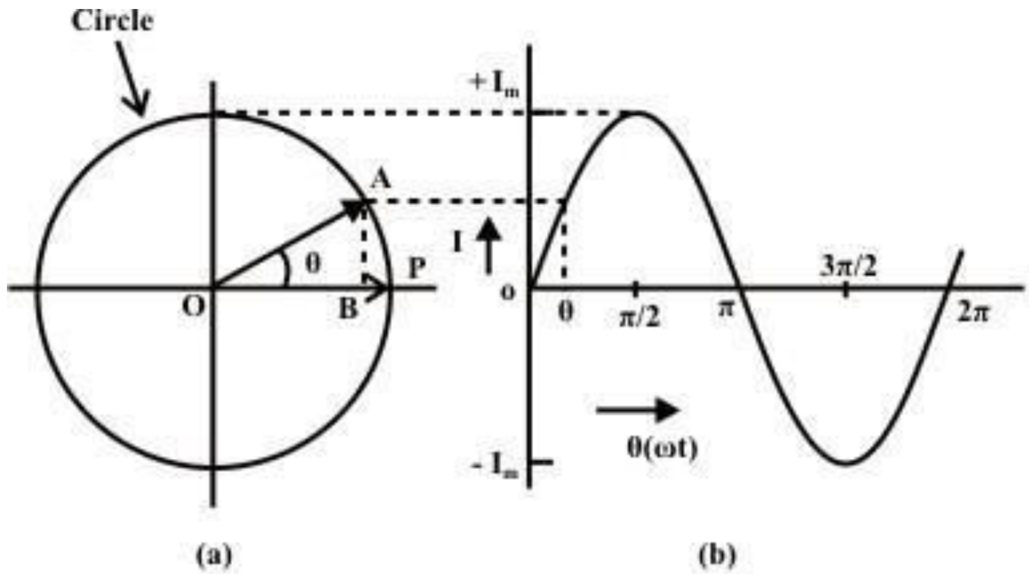


Fig. 13.1(a) Phasor representation of a sinusoidal current, and (b) Waveform

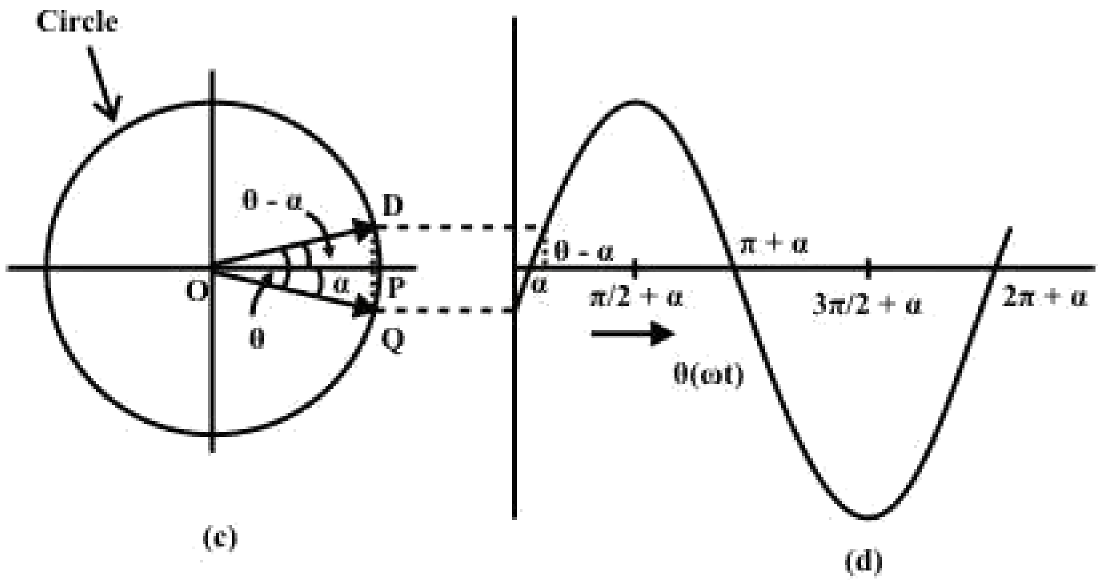


Fig. 13.1 (c) Phasor representation of a phase shifted sinusoidal current, and (d) Waveform

Generalized case

The current can be of the form, $i\ (t) = I_m \sin\ (\omega t - \alpha)$ as shown in Fig. 13.1d. The phasor representation of this current is the line, OQ, at an angle, α (may be taken as negative), with the line, OP along x-axis (Fig. 13.1c). One has to move in clockwise direction to go to OQ from OP (reference line), though the phasor, OQ is assumed to move in anti-clockwise direction as given earlier. After a time t, OD will be at an angle θ with OQ, which is at an angle $(\theta - \alpha = \omega t - \alpha)$, with the line, OP along x-axis. The vertical projection of OD along y-axis gives the instantaneous value of the current,

$$i = \sqrt{2}\, I \sin\ (\omega t - \alpha) = I_m \sin\ (\omega t - \alpha)\ .$$

Phasor representation of Voltage and Current

The voltage and current waveforms are given as,

$v = \sqrt{2}\ V \sin\theta$, and $i = \sqrt{2}\ I \sin\ (\theta + \varphi)$

It can be seen from the waveforms (Fig. 13.2b) of the two sinusoidal quantities – voltage and current, that the voltage, V lags the current I, which means that the positive maximum value of the voltage is reached earlier by an angle, φ, as compared to the positive maximum value of the current. In phasor notation as described earlier, the voltage and current are represented by OP and OQ (Fig. 13.2a) respectively, the length of which are proportional to voltage, V and current, I in different scales as applicable to each one. The voltage phasor, OP (V) lags the current phasor, OQ (I) by the angle φ, as two phasors rotate in the anticlockwise direction as stated earlier, whereas the angle φ is also measured in the anticlockwise direction. In other words, the current phasor (I) leads the voltage phasor (V).

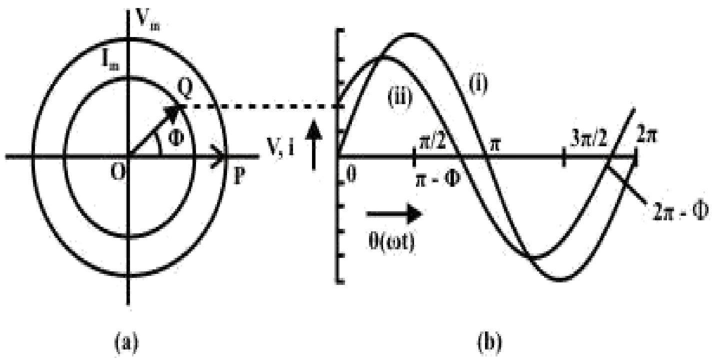


Fig. 13.2 (a) Phasor representation of a sinusoidal (i) voltage and (ii) current, and (b) Waveforms

Mathematically, the two phasors can be represented in polar form, with the voltage phasor ($\bar{V}$) taken as reference, such as $\bar{V} = V \angle 0^0$, and $\bar{I} = I \angle \varphi$.

In Cartesian or rectangular form, these are,

$\bar{V} = V \angle 0^0 = V + j\,0$, and $\bar{I} = I \angle \varphi = I \cos \varphi + j\, I \sin \varphi$,

where, the symbol, j is given by $j = \sqrt{-1}$.

Of the two terms in each phasor, the first one is termed as real or its component in x-axis, while the second one is imaginary or its component in y-axis, as shown in Fig. 13.3a. The angle, φ is in degree or rad.

Phasor Algebra

Before discussing the mathematical operations, like addition/subtraction and multiplication/division, involving phasors and also complex quantities, let us take a look at the two forms – polar and rectangular, by which a phasor or complex quantity is represented. It may be observed here that phasors are also taken as complex, as given above.

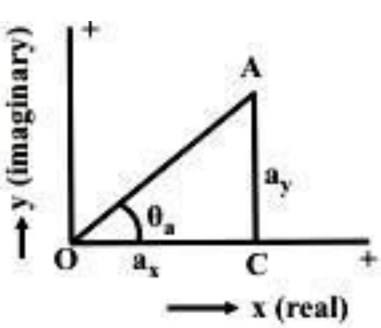


Fig. 13.3 Representation of a phasor, both in rectangular and polar forms

Representation of a phasor and Transformation

A phasor or a complex quantity in rectangular form (Fig. 13.3) is,

$$\bar{A} = a_x + j\,a_y$$

where a_x and a_y are real and imaginary parts, of the phasor respectively.

In polar form, it is expressed as

$$\bar{A} = A\angle\theta_a = A\cos\theta_a + j\,A\sin\theta_a$$

where A and θ_a are magnitude and phase angle of the phasor.

From the two equations or expressions, the procedure or rule of transformation from polar to rectangular form is

$$a_x = A\cos\theta_a \text{ and } a_y = A\sin\theta_a$$

From the above, the rule for transformation from rectangular to polar form is

$$A = \sqrt{a_x^2 + a_y^2} \text{ and } \theta_a = \tan^{-1}\left(a_y / a_x\right)$$

The examples using numerical values are given at the end of this lesson.

Addition/Subtraction of Phasors

Before describing the rules of addition/subtraction of phasors or complex quantities, everyone should recall the rule of addition/subtraction of scalar quantities, which may be positive or signed (decimal/fraction or fraction with integer). It may be stated that, for the two operations, the quantities must be either phasors, or complex. The example of phasor is voltage/current, and that of complex quantity is impedance/admittance, which will be explained in the next lesson. But one phasor and another complex quantity should not be used for addition/subtraction operation.

For the operations, the two phasors or complex quantities must be expressed in rectangular form as

$$\bar{A} = a_x + j\,a_y\,; \quad \bar{B} = b_x + j\,b_y$$

If they are in polar form as

$$\bar{A} = A\angle\theta_a\,; \quad \bar{B} = B\angle\theta$$

In this case, two phasors are to be transformed to rectangular form by the procedure or rule given earlier.

The rule of addition/subtraction operation is that both the real and imaginary parts have to be separately treated as

$$\bar{C} = \bar{A} \pm \bar{B} = (a_x \pm b_x) + j\,(a_y \pm b_y) = c_x + j\,c_y$$

where $c_x = (a_x \pm b_x)$; $c_y = (a_y \pm b_y)$

Say, for addition, real parts must be added, so also for imaginary parts. Same rule follows for subtraction. After the result is obtained in rectangular form, it can be transformed to polar one. It may be observed that the six values of $a's$, $b's$ and $c's$ – parts of the two phasors and the resultant one, are all signed scalar quantities, though in the example, $a's$ and $b's$ are taken as positive, resulting in positive values of $c's$. Also the phase angle $\theta's$ may lie in any of the four quadrants, though here the angles are in the first quadrant only.

This rule for <u>addition</u> can be extended to three or more quantities, as will be illustrated through example, which is given at the end of this lesson.

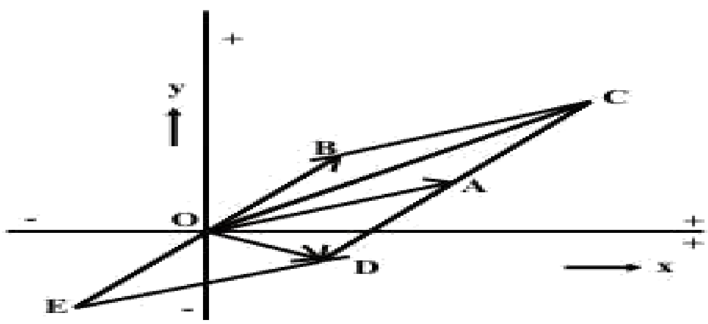


Fig. 13.4 Addition and subtraction of two phasors, both represented in polar form

The addition/subtraction operations can also be performed using the quantities as phasors in polar form (Fig. 13.4). The two phasors are $A(OA)$ and $\bar{B}\,(OB)$. The find the sum $\bar{C}\,(OC)$, a line AC is drawn equal and parallel to OB. The line BC is equal and parallel to OA. Thus, $C = OC = OA + AC = OA + OB = \bar{A} + \bar{B}$. Also,

$OC = OB + BC = OB + OA$

To obtain the difference $\bar{D}\,(OD)$, a line AD is drawn equal and parallel to OB, but in opposite direction to AC or OB. A line OE is also drawn equal to OB, but in opposite direction to OB. Both AD and OE represent the phasor ($-\bar{B}$). The line, ED is equal to OA. Thus, $D = OD = OA + AD = OA - OB = \bar{A} - \bar{B}$. Also $OD = OE + ED = -OB + OA$.
The examples using numerical values are given at the end of this lesson.

Multiplication/Division of Phasors

Firstly, the procedure for multiplication is taken up. In this case no reference is being made to the rule involving scalar quantities, as everyone is familiar with them. Assuming that the two phasors are available in polar from as $\bar{A} = A\angle\theta_a$ and $\bar{B} = B\angle\theta_b$.

Otherwise, they are to be transformed from rectangular to polar form. This is also valid for the procedure of division. Please note that a phasor is to be multiplied by a complex quantity only, to obtain the resultant phasor. A phasor is not normally multiplied by another phasor, except in special case. Same is for division. A phasor is to be divided by a complex quantity only, to obtain the resultant phasor. A phasor is not normally divided by another phasor.

To find the magnitude of the product $\bar{C}$, the two magnitudes of the phasors are to be multiplied, whereas for phase angle, the phase angles are to added. Thus,

$$\bar{C} = C\angle\theta_c = \bar{A}\cdot\bar{B} = A\angle\theta_A \cdot B\angle\theta_B = (A\cdot B)\angle\left(\theta_a+\theta_b\right)$$

where $C = A\cdot B$ and $\theta_c = \theta_a + \theta_b$

Please note that the same symbol, $\bar{C}$ is used for the product in this case.

To divide $\bar{A}$ .by $\bar{B}$ to obtain the result $\bar{D}$., the magnitude is obtained by division of the magnitudes, and the phase is difference of the two phase angles. Thus,

$$\bar{D} = D\angle\theta_d = \frac{\bar{A}}{\bar{B}} = \frac{A\angle\theta_a}{B\angle\theta_b} = \frac{A}{B}\angle\left(\theta_a-\theta_b\right)$$

where $D = A/B$ and $\theta_d = \theta_a - \theta_b$

If the phasors are expressed in rectangular form as

$\bar{A} = a_x + j\,a_y$ and $\bar{B} = b_x + j\,b_y$

where $A = \left(\sqrt{a_x^2 + a_y^2}\right)$; $\theta_a = \tan^{-1}\left(a_y/a_x\right)$

The values of $\bar{B}$ are not given as they can be obtained by substituting $b's$ for $a's$.

To find the product,

$$\bar{C} = C\angle\theta_c = \bar{A}\cdot\bar{B} = \left(a_x + j\,a_y\right)\cdot\left(b_x + j\,b_y\right) = \left(a_x b_x - a_y b_y\right) + j\left(a_x b_y + a_y b_x\right)$$

Please note that $j^2 = -1$ .The magnitude and phase angle of the result (phasor) are,

$$C = \left[\left(a_x b_x - a_y b_y\right)^2 + \left(a_x b_y + a_y b_x\right)^2\right]^{\frac{1}{2}} = \sqrt{\left(a_x^2 + a_y^2\right)}\cdot\sqrt{\left(b_x^2 + b_y^2\right)} = A\cdot B \text{, and}$$

$$\theta_c = \tan^{-1}\frac{a_x b_y + a_y b_x}{a_x b_x - a_y b_y}$$

The phase angle,

$$\theta_c = \theta_a + \theta_b = \tan^{-1}\frac{a_y}{a_x} + \tan^{-1}\frac{b_y}{b_x} = \tan^{-1}\frac{\left(a_y/a_x\right)+\left(b_y/b_x\right)}{1-\left(a_y/a_x\right)\cdot\left(b_y/b_x\right)}$$

$$= \tan^{-1}\frac{a_x b_y + a_y b_x}{a_x b_x - a_y b_y}$$

The above results are obtained by simplification.

To divide $\bar{A}$ by $\bar{B}$ to obtain $\bar{D}$ as

$$\bar{D} = \frac{\bar{A}}{\bar{B}} = \frac{a_x + j\,a_y}{}$$

$$D = d_x + j\,d_y = \frac{\quad}{\bar{B}} = \frac{\quad}{b_x + j\,b_y}$$

To simplify D, i.e. to obtain real and imaginary parts, both numerator and denominator, are to be multiplied by the complex conjugate of $\bar{B}$, so as to convert the denominator into real value only. The complex conjugate of $\bar{B}$ is

$$B^* = b_x + j\,b_y = B \angle -\theta_b$$

In the complex conjugate, the sign of the imaginary part is negative, and also the phase angle is negative.

$$\bar{D} = d_x + j\,d_y = \frac{(a_x + j\,a_y)\cdot(b_x - j\,b_y)}{(b_x + j\,b_y)\cdot(b_x - j\,b_y)} = \frac{a_x b_x + a_y b_y}{b_x^2 + b_y^2} + j\frac{a_y b_x - a_x b_y}{b_x^2 + b_y^2}$$

The magnitude and phase angle of the result (phasor) are,

$$D = \frac{\left[(a_x b_x + a_y b_y)^2 + (a_y b_x - a_x b_y)^2\right]^{\frac{1}{2}}}{(b_x^2 + b_y^2)} = \frac{\sqrt{(a_x^2 + a_y^2)}}{\sqrt{(b_x^2 + b_y^2)}} = \frac{A}{B}$$, and

$$\theta_d = \tan^{-1}\frac{a_y b_x - a_x b_y}{a_x b_x + a_y b_y}$$

The phase angle,

$$\theta_d = \theta_a - \theta_b = \tan^{-1}\frac{a_y}{a_x} - \tan^{-1}\frac{b_y}{b_x} = \tan^{-1}\frac{a_y b_x - a_x b_y}{a_x b_x + a_y b_y}$$

The steps are shown here in brief, as detailed steps have been given earlier.

Example

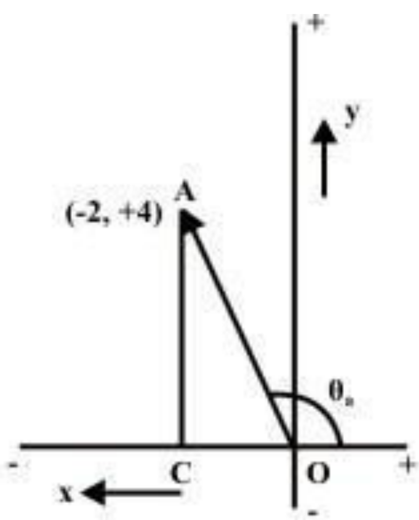


Fig. 13.5 Representation of phasor as an example, both in rectangular and polar forms

The phasor, $\bar{A}$ in the rectangular form (Fig. 13.5) is,

$$\bar{A} = A\angle\theta_a = A\cos\theta_a + j\,A\sin\theta_a = a_x + j\,a_y = -2 + j\,4$$

where the real and imaginary parts are $a_x = -2$; $a_y = 4$

To transform the phasor, $\bar{A}$ into the polar form, the magnitude and phase angle are

$$A = \sqrt{a_x^2 + a_y^2} = \sqrt{(-2)^2 + 4^2} = 4.472$$

$$\theta_a = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}\frac{4}{-2} = 116.565° = 2.034\ rad$$

Please note that θ_a is in the second quadrant, as real part is negative and imaginary part is positive.

Transforming the phasor, $\bar{A}$ into rectangular form, the real and imaginary parts are

$a_x = A\cos\theta_a = 4.472 \cdot \cos 116.565° = -2.0$

$a_y = A\sin\theta_a = 4.472 \cdot \sin 116.565° = 4.0$

Phasor Algebra

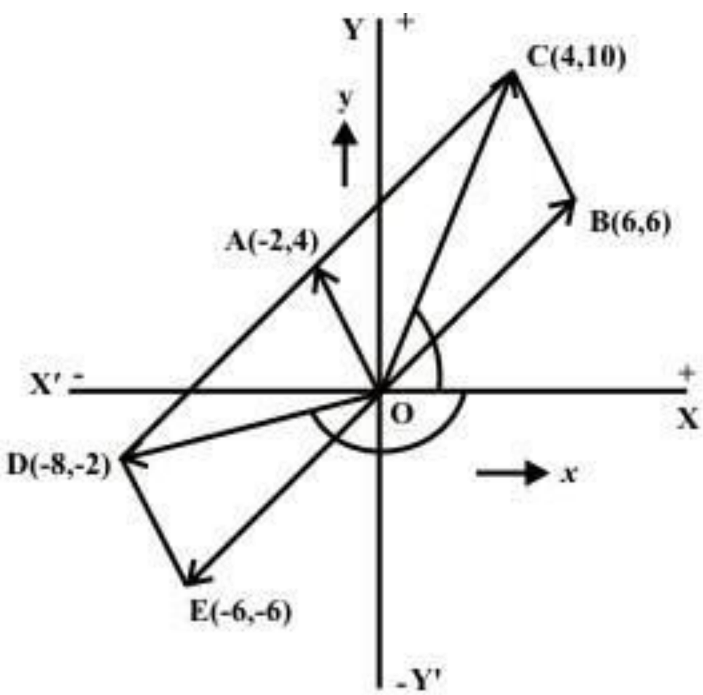


Fig.13.6 Addition and subtraction of two phasors represented in polar form, as an example

Another phasor, $\bar{B}$ in rectangular form is introduced in addition to the earlier one, $\bar{A}$

$\bar{B} = 6 + j\,6 = 8.485\angle 45°$

Firstly, let us take the addition and subtraction of the above two phasors. The sum and difference are given by the phasors, $\bar{C}$ and $\bar{D}$ respectively (Fig. 13.6).

$\bar{C} = \bar{A} + \bar{B} = (-2 + j\,4) + (6 + j\,6) = (-2 + 6) + j\,(4 + 6) = 4 + j10 = 10.77\ \angle 68.2°$

$\bar{D} = \bar{A} - \bar{B} = (-2 + j\,4) - (6 + j\,6) = (-2 - 6) + j\,(4 - 6) = -8 - j\,2 = 8.246\ \angle -166.0°$

It may be noted that for the addition and subtraction operations involving phasors, they should be represented in rectangular form as given above. If any one of the phasors is in polar form, it should be transformed into rectangular form, for calculating the results as shown.

If the two phasors are both in polar form, the phasor diagram (the diagram must be drawn to scale), or the geometrical method can be used as shown in Fig 13.6. The result obtained using the diagram, as shown are the same as obtained earlier.

[$\bar{C}$ (OC) = 10.77, $\angle COX = 68.2°$; and $\bar{D}$ (OD) = 8.246, $\angle DOX = 166.0°$]

Now, the multiplication and division operations are performed, using the above two phasors represented in polar form. If any one of the phasors is in rectangular form, it may be transformed into polar form. Also note that the same symbols for the phasors are used here, as was used earlier. Later, the method of both multiplication and division using rectangular form of the phasor representation will be explained.

The resultant phasor $\bar{C}$, i.e. the product of the two phasors is

$$\bar{C} = \bar{A} \cdot \bar{B} = 4.472\angle 116.565° \cdot 8.485\angle 45° = (4.472 \cdot 8.485)\ \angle(116.565° + 45°)$$

$$= 37.945\angle 161.565° = -36 + j12$$

The product of the two phasors in rectangular form can be found as

$$\bar{C} = (-2 + j\,4) \cdot (6 + j\,6) = (-12 - 24) + j\,(24 - 12) = -36 + j12$$

The result ($\bar{D}$) obtained by the division of $\bar{A}$ by $\bar{B}$ is

$$\bar{D} = \frac{\bar{A}}{\bar{B}} = \frac{4.472\ \angle 116.565°}{8.485\angle 45°} = \frac{4.472}{8.485}\ \angle\left(116.565° - 45°\right) = 0.527\ \angle 71.565°$$

$$= 0.167 + j\,0.5$$

The above result can be calculated by the procedure described earlier, using the rectangular form of the two phasors as

$$\bar{D} = \frac{\bar{A}}{\bar{B}} = \frac{-2 + j\,4}{6 + j\,6} = \frac{(-2 + j\,4) \cdot (6 - j\,6)}{(6 + j\,6) \cdot (6 - j\,6)} = \frac{(-12 + 24) + j\,(24 + 12)}{6^2 + 6^2}$$

$$= \frac{12 + j\,36}{72} = 0.167 + j\,0.5$$

The procedure for the elementary operations using two phasors only, in both forms of representation is shown. It can be easily extended, for say, addition/multiplication, using three or more phasors. The simplification procedure with the scalar quantities, using the different elementary operations, which is well known, can be extended to the phasor quantities. This will be used in the study of ac circuits to be discussed in the following lessons.

The background required, i.e. phasor representation of sinusoidal quantities (voltage/current), and algebra – mathematical operations, such as addition/subtraction and multiplication/division of phasors or complex quantities, including transformation of phasor from rectangular to polar form, and vice versa, has been discussed here. The study of ac circuits, starting from series ones, will be described in the next few lessons.

Problems

13.1 Use plasor technique to evaluate the expression and then find the numerical value at t = 10 ms.

$$i(t) = 150\cos\left(100t - 45^0\right) + 500\sin\left(100t\right) + \frac{d}{dt}\cos\left(100t - 30^0\right)$$

13.2 Find the result in both rectangular and polar forms, for the following, using complex quantities:

a) $\dfrac{5 - j12}{15\angle 53.1^\circ}$

b) $\left(5 - j12\right) + 15\angle -53.1^\circ$

c) $\dfrac{2\angle 30^\circ - 4\angle 210^\circ}{5\angle 450^\circ}$

d) $5\angle 0^\circ + \dfrac{1}{3\sqrt{2}\angle -45^\circ} \cdot 2\angle 210^\circ$

List of Figures

UNIT-2

R-L & R-C Transients

Study of DC transients in R-L and R-C circuits

Objectives

- Definition of inductance and continuity condition for inductors.
- To understand the rise or fall of current in a simple series $R-L$ circuit excited with dc source.
- Meaning of '*Time Constamt* (τ)' for $R-L$ circuit and explain its relationship to the performance of the circuit.
- Energy stored in an inductor
- Definition of capacitance and Continuity condition for capacitors.
- To understand the rise or fall voltage across the capacitor in a simple series $R-C$ circuit excited with dc source.
- Meaning of '*Time Constamt* (τ)' for $R-C$ circuit and explain its relationship to the performance of the circuit.
- Energy stored in a capacitor

L.10.1 Introduction

So far we have considered dc resistive network in which currents and voltages were independent of time. More specifically, Voltage (cause $\rightarrow$ input) and current (effect $\rightarrow$ output) responses displayed simultaneously except for a constant multiplicative factor R $(V=R\cdot I)$. Two basic passive elements namely, inductor (L) and capacitor (C) are introduced in the dc network. Automatically, the question will arise whether or not the methods developed in lesson-3 to lesson-8 for resistive circuit analysis are still valid. The voltage/current relationship for these two passive elements are defined by the derivative (voltage across the inductor $v_L(t) = L\frac{di_L(t)}{dt}$, where $i_L(t)$ =current flowing through the inductor ; current through the capacitor $i_C(t) = C\frac{dv_C(t)}{dt}$, $v_C(t)$ = voltage across the capacitor) or in integral form as $i_L(t) = \frac{1}{L}\int_0^t v_L(t)\,dt + i_L(0)$ *or* $v_C(t) = \frac{1}{C}\int_0^t i_C(t)\,dt + v_C(0)$ rather than the algebraic equation $(V = IR)$ for all resistors. One can still apply the KCL, KVL, Mesh-current method, Node-voltage method and all network theorems but they result in differential equations rather than the algebraic equations that we have considered in resistive networks (see Lession-3 to lesson-8).

An electric switch is turned on or off in some circuit (for example in a circuit consisting of resistance and inductance), transient currents or voltages (quickly changing current or voltage) will occur for a short period after these switching actions. After the transient has ended, the current or voltage in question returns to its steady state situation (or normal steady value). Duration of transient phenomena are over after only a few micro or milliseconds, or few seconds or more depending on the values of circuit parameters (like R, L , *and* C).The situation relating to the sudden application of dc voltage to circuits possessing resistance (R), inductance (L), and capacitance (C) will now be investigated in this lesson. We will continue our discussion on transients occurring in a dc circuit. It is needless to mention that transients also occur in ac circuit but they are not included in this lesson.

L.10.2 Significance of Inductance of a coil and dc transients in a simple R-L circuit

Fig.10.1 shows a coil of wire forming an inductance and its behavior is to resist any change of electric current through the coil. When an inductor carries current, it produces a certain amount of magnetic flux (Φ) in the core or space around it. The product of the magnetic flux (Φ) and the number of turns of a coil (an inductor) is called the 'flux linkage' of the coil.

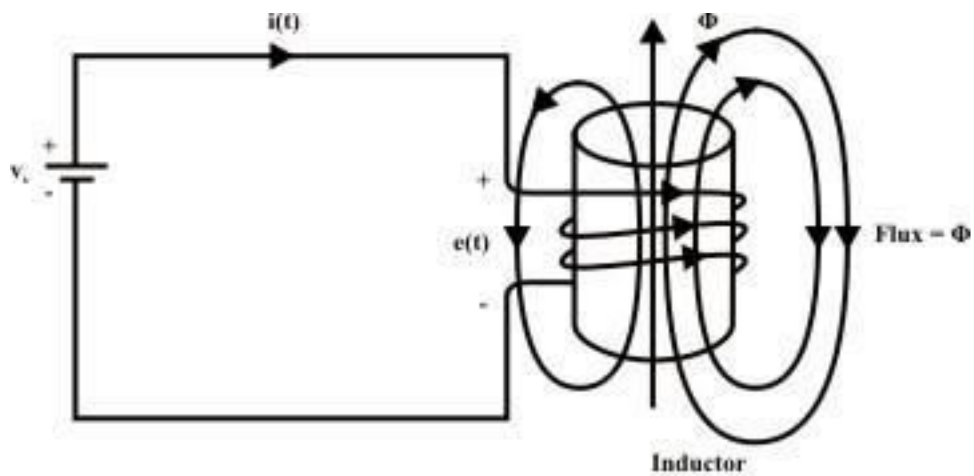


Fig. 10.1

Considering the physical fact that the voltage across the coil is directly proportional to the rate of change of current through the inductor and it is expressed by the equation

$$emf = e(t) = L\,\frac{di(t)}{dt} \quad L = \frac{e(t)}{di(t)/dt} \tag{10.1}$$

where L is the constant of proportionality called inductance of coil and it is measured in $\frac{volt\_sec\,ond}{current\ ampere} = henry\ (H)$. The direction of induced emf is opposite to that of increases or decreases (Lenz's Law)

$$e(t) = -L\frac{di(t)}{dt} \tag{10.2}$$

Let us assume that the coil of wire has $'N'$ turns and the core material has a relatively high permeability (or magnetic path reluctance is very low), so that the magnetic flux (Φ) produced due to current flowing through the coil is concentrated within the core area. The basic fundamental principle according to Faraday, the changing flux through the coil creates an induced emf (e) and it is expressed as

$$e(t) = -N\frac{d\,\Phi(t)}{dt} \tag{10.3}$$

In words, Faraday's law states that the voltage induced in a coil (inductor) is proportional to the number of turns that the coil has, and also to the rate of change of the magnetic flux passing through its coils. From equations (10.2) and (10.3), one can write the following relation

$$L = \frac{N\,d\Phi(t)}{di} = \frac{change\ in\ \ flux\ linkage}{change\ \ in\ current} = \frac{N\,\Phi}{I} \tag{10.4}$$

The inductance of a coil can also be defined as flux (Φ) linkage per unit of current flowing through the coil and it is illustrated through numerical example.

Example-L.10.1: Consider two coils having the same number of turns 'N'. One coil is wrapped in a nonmagnetic core (say, air) and the other is placed on a core of magnetic material as shown in fig.10.2. Calculate the inductances of both coils for same amount of current flowing through them.

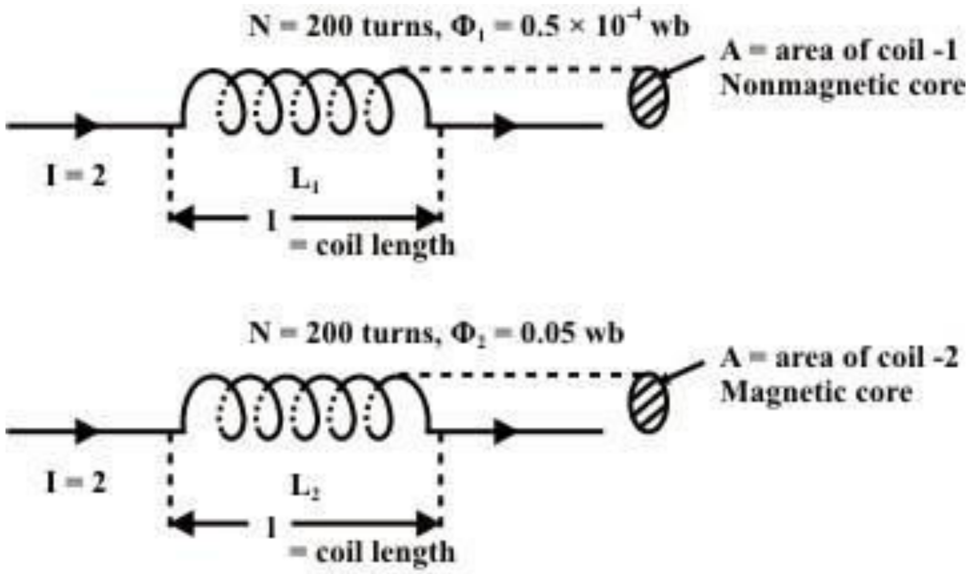


Fig. 10.2: Inductance of coil depends on the surrounding media

Case-A: Nonmagnetic material

Inductance of nonmagnetic material = $L_1 = \dfrac{N\,\Phi_1}{I} = \dfrac{200\cdot(0.5\cdot 10^{-4})}{2} = 5\ m\,H$

Case-B: Magnetic material

Inductance of magnetic material $L_2 = \dfrac{N\cdot\Phi_2}{I} = \dfrac{200\cdot(0.05)}{2} = 5H$ (Note: $L_2 > L_1$)

L.10.2.1 Inductance calculation from physical dimension of coil

A general formula for the inductance of a coil can be found by using an equivalent *Ohm's law* for magnetic circuit and the formula for reluctance. This topic will be discussed in detail in Lesson-21. Consider a solenoid-type electromagnet/toroid with a length much greater than its diameter (at least the length is ten times as great as its diameter). This will produce an uniform magnetic field inside the toroid. The length 'l'of a toroid is the distance around the center axis of its core , as indicated in fig.10.3 by dotted line. Its area ' A ' is the cross-sectional area of the toroid, also indicated in that figure.

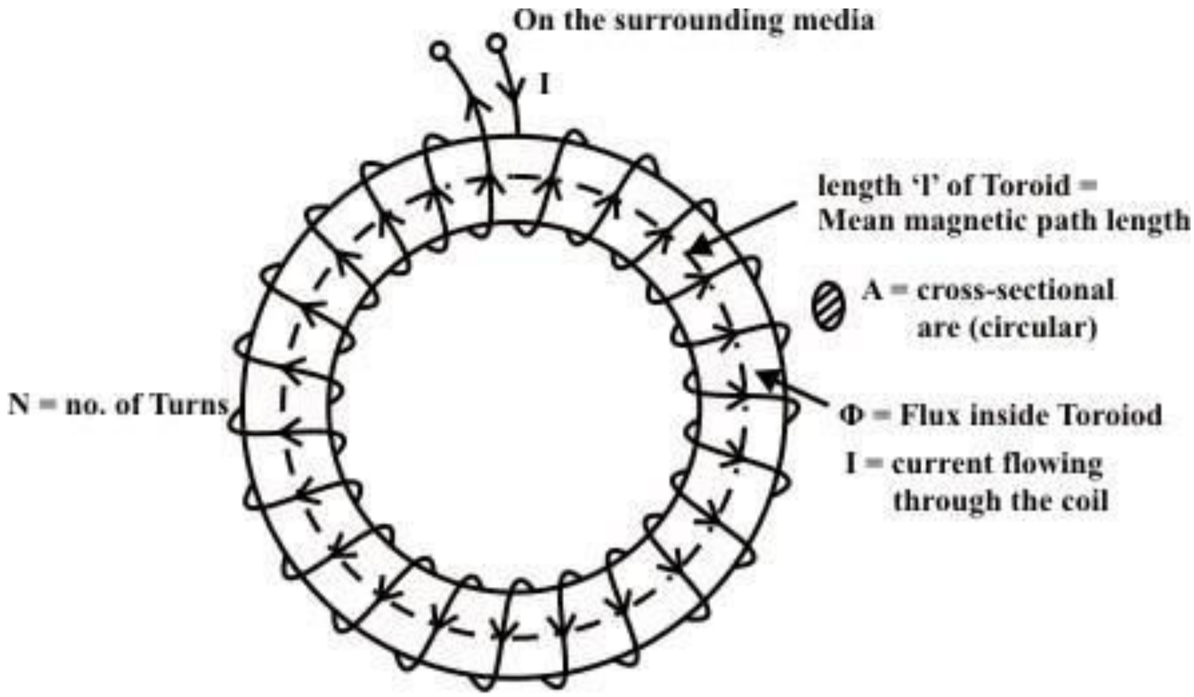


Fig. 10.3: A toroidal inductor

Appling ampere-circuital law for magnetic circuit (see Lesson-21) one can write the following relation

$$N\,I = H\,l \quad H = \frac{N\,I}{l} At/m \tag{10.5}$$

We know, flux is always given by the product of flux density (B) and area (A) through which flux density ids uniform. That is,

$$\Phi = B \cdot A = \mu\,H\,A = \mu\,\frac{N\,I}{l}A \quad (\,note,\ \mu = \mu_0 \cdot \mu_r\ ) \tag{10.6}$$

where $B = \mu H$ and H is the uniform field intensity around the mean magnetic path length ' l '. Substituting the equation (10.6) into the defining equation for inductance, equation (10.4) gives

$$L = \frac{N\,\Phi}{I} = \frac{\mu N^2 I A}{I\,l} = \frac{\mu N^2 A}{l} \tag{10.7}$$

Remark-1: The expression (10.7) is derived for long solenoids and toroids, computation of inductance is valid only for those types.

L.10.2.2 Continuity condition of Inductors

The current that flows through a linear inductor must always be a continuous. From the expression (10.1), the voltage across the inductor is not proportional to the current flowing through it but to the rate of change of the current with respect to time, $\frac{di(t)}{dt}$. The voltage across the inductor (v_L) is zero when the current flowing through an inductor does not change with time. This observation implies that the inductor acts as a short circuit under steady state dc current. In other words, under the steady state condition, the inductor terminals are shorted through a conducting wire. Alternating current (ac), on the other hand, is constantly changing; therefore, an inductor will create an opposition voltage polarity that tends to limit the changing current. If current changes very rapidly with time, then inductor causes a large opposition voltage across its terminals. If current changes through the inductor from one level to another level instantaneously i.e. in dt 0 sec., then the voltage across it would become infinite and this would require infinite power at the terminals of the inductor. Thus, instantaneous changes in the current through an inductor are not possible at all in practice.

Remark-2: (i) The current flowing through the inductor cannot change instantaneously (i.e. $i(0^-)$ just right before the change of current = $i(0^+)$ just right after the change of current). However, the voltage across an inductor can change abruptly. (ii) The inductor acts as a short circuit (i.e. inductor terminals are shorted with a conducting wire) when the current flowing through the inductor does not change (constant). (iii) These properties of inductor are important since they will be used to determine "boundary conditions".

L.10.3 Study of dc transients and steady state response of a series R-L circuit.

Ideal Inductor: Fig.10.4 shows an ideal inductor, like an ideal voltage source, has no resistance and it is excited by a dc voltage source V_S .

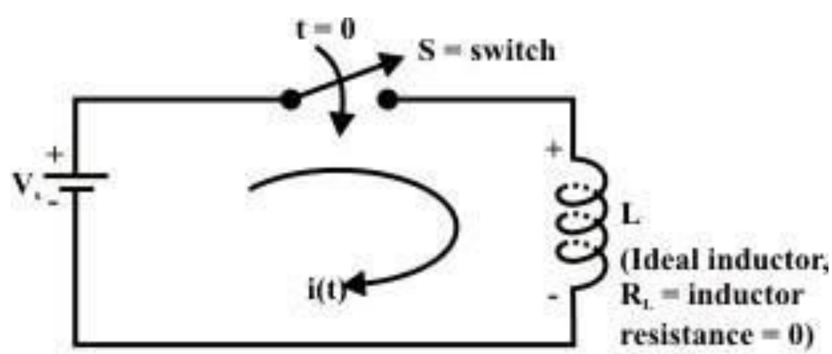


Fig. 10.4: An ideal inductor connected to a constant voltage source

The switch ' S ' is closed at time ' $t = 0$ ' and assumed that the initial current flowing through the ideal inductor $i(0)$ just before closing the switch is equal to zero. To find the system response ($i(t) - vs - t$), one can apply KVL around the closed path.

KVL

$$V_S - L\frac{di(t)}{dt} = 0 \qquad \frac{d\,i(t)}{dt} = \frac{V_S}{L} \tag{10.8}$$

$$\int_0^{i(t)} d\,i(t) = \frac{V_s}{L}\int_0^t dt \quad i(t) = \frac{V_s}{L}t + i(0) \qquad i(t) = \frac{V_s}{L}t \quad (\text{note } i(0) = 0) \tag{10.9}$$

Equation (10.9) implies that the current through inductor increases with increase in time and theoretically it approaches to infinity as $t \rightarrow \infty$ but in practice, this is not really the case.

Real or Practical inductor:

Fig.10.5 shows a real or practical inductor has some resistance and it is exactly equal to the resistance of the wire used to wind the coil.

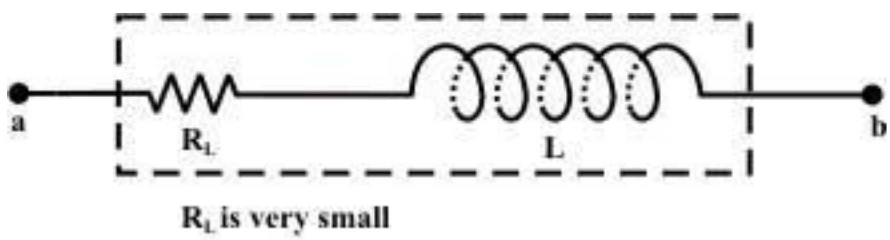


Fig. 10.5: Representation of an practical inductor

Let us consider a practical inductor is connected in series with an external resistance R_1 and this circuit is excited with a dc voltage V_S as shown in fig.10.6(a).

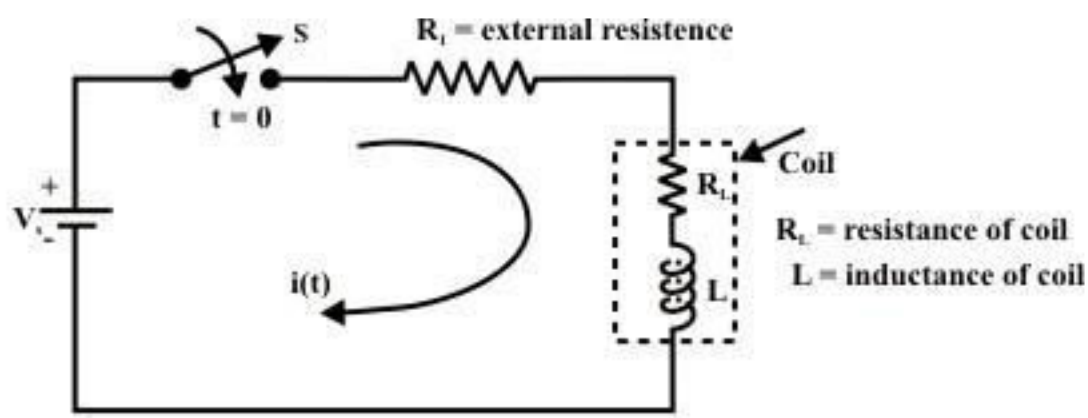


Fig. 10.6(a): a practical inductor connected to a constant voltage source

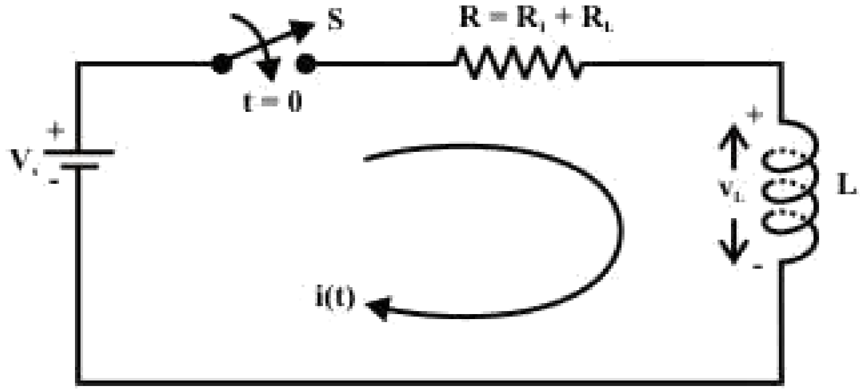


Fig. 10.6(b): Equivalent representation circuit shown in fig. L.10.6(a)

Our problem is to study the growth of current in the circuit through two stages, namely; (i) dc transient response (ii) steady state response of the system.

D.C Transients: The behavior of the current ($i(t)$) ; charge ($q(t)$) and the voltage ($v(t)$) in the circuit (like $R-L$; $R-C$: $R-L-C$ circuit) from the time ($t(0^+)$) switch is closed until it reaches its final value is called dc transient response of the concerned circuit. The response of a circuit (containing resistances, inductances, capacitors and switches) due to sudden application of voltage or current is called transient response. The most common instance of a transient response in a circuit occurs when a switch is turned on or off – a rather common event in an electric circuit.

L.10.3.1 Growth or Rise of current in R-L circuit

To find the current expression (response) for the circuit shown in fig. 10.6(a), we can write the KVL equation around the circuit

$$V_S - (R_1 + R_L) i(t) - v_L(t) = 0 \quad V_S = R\, i(t) + L\frac{d\,i(t)}{dt} \tag{10.10}$$

where V_S is the applied voltage or forcing function , R_L is the resistance of the coil, R_1 is the external resistance. One can combine the resistance of coil R_L to the external resistance R_1 in order to obtain a simplified form of differential equation. The circuit configuration shown in fig. 10.6(a) is redrawn equivalently in fig.10.6(b) for our convenience. The equation (10.10) is the standard first order differential equation and its solution can be obtained by classical method. The solution of first or second order differential equation is briefly discussed in Appendix (at the end of this lesson-10). The following relation gives the solution of equation (10.10)

$$i(t) = i_n(t) + i_f(t) = A_1 e^{\alpha t} + A \tag{10.11}$$

Here, $i_n(t)$ is the complementary solution/natural solution of differential equation (10.10). It is also sometimes called as transient response of system (i.e. the first part of response is due to an initial condition of the system or force free response). The second part $i_f(t)$ of eq. (10.11) is the particular integral solution/force response or steady state response of the system due to the forcing function ($f(t) = V_S$) or input signal to the series $R - L$ circuit. It may be noted the term A provide us the steady state solution of the first order differential equation while the forcing function (or input to the system) is step function (or constant input). More specifically, for a linear system, the steady state solution of any order differential equation is the same nature of forcing function ($f(t)$) or input signal but different in magnitude. We have listed in tabular form the nature of steady state solution of any order differential equation for various types of forcing functions (see in Appendix). To get the complete solution of eq. (10.10), the values of α, A_1 *and* A are to be computed following the steps given below:

Step-1: How to find the value of α?

Assigning $V_S = 0$ and introducing an operator $\alpha = \frac{d}{dt}$ in eq.(10.10) , we get a characteristic equation that will provide us the numerical value of α. This in turn, gives us the transient response of the system provided the constant A_1 is known to us.

The Characteristic equation of (10.10) is $R + \alpha L = 0 \quad \alpha = -\frac{R}{L}$.

Step-2: How to obtain the constants A_1 *and* A ?

It may be noted that the differential eq. (10.10) must be satisfied by the particular integral solution or steady state solution $i_f(t)$. The value of $i_f(t)$ at steady state condition (i.e. $t \to \infty$) can be found out using the eq.(10.11) and it is given below.

Using final condition ($t \to \infty$)

$$V_S = R\, i_f(t) + L\frac{d\, i_f(t)}{dt} \tag{10.12}$$

(note: at steady state ($t \to \infty$) $i_f(t) = A = cons\tan t$ from eq. (10.11))

$$V_S = R\, A + L\frac{d\, A}{dt} \quad A = \frac{V_s}{R} \tag{10.13}$$

Using initial condition (t = 0)

Case-A: Assume current flowing through the inductor just before closing the switch 'S" (at $t = 0^-$) is $i(0^-) = 0$.

$$i(0) = i(0^-) = i(0^+) = A_1 + A \tag{10.14}$$

$$0 = A_1 + A \quad A_1 = -A = -\frac{V_s}{R}$$

Using the values of α, A_1 *and* A in equation (10.11), we get the current expression as

$$i(t) = \frac{V_s}{R}\left(1 - e^{-\frac{R}{L}t}\right) \tag{10.15}$$

The table shows how the current i(t) builds up in a R-L circuit.

Actual time (t) in sec	Growth of current in inductor (Eq.10.15)
$t = 0$	$i(0) = 0$
$t = \tau = \frac{L}{R}$	$i(\tau) = 0.632 \cdot \frac{V_s}{R}$
$t = 2\tau$	$i(2\tau) = 0.865 \cdot \frac{V_s}{R}$
$t = 3\tau$	$i(3\tau) = 0.950 \cdot \frac{V_s}{R}$
$t = 4\tau$	$i(4\tau) = 0.982 \cdot \frac{V_s}{R}$
$t = 5\tau$	$i(5\tau) = 0.993 \cdot \frac{V_s}{R}$

Note: Theoretically at time $t \to \infty$ the current in inductor reaches its steady state value but in practice the inductor current reaches 99.3% of its steady state value at time $t = 5\tau$ (sec.) .

The expression for voltage across the external resistance R_1 (see Fig. 10.6(a))

$$= v_{R1} = i(t)R_1 = \frac{V_s}{R}\left(1 - e^{-\frac{R}{L}t}\right)R_1 \tag{10.16}$$

The expression for voltage across the inductor or coil

$$v_{coil}(t) = v_{inductor}(t) = V_S - v_{R_1}(t) = V_S - \frac{V_s}{R}(R_1)\left(1 - e^{-\frac{R}{L}t}\right) \tag{10.17}$$

Graphical representation of equations (10.15)-(10.17) are shown in Fig.10.7 for different choices of circuit parameters (i.e., L & R)

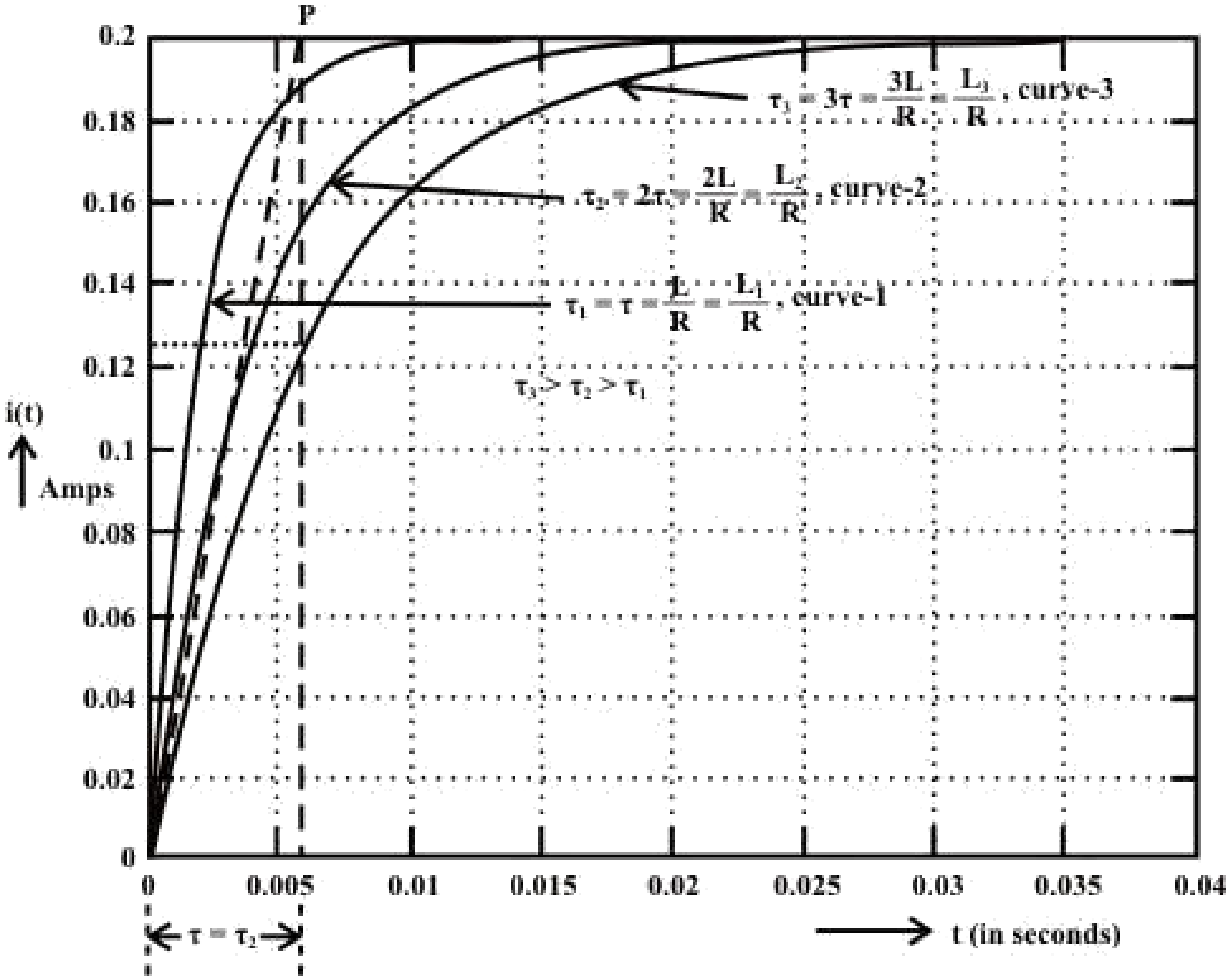


Fig. 10.7(a): Growth of current in R-L circuit (assumed initial current through inductor is zero).

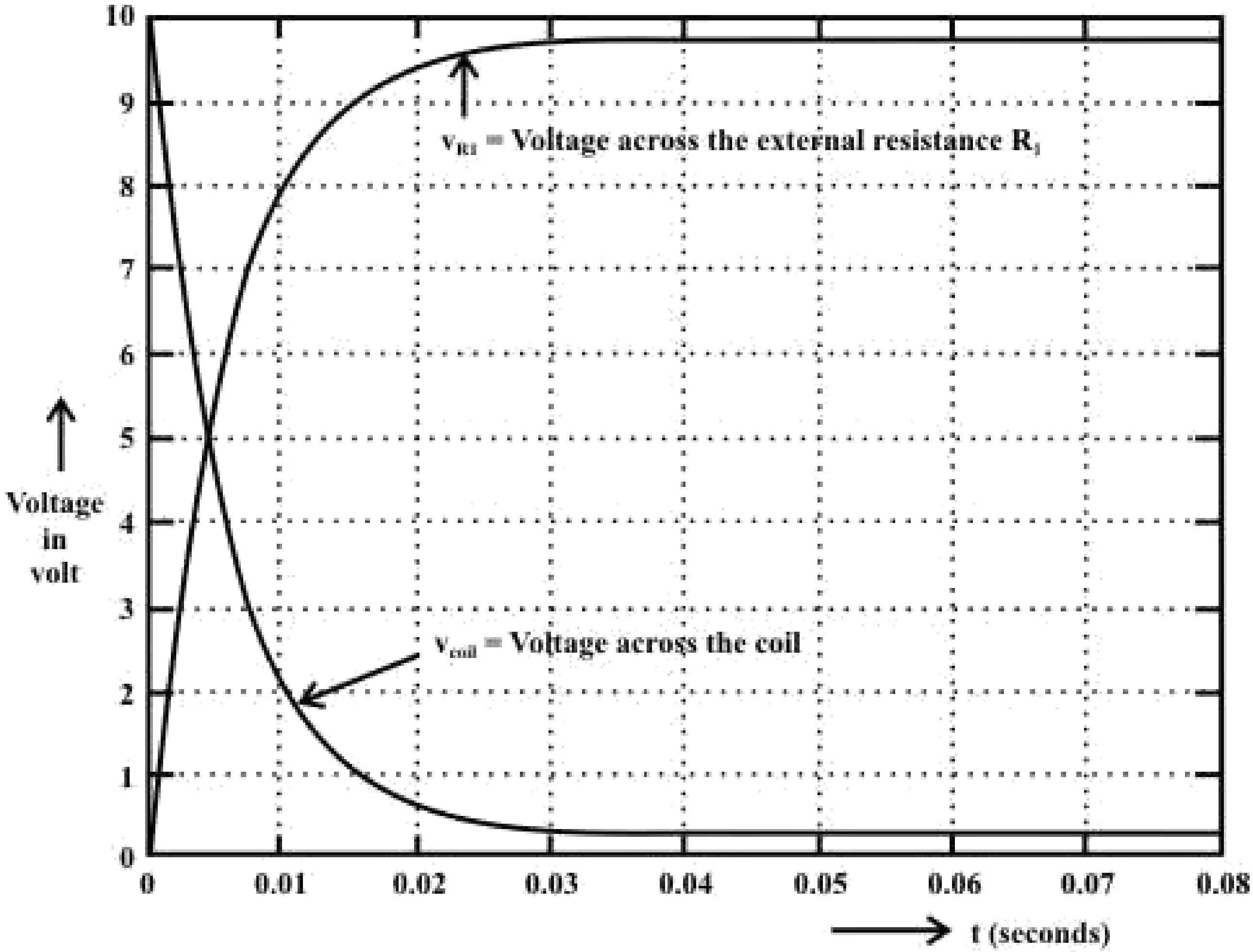


Fig. 10.7(b): Voltage response in different elements of R-L circuit (assumed i_o = o)

Case-B: Assume current flowing through the inductor just before closing the switch 'S" (at $t = 0^-$) is $i(0^-) = i_0 \neq 0$.

Using equations (10.13) and (10.14), we get the values of $A = \frac{V_S}{R}$ *and* $A_1 = i(0) - A = i_0 - \frac{V_S}{R}$. Using these values in equation (10.11), the expression for current flowing through the circuit is given by

$$i(t) = \frac{V_s}{R}\left(1 - e^{-\frac{R}{L}t}\right) + i_0 e^{-\frac{R}{L}t} \qquad (10.18)$$

The second part of the right hand side of the expression (10.18) indicates the current flowing to the circuit due to initial current i_0 of inductor and the first part due to the forcing function V_S applied to the circuit. This means that the complete response of the circuit is the algebraic sum of two outputs due to two inputs; namely (i) due to forcing function V_S (ii) due to initial current i_0 through the inductor. This implies that the superposition theorem is also valid for such type of linear circuit. Fig.10.8 shows the response of inductor current when the circuit is excited with a constant voltage source V_S and the initial current through inductor is i_0 .

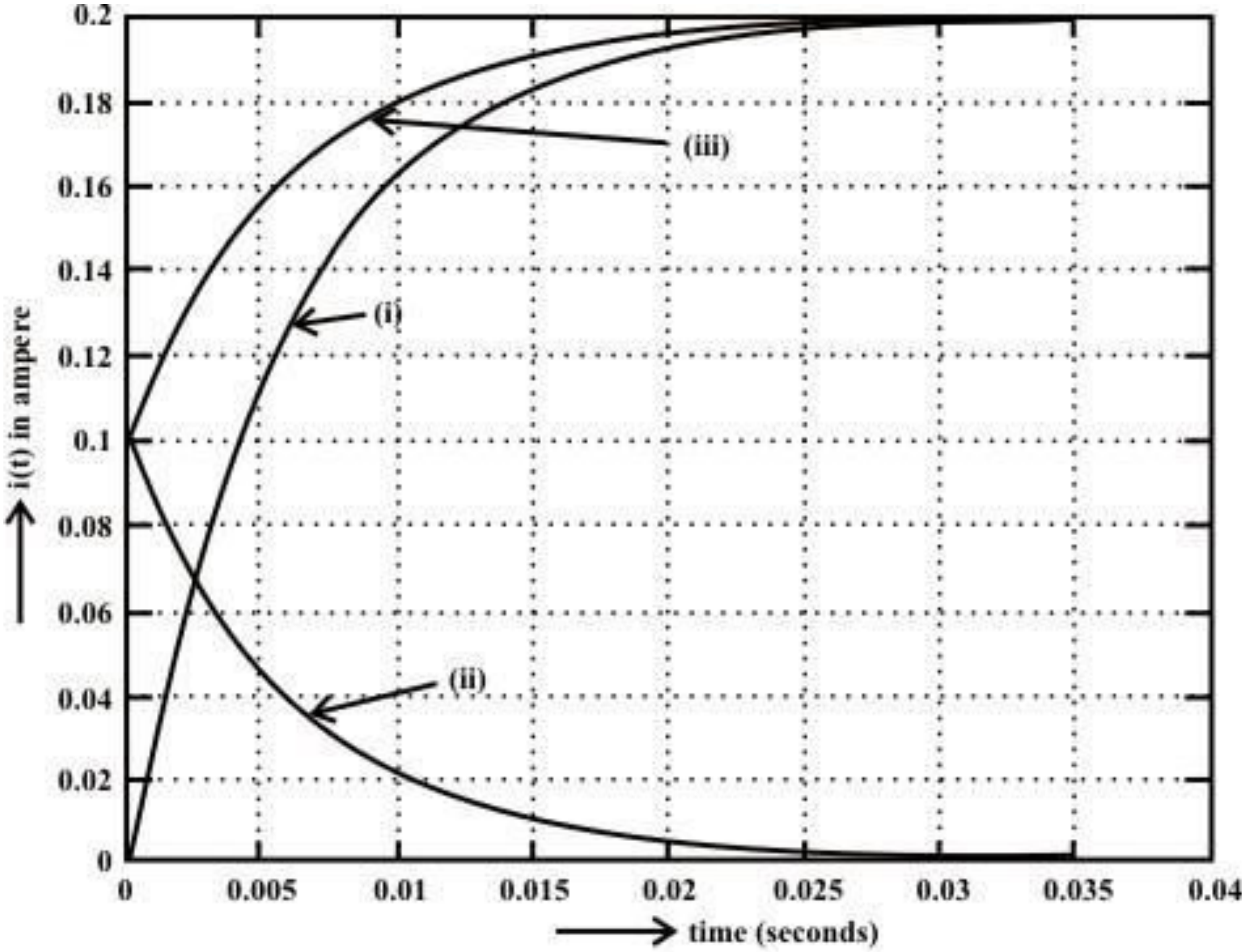


Fig. 10.8: Current through inductor due to (i) forcing function V_s only, (ii) initial condition i_o only, (iii) combined effect of (i) and (ii)

Remark-3: One can also solve this differential equation by separating the variables and integrating.

Time constant (τ) for exponential growth response (τ): We have seen that the current through inductor is represented by

$$i(t) = \frac{V_S}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

when a series $R - L$ circuit is excited by a constant voltage source (V_S) and an initial current through the inductor i_0 is assumed to be zero. Further it may be noted that the current through the inductor (see fig.10.7) increases as time increases. The shape of growing current before it reaches to a steady state value $\frac{V_S}{R}$ entirely depends on the parameters of $R-L$ circuit (i.e. R & L) that associated with the exponential term $e^{-\frac{R}{L}t}$. As 't' grows larger and larger the transient, because of its negative exponential factor, diminishes and disappears, leaving only the steady state.

Definition of Time Constant (τ) of $R-L$ Circuit: It is the time required for any variable or signal (in our case either current ($i(t)$) or voltage $(v_{R1}\ or\ v_L)$) to reach 63.2% (i.e the time at which the factor $-e^{-\frac{R}{L}t}\times 100$ in eq.(10.15) becomes $(1-e^{-1})\times 100 = 63.2\ \%$) of its final value. It is possible to write an exact mathematical expression to calculate the time constant (τ) of any first-order differential equation.

Let 't' is the time required to reach 63.2% of steady-state value of inductor current (see fig. 10.6(a)) and the corresponding time 't' expression can be obtained as

$$i(t) = 0.632*\frac{V}{R} = \frac{V}{R}\left(1-e^{-\frac{R}{L}t}\right) \quad 0.632 = 1-e^{-\frac{R}{L}t} \quad 0.368 = e^{-\frac{R}{L}t} = e^{-1} \quad t = \frac{L}{R} = \tau\ (\text{sec.})$$

The behavior of all circuit responses (for first-order differential equation) is fixed by a single time constant τ (for $R-L$ circuit $\tau = \frac{L}{R}$) and it provides information about the speed of response or in other words, it indicates how first or slow the system response reaches its steady state from the instant of switching the circuit. Observe the equation (10.15) that the smaller the time constant (τ), the more rapidly the current increases and subsequently it reaches the steady state (or final value) quickly. On the other hand, a circuit with a larger time constant (τ) provides a slow response because it takes longer time to reach steady state. These facts are illustrated in fig.10.7(a). In accordance with convenience, the time constant of an exponential term $\left(say\ p(t) = p_0\left(1-e^{-at}\right)\right)$ is the reciprocal of the coefficient 'a' associated with the 't' in the power of exponential term.

Remark -4: An interesting property of exponential term is shown in fig. 10.7(a) . The time constant τ of a first order differential equation may be found graphically from the response curve. It is necessary to draw a tangent to the exponential curve at time '$t = 0$' and maintained the same slope until it intersects the steady state value of current curve at P point. A perpendicular is drawn from the point P to the time axis and it intersects the time axis at $t = \tau$ (see fig. 10.7(a)). Mathematically, this can be easily verified by considering the equation of a straight line tangent to the current curve at $t = 0$, given by $y = mt$ where m is the slope of the straight line, expressed as

$$m = \frac{d\,i(t)}{dt}\Big|_{t=0} = \frac{d\left(\frac{V_s}{R}\left(1-e^{-\frac{R}{L}t}\right)\right)}{dt}\Big|_{t=0} = \frac{V_s}{L} \tag{10.19}$$

Here, we designated the value of time ' t ' required to reach y from ' 0 ' to $\frac{V_S}{R}$ *units* , assuming a constant rate (slope) of growth. Thus,

$$\frac{V_S}{R} = \frac{V_S}{L}\,t \quad t = \frac{L}{R} = \tau \text{ (sec.)} \tag{10.20}$$

It is often convenient way of approximating the time constant (τ) of a circuit from the response curve (see fig.10.7(a) for curve-2).

L.10.3.2 Fall or Decay of current in a R-L circuit

Let us consider the circuit shown in fig. 10.9(a). In this circuit, the switch 'S' is closed sufficiently long duration in position '1'. This means that the current through the inductor reaching its steady-state value ($I = \frac{V_S}{R} = \frac{V_S}{R_1 + R_L} = I_0$) and it acts, as a short circuit i.e. the voltage across the inductor is nearly equal to zero since resistance R_L R_1 . If the switch 'S' is opened at time 't'=0 and kept in position '2' for $t > 0$ as shown in fig. 10.9(b), this situation is referred to as a source free circuit.

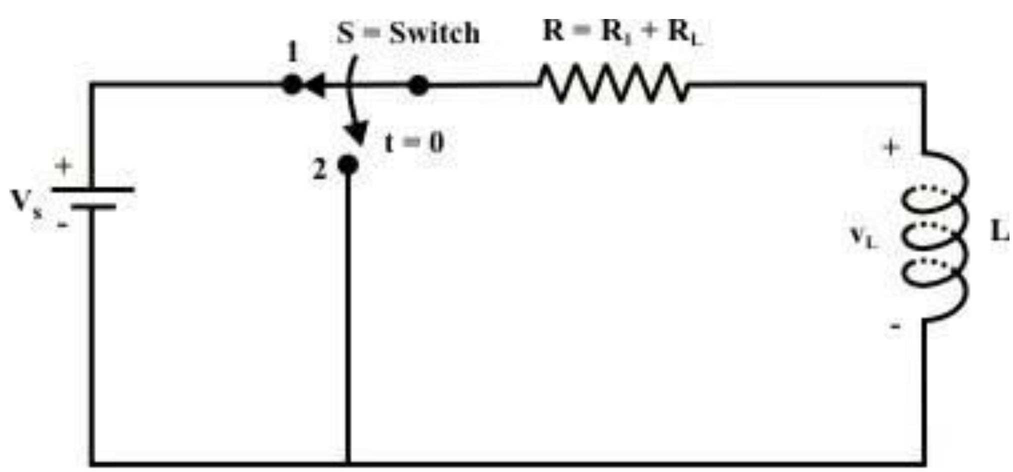


Fig. 10.9(a)

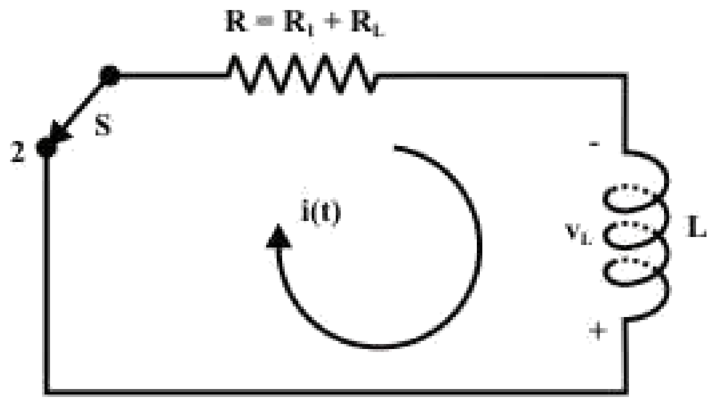


Fig. 10.9(b): Decay of current in R - L circuit

Since the current through an inductor cannot change instantaneously, the current through the inductor just before ($i(0^-)$) and after ($i(0^+)$) opening the switch 'S' must be same. Because there is no source to sustain the current flow in inductor, the magnetic field in inductor starts to collapse and this, in turn, will induce a voltage across the inductor. The polarity of this induced voltage across the inductor is just in reverse direction compared to the situation that occurred during the growth of current in inductor (i.e. when the switch 'S' is kept in position '1'). This is illustrated in fig. 10.9(b), where the voltage induced in inductor is positive at the bottom of the inductor terminal and negative at the top. This implies that the current through inductor will still flow in the same direction, but with a magnitude decaying toward zero. Appling KVL around the closed circuit in fig. 10.9(b), we obtain

$$L\frac{d\,i(t)}{dt} + R\,i(t) = 0 \tag{10.21}$$

The solution of the homogeneous (input free), first-order differential equation with constant coefficients subject to the initial (boundary) inductor current (initial condition, $i(0^-) = i(0^+) = \frac{V_S}{R} = I$) is given by

$$i(t) = i_n(t) = A_1\, e^{\alpha t} \tag{10.22}$$

where α can be found from the characteristic equation of eq.(10.21) described by

$$L\,\alpha + R = 0 \quad \alpha = -\frac{R}{L} \tag{10.23}$$

At time $t = 0$, the initial condition $i(0^-) = i(0^+) = \frac{V_S}{R}$ is used in equation (10.22) to compute the constant A_1 and it is given below.

$$i(0) = A_1 \qquad A_1 = \frac{V_S}{R} = I$$

Using the values of A_1 *and* α in equation (10.22), we get final expression as

$$i(t) = \frac{V_S}{R} e^{-\frac{R}{L}t} \qquad for \quad t \geq 0 \tag{10.24}$$

A sketch of $i(t)$ for $t \geq 0$ is shown in fig.10.10. Here, transient has ended and steady state has been reached when both current in inductor $i(t)$ and voltage across the inductor including its internal resistance are zero.

Time Constant (τ) for exponential decay response: For the source free circuit, it is the time τ by which the current falls to 36.8 percent of its initial value. The initial condition in this case (see fig. 10.9(a) is considered to be the value of inductor's current at the moment the switch S is opened and kept in position ' 2 '. Mathematically, τ is computed as

$$i(t) = 0.368 \cdot \frac{V_S}{R} = \frac{V_S}{R} e^{-\frac{R}{L}t} \qquad t = \tau = \frac{L}{R} \tag{10.25}$$

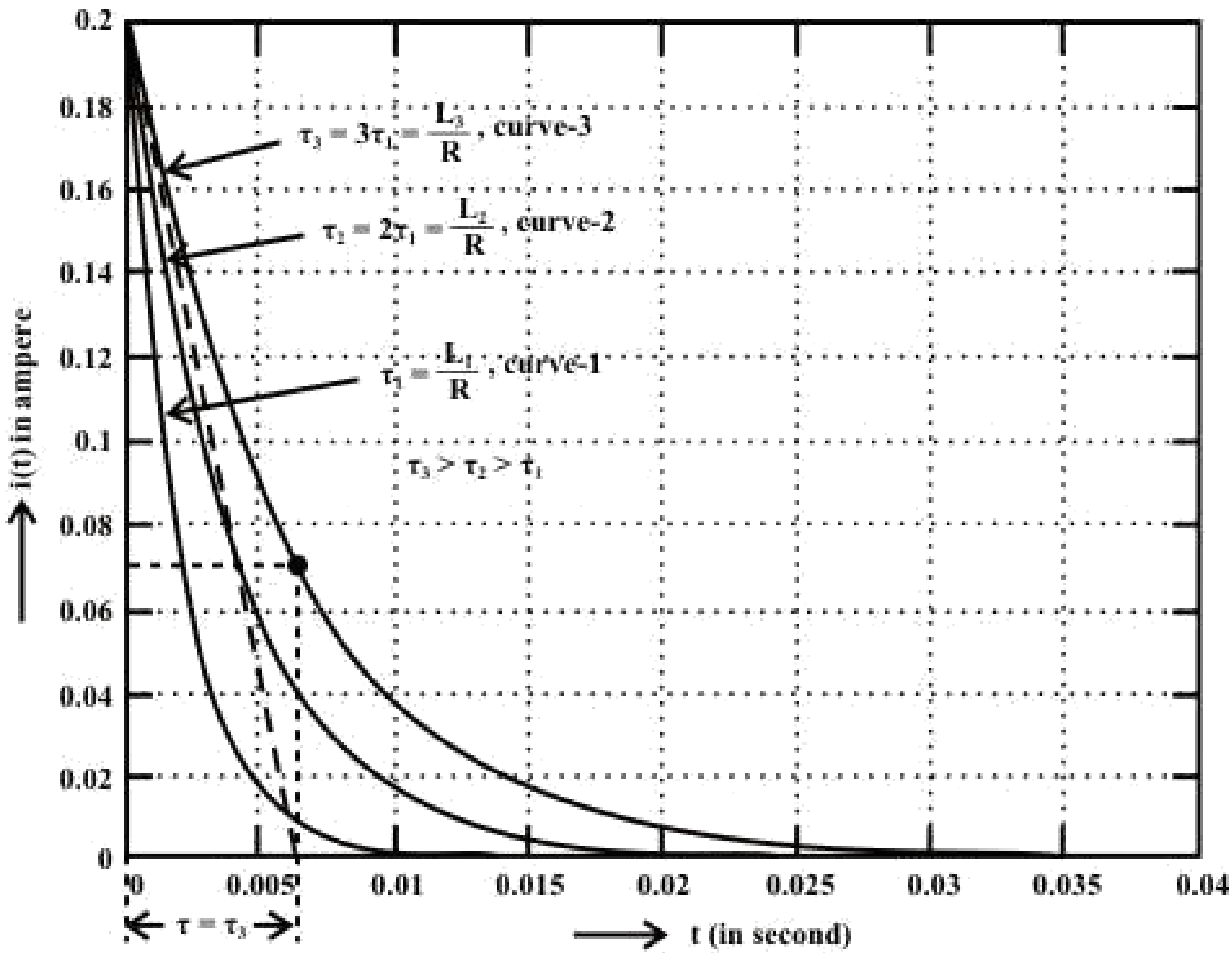


Fig. 10.10: Fall of current in R-L circuit (assumed initial current through inductor is I).

Alternatively, the time constant for an exponential decay response of a circuit may be computed graphically by adopting the steps (see equations (10.19) and (10.20)) as discussed before. In fig.10.10, a tangent is drawn to the exponential decay curve at time ' $t = 0$ ' and maintained the same slope until the straight line intercepts time axis at time $t = \tau$. Approximately, the value of τ can thus be found directly from graphical representation of exponential decay curve.

L.10.3.3 Energy stored in an inductor

Let us turn our attention to power and energy consideration for an inductor. The instantaneous power absorbed by the inductor is expressed by product of the current through inductor $i(t)$ and the voltage across it $v(t)$.

$$p(t) = v(t)\, i(t) = i(t)\, L \frac{d\, i(t)}{dt} \qquad (10.26)$$

Since the energy is the product of power and time, the energy absorbed by an inductor over a period is expressed as

$$W_L = \int_{t_0}^{t} p(t)\,dt = \int_{t_0}^{t} i(t)\, L \frac{d\,i(t)}{dt}\, dt = \frac{1}{2} L \left[i^2(t) - i^2(t_0) \right] \qquad (10.27)$$

where we select the current through inductor at time ‘ $t_0 = -\infty$ ’ is $i(-\infty) = 0$. Then, we have $W_L = \frac{1}{2} L\, i^2(t)$ and from this relation we see that the energy stored in an inductor is always non-negative. At any consequent time at which the current is zero, no energy is stored in the inductor. The ideal inductor ($R_L = 0\ \Omega$) never dissipates energy, but only stores. In true sense, a physical or practical inductor dissipates a very small amount of stored energy due to its small series resistance.

Example-L.10.2 Fig.10.11 shows the plot of current $i(t)$ through a series $R - L$ circuit when a constant forcing function of magnitude $V_S = 50\ V$ is applied to it. Calculate the values of resistance R and inductance L .

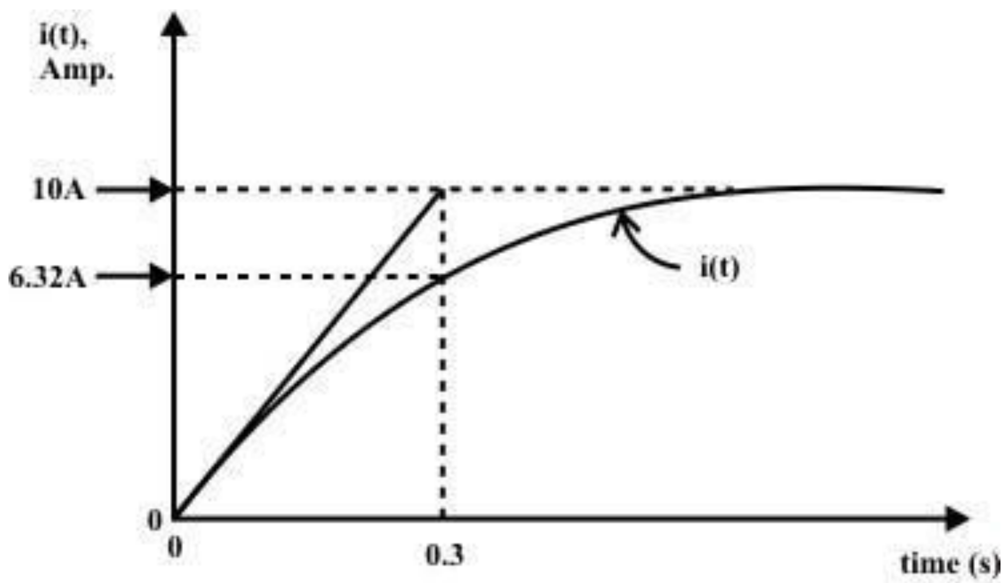


Fig. 10.11: Current -y$_s$-time characteristic

Solution: From fig.10.11 one can easily see that the steady state current flowing through the circuit is 10 A and the time constant of the circuit $\tau = 0.3$ sec. The following relationships can be written as

$$i_{steady\ state} = \frac{V_S}{R} \quad 10 = \frac{50}{R} \quad R = 5\ \Omega$$

and $\tau = \frac{L}{R} \quad 0.3 = \frac{L}{5} \quad L = 1.5\ H$

Example-L.10.3 For the circuit shown in Fig.10.12, the switch ‘ S ’ has been closed for a long time and then opens at $t = 0$.

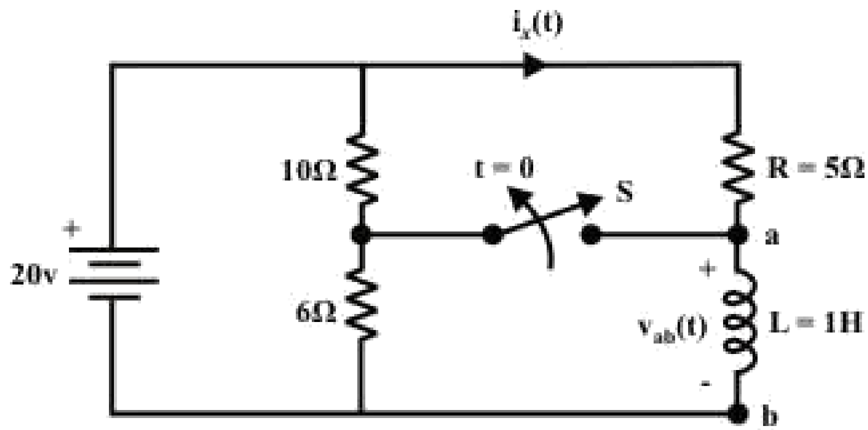


Fig. 10.12

Find,
(i) $v_{ab}(0^-)$ (ii) $i_x(0^-)$, $i_L(0^-)$ (iii) $i_x(0^+)$ (iv) $v_{ab}(0^+)$ (v) $i_x(t=\infty)$ (vi) $v_{ab}(t=\infty)$
(vii) $i_x(t)$ $for\ \ t>0$

Solution: When the switch S was in closed position for a long time, the circuit reached in steady state condition i.e. the current through inductor is constant and hence, the voltage across the inductor terminals *a and b* is zero or in other words, inductor acts as short circuit i.e., (i) $v_{ab}(0^-) = 0\ V$. It can be seen that the no current is flowing through 6 Ω resistor. The following are the currents through different branches just before the switch ' S ' is opened i.e., at $t = 0^-$.

$i_x(0^-) = \frac{20}{5} = 4\ A$ and the current through $10\ \Omega$ resistor, $i_{10\Omega}(0^-) = \frac{20}{10} = 2\ A$. The algebraic sum of these two currents is flowing through the inductor i.e., (ii) $i_L(0^-) = 2 + 4 = 6\ A$.

When the switch ' S ' is in open position

The current through inductor at time $t = 0^+$ is same as that of current $i_L(0^-)$, since inductor cannot change its current instantaneously .Therefore, the current through $i_x(0^+)$ is given by

$i_x(0^+) = i_L(0^+) = 6\ A$.

Applying KVL around the closed loop at $t = 0^+$ we get,

$20 - i_x(0^+)\cdot R = v_{ab}(0^+)$ $\qquad$ $20 - 6\cdot 5 = v_{ab}(0^+)$ $\ v_{ab}(0^+) = -10V$

The negative sign indicates that inductor terminal ' b ' as +ve terminal and it acts as a source of energy or mathematically, $v_{ba}(0^+) = 10V$.

At steady state condition ($t \to \infty$) the current through inductor is constant and hence inductor acts as a short circuit. This establishes the following relations:

$$v_{ba}(t=\infty) = 0V \quad \text{and}\ i_x(t=\infty) = \frac{20}{5} = 4\ A \qquad (10.28)$$

The circuit expression $i_x(t)$ for $t \geq 0$ can be obtained using the KVL around the closed path (see fig.10.12).

KVL equation:

$$V_S - i_x(t) \cdot 5 - L\frac{di_x(t)}{dt} = 0$$

$$i_x(t) \cdot 5 + L\frac{di_x(t)}{dt} = V_S \tag{10.29}$$

The solution of first order differential equation due to forcing function and initial condition is given by

$$i_x(t) = A_1 e^{-\frac{R}{L}t} + A \tag{10.30}$$

Initial and final conditions are: (i) At $t = 0$, $i_x(0) = i_L(0^-) = i_L(0^+) = 6\,A$ (ii) $t \to \infty$, current through inductor $i_L(t = \infty) = 4\,A$ (see Eq. 10.28). Using initial and final conditions equation (10.30) we get, $A_1 = 6 - A$ *and* $A = 4 A_1 = 2$

From equation (10.30), we get the final expression as $i_x(t) = 4 + 2\,e^{-\frac{5}{1}t}$ for $t \geq 0$.

Example: L.10.4 The switch 'S' is closed in position '1' sufficiently long time and then it is kept in position '2' as shown in fig.10.13. Compute the value of v_L *and* i_L (i) the instant just prior to the switch changing; (ii) the instant just after the switch changes. Find also the rate of change of current through the inductor at time $t = 0^+$ *i.e.,* $\left.\frac{di(t)}{dt}\right|_{t=0^+}$.

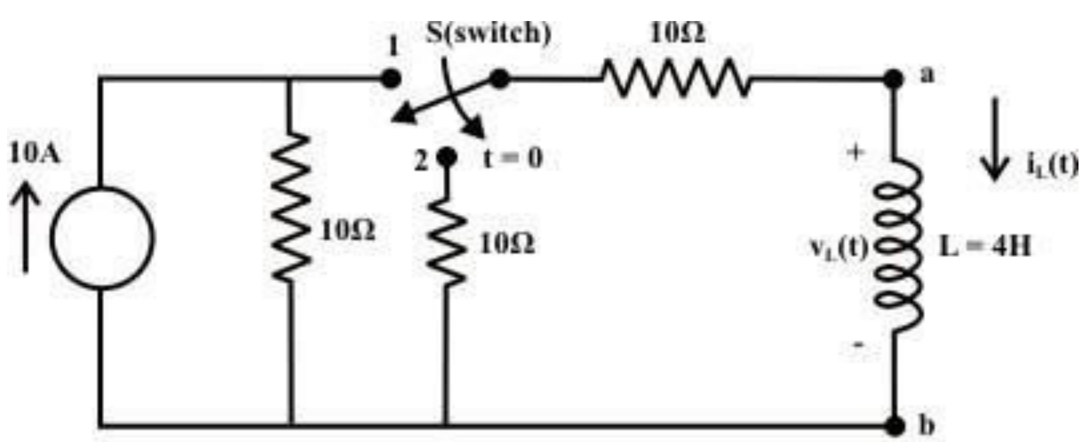


Fig. 10.13

Solution: We assume that the circuit has reached at steady state condition when the switch was in position '1'. Note, at steady state the inductor acts as short circuit and voltage across the inductor is zero.

At $t = 0^-$, the current through and the voltage across the inductor are $i_L(0^-) = \frac{10}{10+10} \cdot 10 = 5\,A$ and $v_L(0^-) = 0V$ respectively. When the switch is kept in position ' 2 ', current through the inductor cannot change instantaneously but this is not true for the voltage across the inductor. At $t = 0^+$, one can write the following expressions:

$i_L(0^+) = 5A$ and $v_L(0^+) = -(10+10) \cdot 5 = -100V$ (' b ' is more + ve potential than ' a ' terminal). Note that the stored energy in inductor is dissipated in the resistors. Now, the rate of change of current through inductor at time $t = 0^+$ is obtained as

$$L\left.\frac{di_l(t)}{dt}\right|_{t=0^+} = -100V \qquad \left.\frac{di_l(t)}{dt}\right|_{t=0^+} = \frac{-100}{4} = -25\ amp./\sec.$$

Example: L.10.5 Fig. 10.14(a) shows that a switch ' S ' has been in position '1 ' for a long time and is moved in position ' 2 ' at time ' $t = 0$ '. Find the expression $v(t)$ for $t \geq 0$.

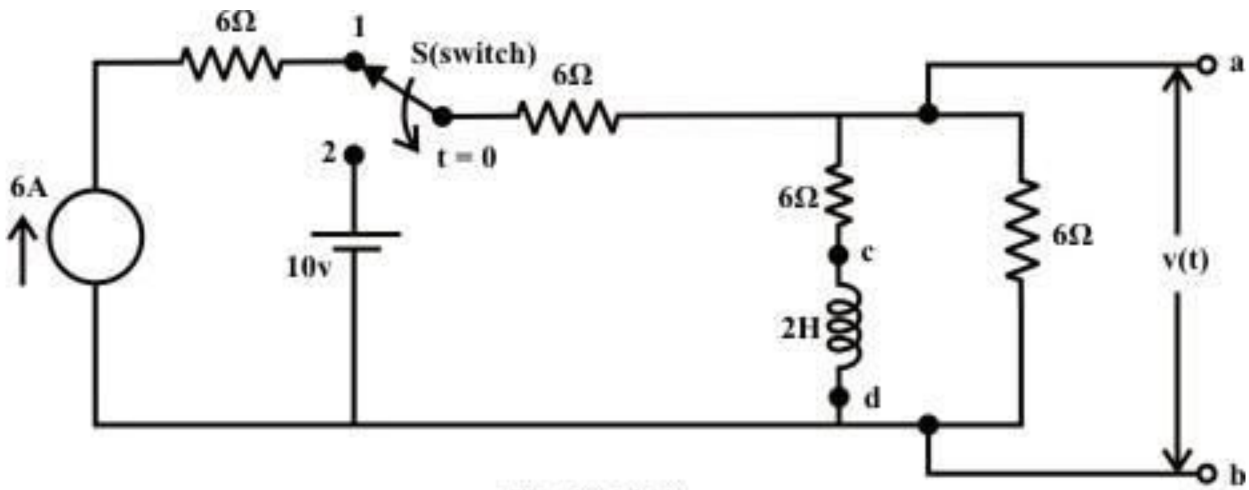


Fig. 10.14(a)

Solution: When the switch ' S ' is in position '1', the current through inductor (using the fundamental property of inductor currents) at steady state condition (see fig.10.14(b)) is

$$I_L = \frac{6}{6+6} \cdot 6 = 3\,A I_L(0^-) = I_L(0^+) = 3\,A \tag{10.31}$$

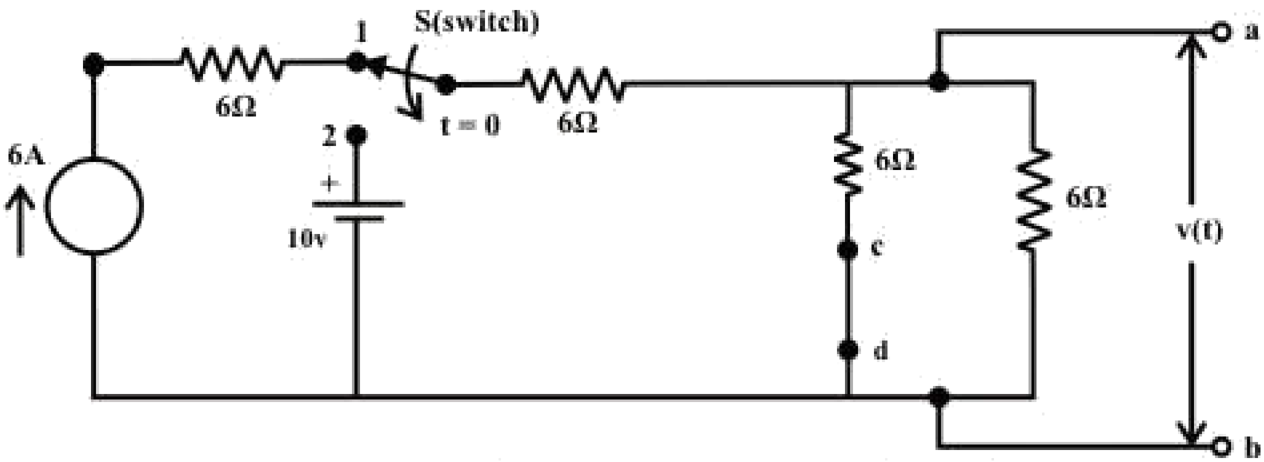


Fig. 10.14(b): Circuit diagram for the switch in position '1'

The circuit for the switch ' *S* is in position ' 2 ' is shown in fig.10.14 (c). The current in inductor can be computed using following two different methods.

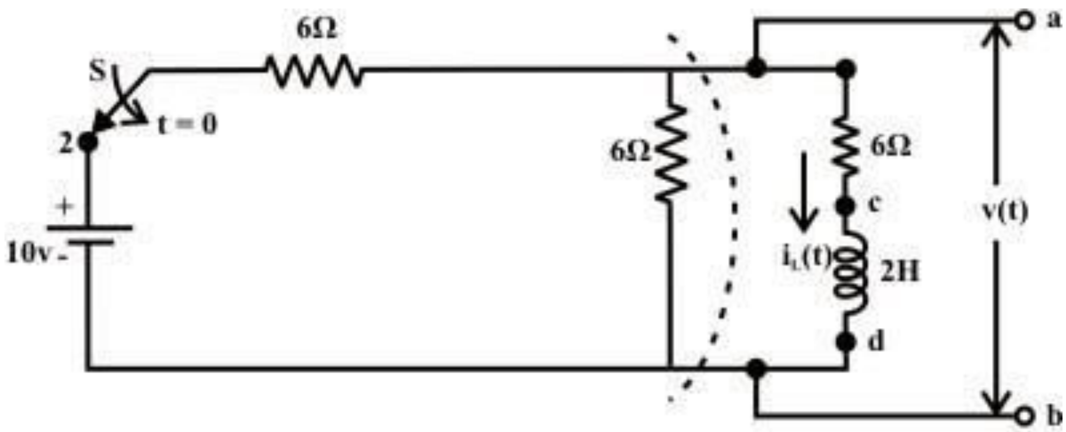


Fig. 10.14(c)

Method-1: Using Thevenin's theorem

Convert the part of a circuit **containing independent sources and resistances into** an equivalent Thevenin's voltage source as shown in fig.10.14.(d).

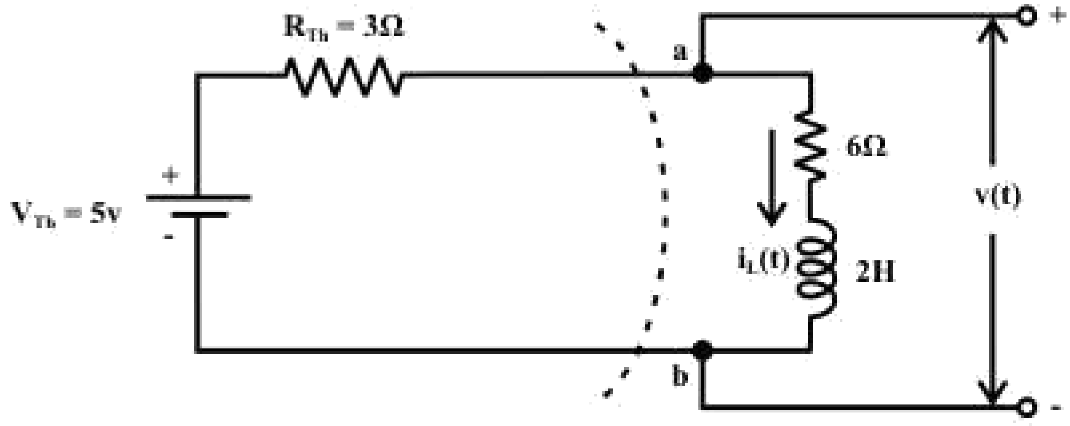


Fig. 10.14(d)

Using the KVL around the closed path is

$$9\,i_L(t) + 2\cdot\frac{di_L(t)}{dt} = 5 \tag{10.32}$$

The solution of the above equation is given by

$$i_L(t) = A_1\,e^{-\frac{9}{2}t} + A = i_n(t) + i_f(t) \tag{10.33}$$

where, $i_n(t)$ = complementary/natural/transient solution of eq.(10.32)

$i_f(t)$ = particular/ steady state/final solution of eq.(10.32)

The constants A_1 *and* A are computed using the initial and final conditions of the circuit when the switch is kept in position ' 2 '.

At time $t = 0$,

$$i_L(0) = i_L(0^+) = 3 = A_1 + A \tag{10.34}$$

At time $t \to \infty$, the current in inductor reached its steady state condition and acts as a short circuit in a dc source network. The current through inductor is

$$i_L(t=\infty) = \frac{5}{3+6} = 0.555\,amp. = i_f = A \tag{10.35}$$

Using the above two equations in (10.33), one can obtain the final voltage expression for voltage $v(t)$ across the terminals ' a ' and ' b ' as

$$v_{ab}(t) = v(t) = 5 - i_L(t)\cdot 3 = 5 - \left(2.445\cdot e^{-\frac{9}{2}t} + 0.555\right)\cdot 3 = 3.339 - 7.335\cdot e^{-\frac{9}{2}t}\ V$$

Method-2: Mesh current method

Assign the loop currents in clockwise directions and redrawn the circuit as shown in Fig. 10.14(e). The voltage across the terminals ' a ' and ' b ' can be obtained by solving the following loop equations.

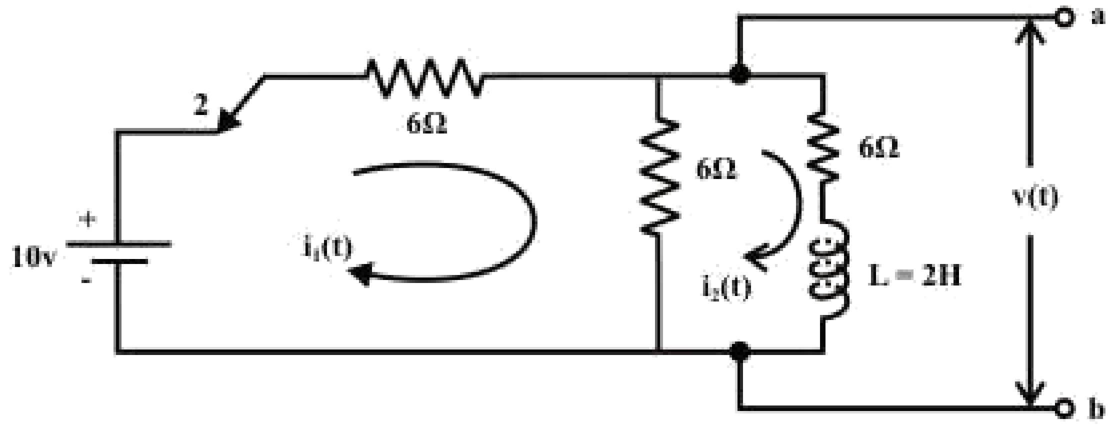


Fig. 10.14(e)

Loop-1:

$$10 - 6\,i_1(t) - 6\left(i_1(t) - i_2(t)\right) = 0 \quad 10 = 12\,i_1(t) - 6\,i_2(t) \quad i_1(t) = \frac{1}{12}\left(10 + 6\,i_2(t)\right) \tag{10.36}$$

Loop-2:

$$-6\,i_2(t) - L\frac{di_2(t)}{dt} - 6\left(i_2(t) - i_1(t)\right) = 0 \quad -6\,i_1(t) + 12\,i_2(t) + 2\frac{di_2(t)}{dt} = 0 \tag{10.37}$$

Using the value of $i_1(t)$ in equation (10.37), we get

$$9\,i_2(t) + 2\cdot\frac{di_2(t)}{dt} = 5 \tag{10.38}$$

To solve the above first order differential equation we must know inductor's initial and final conditions and their values are already known (see, $i_2(0^-) = i_2(0^+) = 3\,A$ and $i_2(t=\infty) = \frac{5}{3+6} = 0.555\ amp.$). The solution of differential equation (10.38) provides an expression of current $i_2(t)$ and this, in turn, will give us the expression of $i_1(t)$. The voltage across the terminals 'a' and 'b' is given by

$$v_{ab} = 10 - 6\cdot i_1(t) = 6i_2(t) + 2\frac{di_2(t)}{dt} = 3.339 - 7.335\cdot e^{-\frac{9}{2}t}\ V$$

where, $i_2(t)$, $i_1(t)$ can be obtained by solving equations (10.38) and (10.36). The expressions for $i_2(t)$ and hence $i_1(t)$ arte given below:

$$i_2(t) = 2.445\cdot e^{-\frac{9}{2}t} + 0.555 \quad \text{and} \quad i_1(t) = \frac{1}{12}\left(10 + 6\,i_2(t)\right) = 1.11 + 1.2225e^{-\frac{9}{2}t}$$

L.10.4 Capacitor and its behavior

Fig.10.15 shows a capacitor consists of two pieces of metal (the plates) separated from each other by a good insulator (the dielectric), with two wires (the leads) attached to the metal plates.

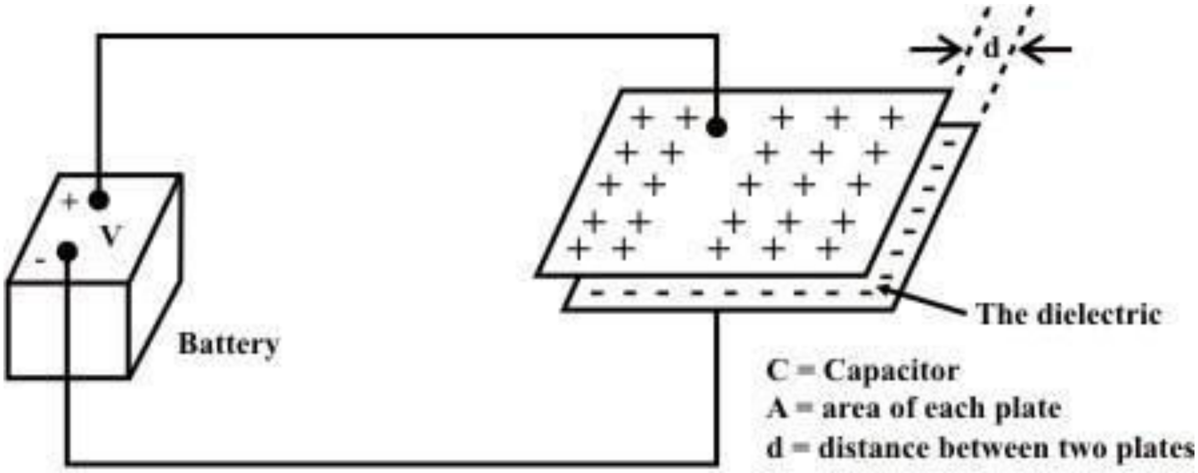


Fig. 10.15: Charging of a Capacitor

A battery is connected across the capacitor to transport charge from one plate to the other until the capacitor charge voltage buildup is equal to the battery voltage V. The voltage across the capacitor depends on how much charge was deposited on the plates and also how much capacitance the capacitor has. In other words, there is a relationship between the voltage (V), charge (Q) and capacitance (C), they are related with a mathematical expression as

$$C = \frac{Q\,(coulumb)}{V\,(volt)} \tag{10.39}$$

where Q = magnitude of charge stored on each plate, V = voltage applied to the plates and the unit of capacitance is in *Farad*. Although the capacitance C of a capacitor is the ratio of charge per plate to the applied voltage but it mainly depends on the physical dimension of the capacitor. If the area of the plates is larger, the more would be the amount of charge stored over the surface of the plates, resulting higher value of capacitance. On the other hand, if the spacing ‘ d ’ between the plates is closer, accumulates more charge over the parallel plates and thus increases the value of the capacitance. The quality of dielectric material has an effect on capacitance between the plates. The good quality of dielectric material indicates that higher the permittivity, resulting greater the capacitance. The value of capacitance can be expressed in terms physical parameters of capacitor as

$C = \frac{\varepsilon A}{d} = \frac{\varepsilon_0\, \varepsilon_r A}{d}$ where A is the area of each plate, d is the distance between the plates, ε_0 (= $8.85 \cdot 10^{-12}$) is the permittivity of free-space, ε_r = relative permittivity of dielectric material and C is the capacitance in Farad. It is important to note that when the applied voltage across the capacitor exceeds a certain value the dielectric material breaks down and loses it insulation property.

L.10.4.1 Continuity condition of capacitors

To find the current-voltage relationship of the capacitor, one can take the derivative of both sides of Eq.(10.39)

$$C\frac{d\,v_c(t)}{dt} = \frac{d\,q(t)}{dt} = i(t)\quad i(t) = C\frac{d\,v_c(t)}{dt} \qquad (10.40)$$

The voltage-current relation can also be represented by another form as $v_c(t) = \frac{1}{C}\int_{t_0}^{t} i(t)\,dt + v_c(t_0)$ where $v_c(t_0)$ is voltage across the capacitor at time ' t_0 '. It can be seen that when the voltage across a capacitor is not changing with time, or, in other words, the capacitor is fully charged and the current through the capacitor is zero (see Eq.10.40**).** This means that the capacitor resembles as an open circuit and blocks the flow of current through the capacitor. Equation (10.40) shows that an instantaneous ($t = 0$) change in capacitance voltage must be accompanied by an infinite current that requiring an infinite power source. In practice, this situation will not occur in any circuits containing energy storing elements. Thus, the voltage across the capacitor (or electric charge $q(t)$) cannot change instantaneously (*i.e.* , $t = 0$) , that is we cannot have any discontinuity in voltage across the capacitor.

Remark-5

(i) The voltage across and charge on a capacitor cannot change instantaneously (i.e. $v_c(0^-)$ just right before the change of voltage = $v_c(0^+)$ just right after the change of voltage). However, current through a capacitor can change abruptly. (ii) The capacitor acts as an open circuit (i.e., when the capacitor is fully charged) when voltage across the capacitor does not change (constant). (iii) These properties of capacitor are important since they will be used to determine "boundary conditions".

L.10.4.2 Study of dc transients and steady state response of a series R-C circuit.

Ideal and real capacitors: An ideal capacitor has an infinite dielectric resistance and plates (made of metals) that have zero resistance. However, an ideal capacitor does not exist as all dielectrics have some leakage current and all capacitor plates have some resistance. A capacitor's leakage resistance is a measure of how much charge (current) it will allow to leak through the dielectric medium. Ideally, a charged capacitor is not supposed to allow leaking any current through the dielectric medium and also assumed not to dissipate any power loss in capacitor plates resistance. Under this situation, the model as shown in fig. 10.16(a) represents the ideal capacitor. However, all real or practical capacitor leaks current to some extend due to leakage resistance of dielectric medium. This leakage resistance can be visualized as a resistance connected in parallel with the capacitor and power loss in capacitor plates can be realized with a resistance connected in series with capacitor. The model of a real capacitor is shown in fig. 10.16(b).

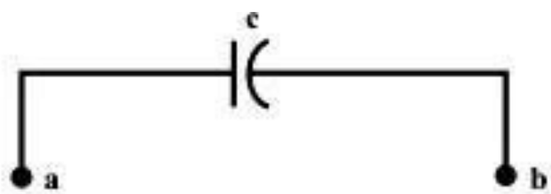


Fig. 10.16(a): Symbolic representation of an ideal capacitor

In present discussion, an ideal capacitor is considered to study the behavior of dc transients in $R - C$ circuit.

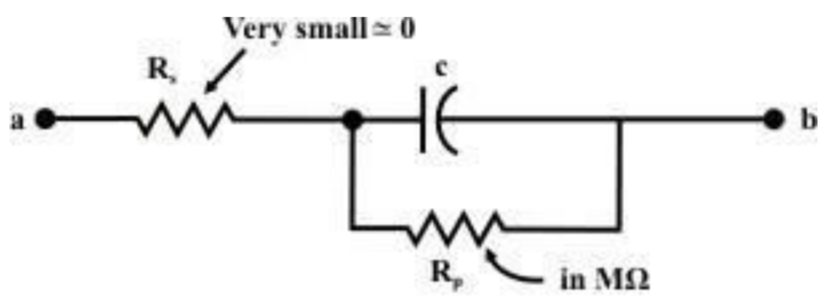


Fig. 10.16(b): Symbolic representation of a real capacitor

L.10.4.3 Charging of a capacitor or Growth of a capacitor voltage in dc circuits

Let us consider a simple series $R - C$ circuit shown in fig. 10.17(a) is connected through a switch 'S' to a constant voltage source V_S .

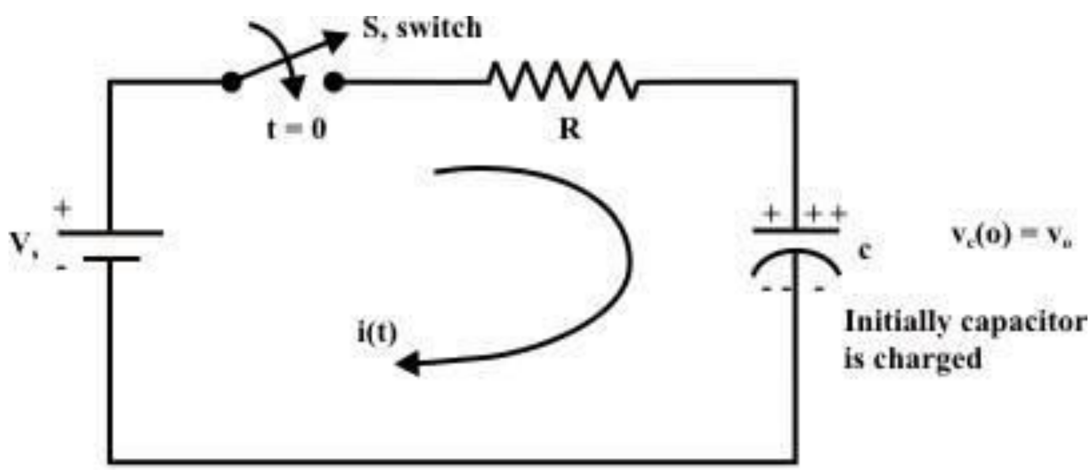


Fig. 10.17(a): Charging of a RC circuit

The switch ' S ' is closed at time ' $t = 0$ ' (see fig. 10.7(a)). It is assumed that the capacitor is initially charged with a voltage $v_c(0) = v_0$ and the current flowing through the circuit at any instant of time ' t ' after closing the switch is $i(t)$.

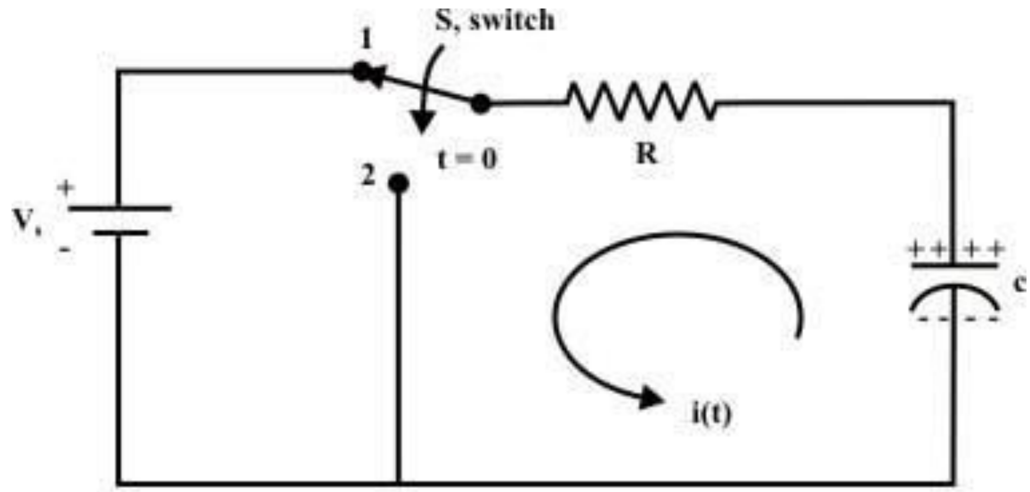


Fig. 10.17(b): Discharging of a RC circuit

The KVL equation around the loop can be written as

$$V_S = R\, i(t) + v_c(t) \quad V_S = R\, C \frac{d\, v_c(t)}{dt} + v_c(t) \tag{10.41}$$

The solution of the above first-order differential equation (10.41) due to forcing function V_s is given by

$v_c(t) = v_{cn}(t)$ (natural response/transient response) + $v_{cf}(t)$ (steady-state response)

$$= A_1 e^{\alpha t} + A \tag{10.42}$$

The constants A_1, *and* A are computed using the initial and boundary conditions. The value of α is obtained from the characteristic equation given by (see in detail in Appendix)

$$R\, C\, \alpha + 1 = 0 \quad \alpha = -\frac{1}{RC}$$

Eq. (10.42) is then rewritten as

$$v_c(t) = A_1 e^{-\frac{1}{RC} t} + A \tag{10.43}$$

At steady state, the voltage across the capacitor is $v_c(\infty) = v_{cf} = A$ which satisfy the original differential equation (10.41). i.e.,

$$V_s = R\, C \frac{d\, v_{cf}}{dt} + v_{cf} \quad R\, C \frac{d\, A}{dt} + A \qquad A = V_S$$

Using the initial condition (at $t = 0$) in equation (10.43), we get

$$v_c(0) = v_0 = A_1 e^{-\frac{1}{RC}\cdot 0} + A \quad A_1 = v_0 - A = v_0 - V_s$$

The values of A_1 , A , and Eq. (10.43) together will give us the final expression for capacitor voltage as

$$v_c(t) = (v_0 - V_s)\, e^{-\frac{1}{RC}t} + V_S \qquad v_c(t) = V_S\left(1 - e^{-\frac{1}{RC}t}\right) + v_0 e^{-\frac{1}{RC}t} \tag{10.44}$$

Thus,

$$v_c(t) = \begin{cases} v_0 & t < 0 \\ v_c(t) = V_S\left(1 - e^{-\frac{1}{RC}t}\right) + v_0 e^{-\frac{1}{RC}t} & t > 0 \end{cases}$$

Response of capacitor voltage with time is shown in fig. 10.18.

Special Case: Assume initial voltage across the capacitor at time ' $t = 0$ ' is zero i.e., $v_c(0) = v_0 = 0$. The voltage expression for capacitor at any instant of time can be written from Eq.(10.44) with $v_c(0) = v_0 = 0$.

Voltage across the capacitance $v_c(t) = V_s\left(1 - e^{-\frac{1}{RC}t}\right)$ (10.45)

Voltage across the resistance $v_R(t) = V_S - v_c(t) = V_S e^{-\frac{1}{RC}t}$ (10.46)

Charging current through the capacitor $i(t) = \frac{v_R}{R} = \frac{V_S}{R} e^{-\frac{1}{RC}t}$ (10.47)

Charge accumulated on either plate of capacitor at any instant of time is given by

$$q(t) = C v_c(t) = C V_S\left(1 - e^{-\frac{1}{RC}t}\right) = Q\left(1 - e^{-\frac{1}{RC}t}\right) \tag{10.48}$$

where Q is the final charge accumulated in the capacitor at steady state (i.e., $t \to \infty$). Once the voltage across the capacitor $v_c(t)$ is known, the other quantities (like, $v_R(t)$, $i(t)$, *and* $q(t)$) can easily be computed using the above expressions. Fig. 10.19(a) shows growth of capacitor voltage $v_c(t)$ for different choices of circuit parameters (assumed that the capacitor is initially not charged). A sketch for $q(t)$ *and* $i(t)$ is shown in fig. 10.19(b).

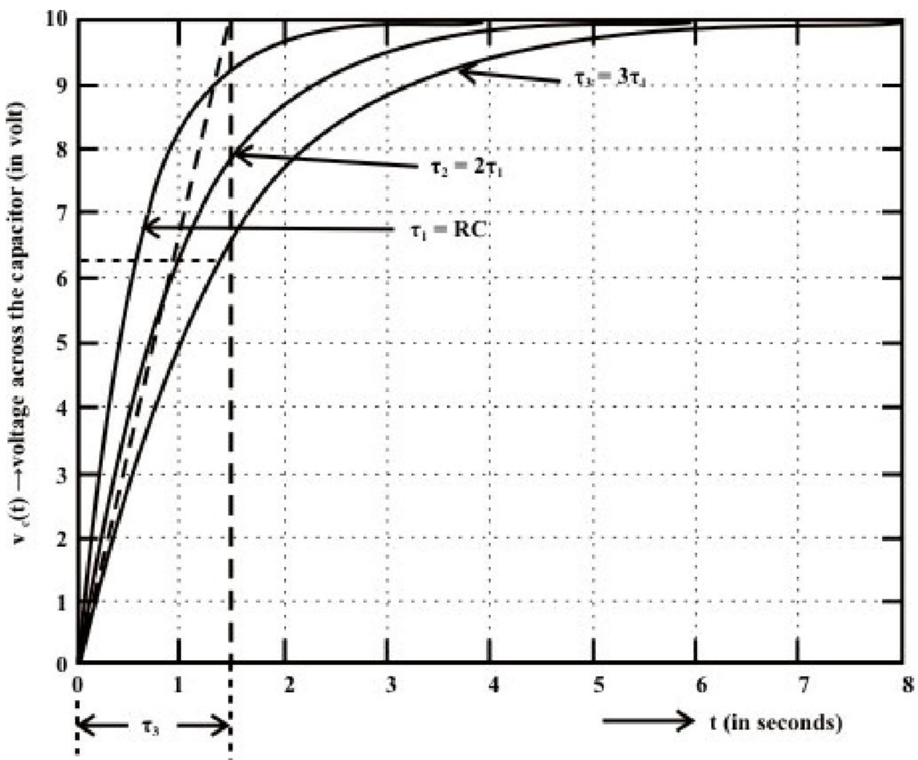


Fig. 10.19(a): Growth of capacitor voltage (assumed initial capacitor voltage is zero)

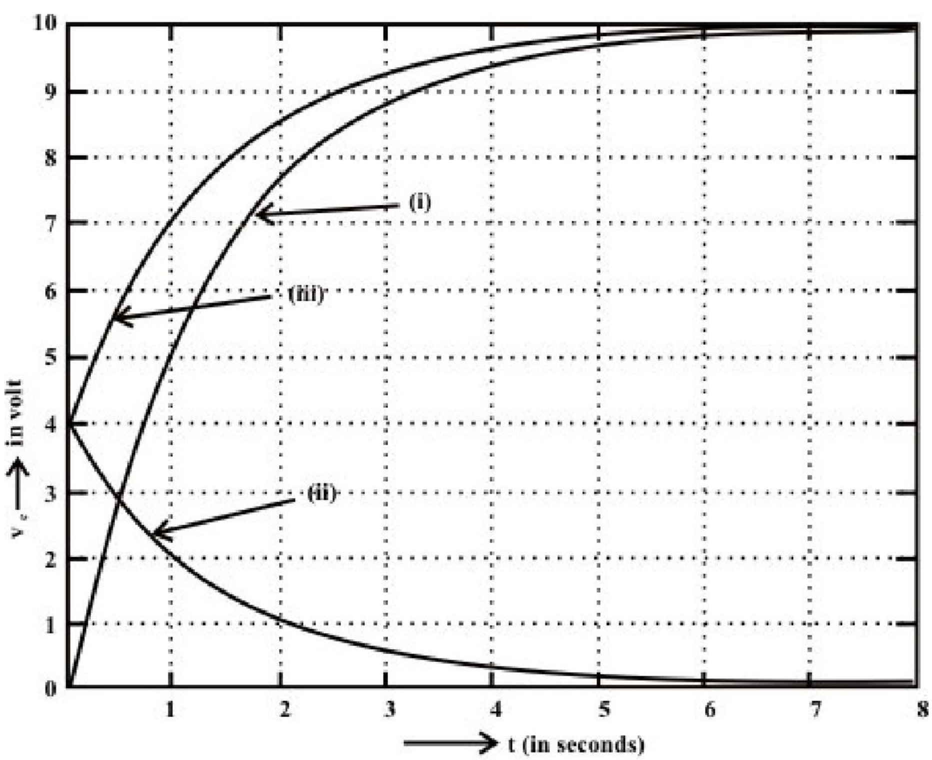


Fig. 10.18: Voltage across the capacitor due to
(i) the forcing function V_s acting alone
(ii) discharge of capacitor initial voltage v_0
(iii) Combine effect of (i) and (ii)

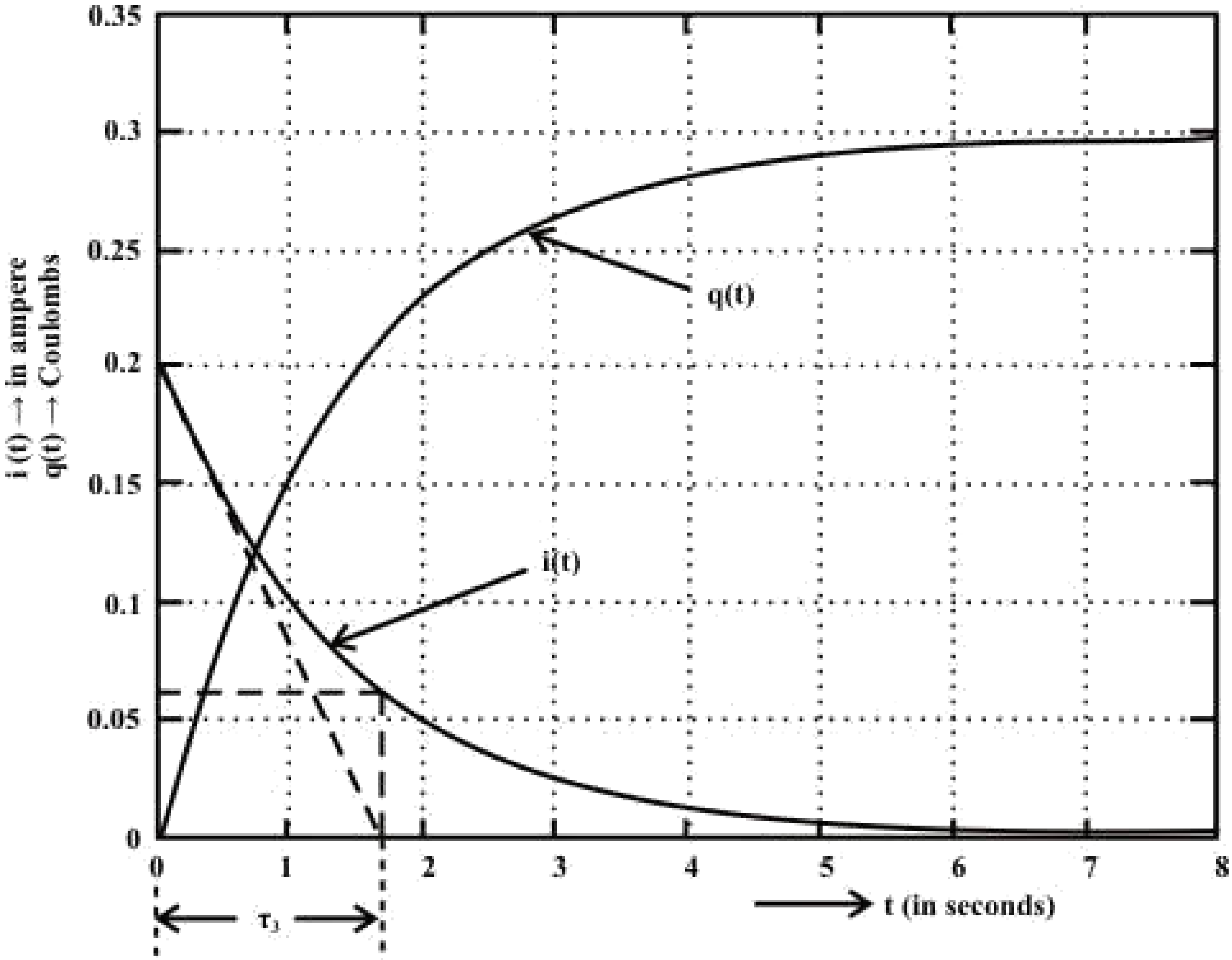


Fig. 10.19(b): System response due to the forcing function V_s (assumed capacitor initial voltage $v_o = 0$)

Following the definition given in section L.10.3.1, time constant of each of the exponential expressions described in Eqs. 10.45 to 10.48 may be found as $\tau = RC$ (for RC circuit).

L.10.4.4 Discharging of a capacitor or Fall of a capacitor voltage in dc circuits

Fig. 10.17(b) shows that the switch 'S' is closed at position '1' for sufficiently long time and the circuit has reached in steady-state condition. At '$t = 0$' the switch' S' is opened and kept in position '2' and remains there. Our job is to find the expressions for (i) voltage across the capacitor (v_c) (ii) voltage across the resistance (v_R) (iii) current ($i(t)$) through the capacitor (discharging current) (iv) discharge of charge ($q(t)$) through the circuit.

Solution: For $t < 0$, the switch ' S ' in position 1. The capacitor acts like an open circuit to dc, but the voltage across the capacitor is same as the supply voltage V_S . Since, the capacitor voltage cannot change instantaneously, this implies that

$v_c(0^-) = v_c(0^+) = V_S$

When the switch is closed in position ' 2 ', the current $i(t)$ will flow through the circuit until capacitor is completely discharged through the resistance R . In other words, the discharging cycle will start at $t = 0$. Now applying KVL around the loop, we get

$$R\,C\frac{d\,v_c(t)}{dt} + v_c(t) = 0 \qquad (10.49)$$

The solution of input free differential equation (10.49) is given by

$$v_c(t) = A_1\,e^{\alpha t} \qquad (10.50)$$

where the value of α is obtained from the characteristic equation and it is equal to $\alpha = -\frac{1}{RC}$. The constant A_1 is obtained using the initial condition of the circuit in Eq.(10.50). Note, at ' $t = 0$ '(when the switch is just closed in position ' 2 ') the voltage across the capacitor $v_c(t) = V_S$. Using this condition in Eq.(10.50), we get

$$v_c(0) = V_S = A_1\,e^{-\frac{1}{RC}\cdot 0} \qquad A_1 = V_s$$

Now the following expressions are written as

Voltage across the capacitance $v_c(t) = V_S\,e^{-\frac{1}{RC}t}$ (10.51)

Voltage across the resistance $v_R(t) = -v_c(t) = -V_S\,e^{-\frac{1}{RC}t}$ (10.52)

Charging current through the capacitor $i(t) = \frac{v_R}{R} = -\frac{V_S}{R}e^{-\frac{1}{RC}t}$ (10.53)

An inspection of the above exponential terms of equations from (10.51) to (10.53) reveals that the time constant of RC circuit is given by

$$\tau = RC \text{ (sec.)}$$

This means that at time $t = \tau$, the capacitor's voltage v_c drops to 36.8% of its initial value (see fig. 10.20(a)). For all practical purposes, the dc transient is considered to end after a time span of 5τ. At such time steady state condition is said to be reached. Plots of above equations as a function of time are depicted in fig. 10.20(a) and fig. 10.20(b) respectively.

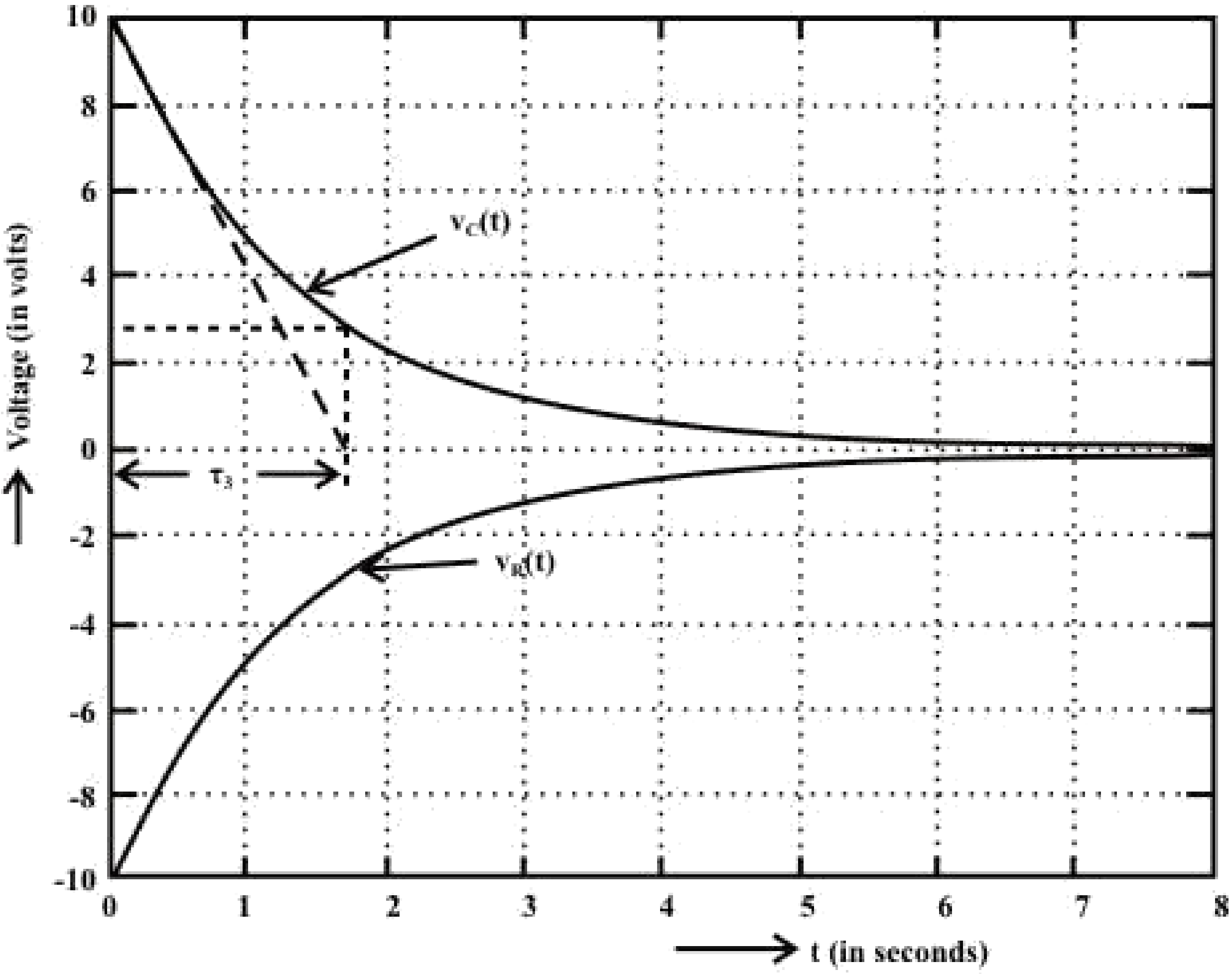


Fig. 10.20(a): Discharge of capacitor voltage with time in R-C circuit

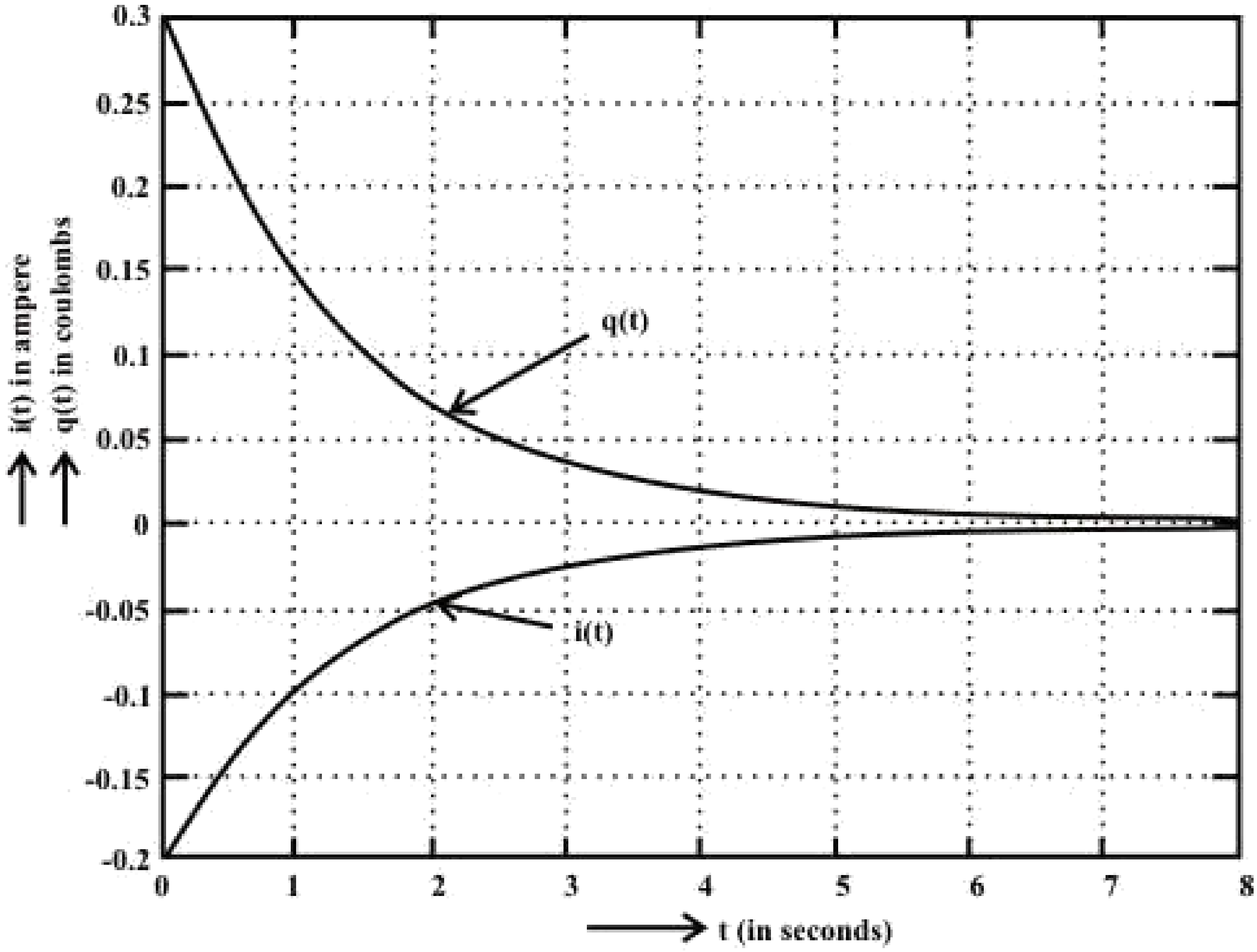


Fig. 10.20(b): System response due to capacitor discharge

L.10.5 Energy stored in a capacitor

The ideal capacitor does not dissipate any of the energy supplied by the source. It stores energy in the form of an electric field between the conducting plates. Let us consider a voltage source V_S is connected to a series $R-C$ circuit and it is assumed that the capacitor is initially uncharged. The capacitor voltage $(v_c(t))$ and current $(i_c(t))$ waveforms during the charging period are shown in fig.10.21 (see the expressions (10.45) and (10.47)) and instantaneous power ($p_c(t) = v_c(t) \cdot i(t)$) supplied to the capacitor is also shown in the same figure.

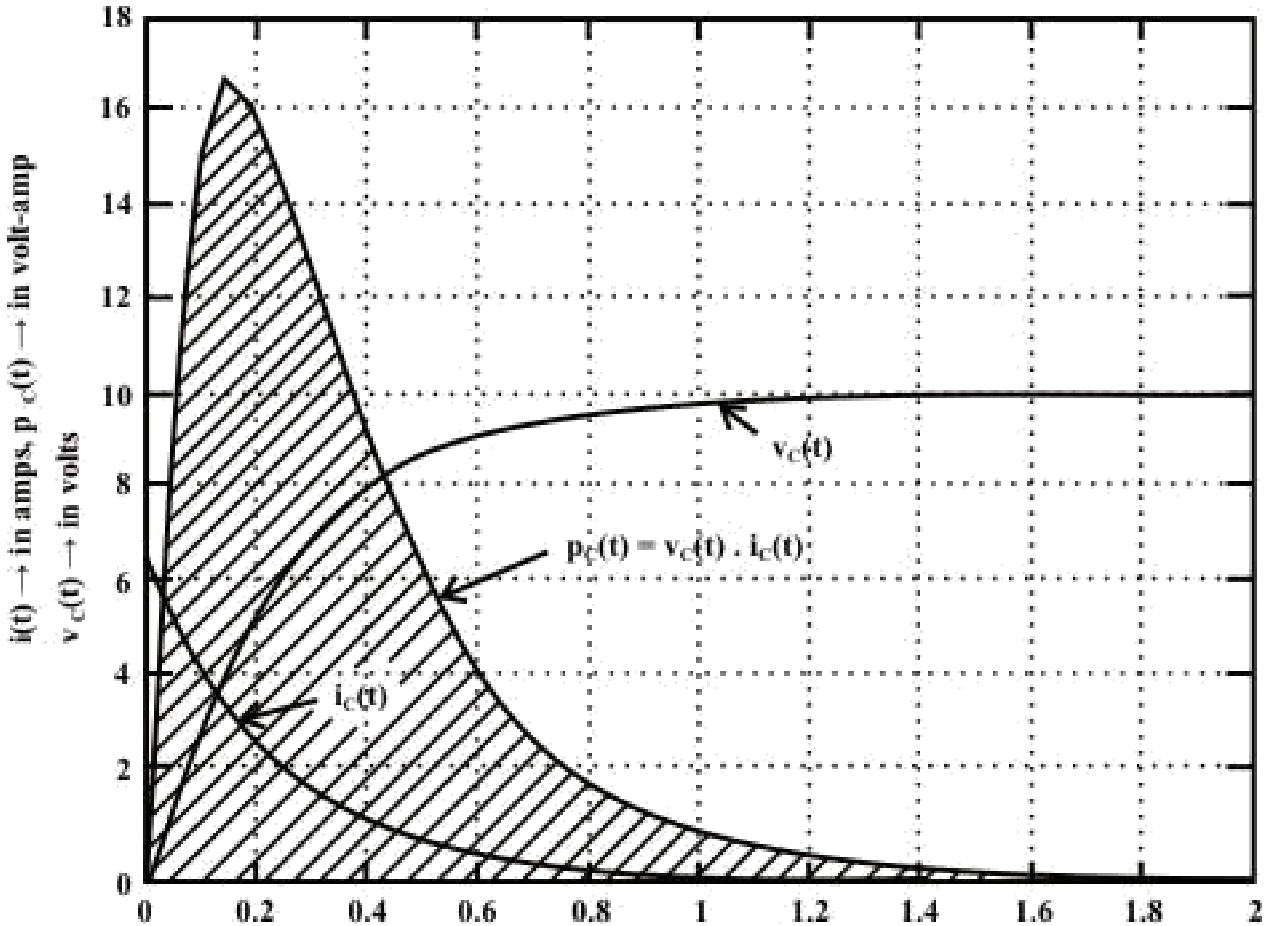


Fig. 10.21: Instantaneous power delivered to the capacitor

Let us consider the instantaneous power supplied to the capacitor is given by

$p_c(t) = v_c(t) \cdot i(t)$ (10.54) Now, the energy supplied to the capacitor in dt second is given by

$$w = p_c(t) \cdot dt = v_c(t)\, C\frac{d\, v_c(t)}{dt} \cdot dt = C\, v_c(t)\, dv_c(t) \qquad (10.55)$$

Total energy supplied to the capacitor in t seconds is expressed as

$$w(t) = C \int_{v_c(0)=0}^{v_c(t)=v} v_c(t)\, dv_c(t) = \frac{1}{2} C\, v_c^2 = \frac{1}{2}\frac{q^2(t)}{C} \; (Joules) \qquad (10.56)$$

(Note initial voltage across capacitor is zero and $q(t)$ is the charge accumulated on each plate at a time t).

When the capacitor is fully charged, its terminal voltage is equal to the source voltage V_S. The amount of energy stored in capacitor in the form of electric field is given by

$$W = \frac{1}{2} C\, V_S^2 = \frac{1}{2}\frac{Q^2}{C} (Joules) \qquad (10.57)$$

where Q is the final charge accumulated on each plate of the capacitor at steady state (i.e., $t \rightarrow \infty$) i.e., when the capacitor is fully charged.

Example: L.10.6 The switch ' S ' shown in fig.L.10.22 is kept open for a long time and then it is closed at time ' $t = 0$ '. Find (i) $v_c(0^-)$ (ii) $v_c(0^+)$ (iii) $i_c(0^-)$ (iv) $i_c(0^+)$ (v) $\left.\frac{dv_c(t)}{dy}\right|_{t=0^+}$ (vi) find the time constants of the circuit before and after the switch is closed (iv) $v_c(\infty)$

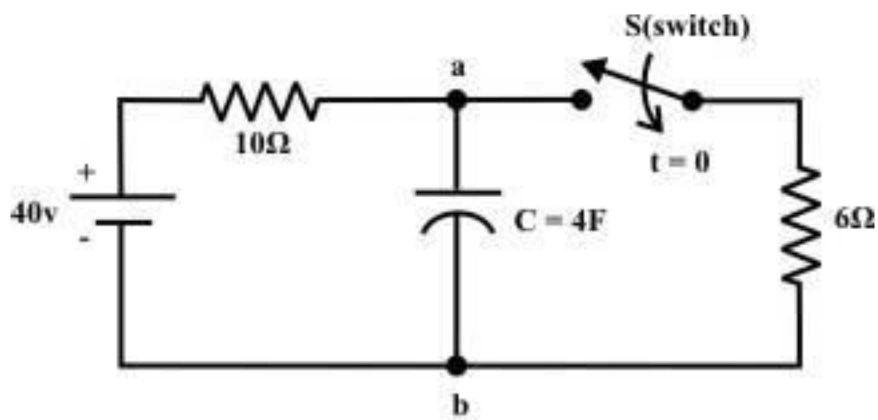


Fig. 10.22

Solution: As we know the voltage across the capacitor $v_c(t)$ cannot change instantaneously due to the principle of conservation of charge. Therefore, the voltage across the capacitor just before the switch is closed $v_c(0^-)$ = voltage across the capacitor just after the switch is closed $v_c(0^+) = 40\ V$ (note the terminal ' a ' is positively charged. It may be noted that the capacitor current before the switch ' S ' is closed is $i_c(0^-) = 0\ A$. On the other hand, at $t = 0$, the current through 10 Ω resistor is zero but the current through capacitor can be computed as

$i_c(0^+) = \frac{v_c(0)}{6} = \frac{40}{6} = 6.66\ A$ (note, voltage across the capacitor cannot change instantaneously at instant of switching). The rate of change of capacitor voltage at time ' $t = 0$ ' is expressed as

$$C \left.\frac{dv_c(t)}{dt}\right|_{t=0^+} = i_c(0) \qquad \frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{6.66}{4} = 1.665\ volt/\sec.$$

Time constant of the circuit before the switch was closed $= \tau = R\,C = 10 \cdot 4 = 40$ sec. Time constant of the circuit after the switch is closed is $\tau = R_{Th}\,C = \frac{10 \cdot 6}{6} \cdot 4 = 15$sec. (replace 10 + the part of the circuit than contains only independent sources and resistive elements by an equivalent Thevenin's voltage source. In this case, we need only to find the Thevenin resistance R_{Th}).

Note: When the switch is kept in closed position, initially the capacitor will be in discharge state and subsequently its voltage will decrease with the increase in time. Finally, at steady state the capacitor is charged with a voltage

$v_c(t=\infty) = \frac{40}{10+6} \cdot 6 = 15\ V$ (theoretically, time required to reach the capacitor voltage at steady value is $5\,\tau = 5 \cdot 15 = 75$ sec.).

Example: L.10.7 The circuit shown in Fig.10.23 has been established for a long time. The switch is closed at time $t = 0$. Find the current (i) $i_1(0^+), i_2(0^+), i_3(0^+)$, *and* $\left.\frac{dv_{de}}{dt}\right|_{t=0^+}$ (ii) at steady state the voltage across the capacitors, $i_1(\infty), i_2(\infty)$ *and* $i_3(\infty)$.

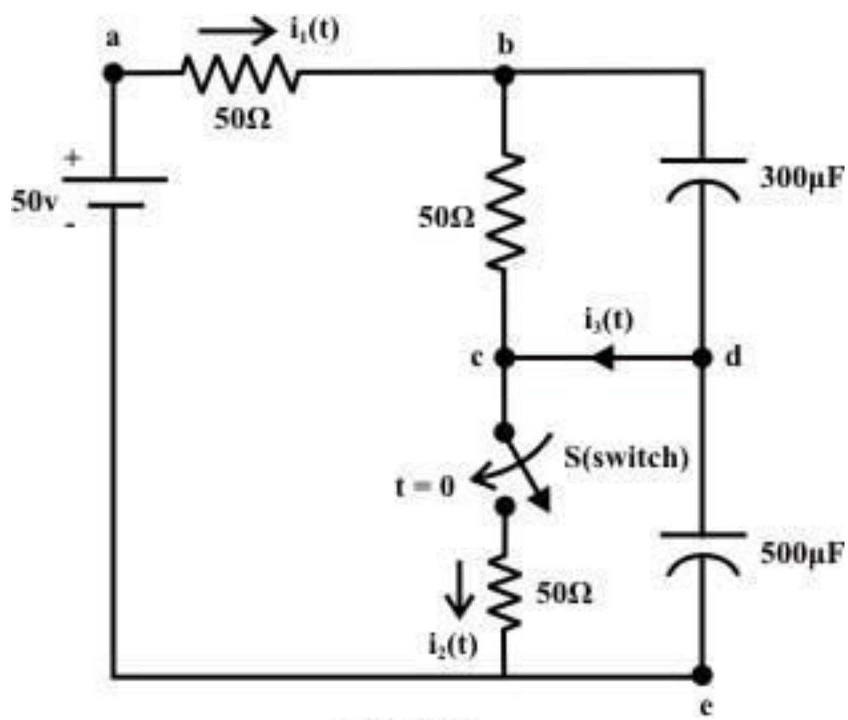


Fig. 10.23

Solution: (i) At $t = 0^-$ no current flowing through the circuit, so the voltage at points ' b ' and ' d ' are both equal to 50 *volt*. When the switch ' S ' closes the capacitor voltage remains constant and does not change its voltage instantaneously. The current $i_1(0^+)$ through $a - b$ branch must then equal to zero, since voltage at terminal ' b ' is equal to $v_b(0^+) = 50$ *volt*. , current through $b - c$ is also zero. One can immediately find out the current through $c-e$ equal to $i_2(0^+) = \frac{50}{50} = 1\,A$. Appling KCL at point ' c ', $i_3(0^+) = 1\,A$ which is the only current flow at $t = 0^+$ around the loop ' $d-c-e-d$. Note the capacitor across ' $d-e$ ' branch acts as a voltage source, the change of capacitor voltage $\left.\frac{dv_{de}}{dt}\right|_{t=0^+} = \frac{1}{500\cdot 10^{-6}} i_3(0^+) = 2\,k\,volt/\sec.$

(ii) at steady state the voltage across each capacitor is given

$= \frac{50}{150} \times 50 = 16.666\,volt.$

At steady state current delivered by the source to the different branches are given by

$i_1(\infty) = \frac{50}{150} = 0.333\,A;\ i_2(\infty) = 0.333\,A\ and\ i_3(\infty) = 0A$

Example: L.10.8 The circuit shown in fig. 10.24(a) is switched on at time $t = 0$. How long it takes for the capacitor to attain 70 % of its final voltage? Assume the capacitor is initially not charged. Find also the time constant (τ) of the circuit after the switch is closed.

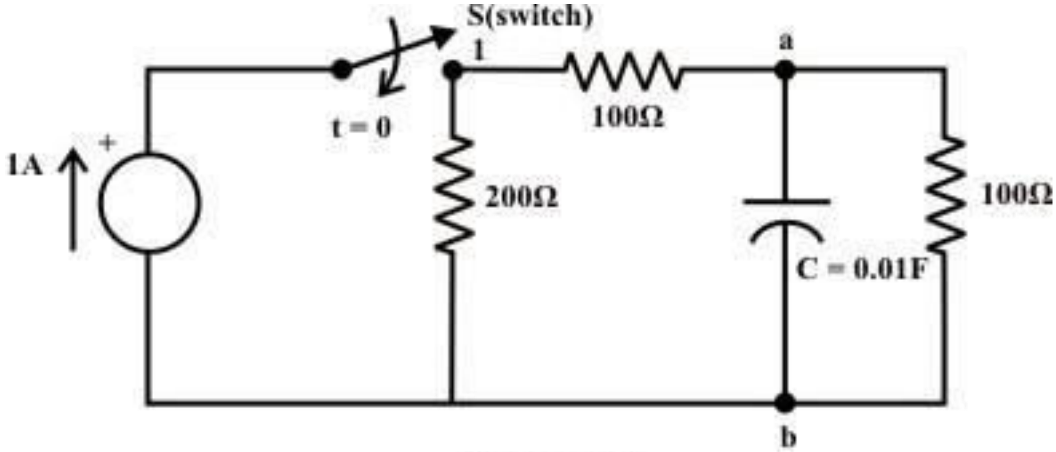


Fig. 10.24(a)

The circuit containing only resistive elements and independent current source (i.e., non-transient part of the circuit) is converted to an equivalent voltage source which is shown in fig.10.24(b).

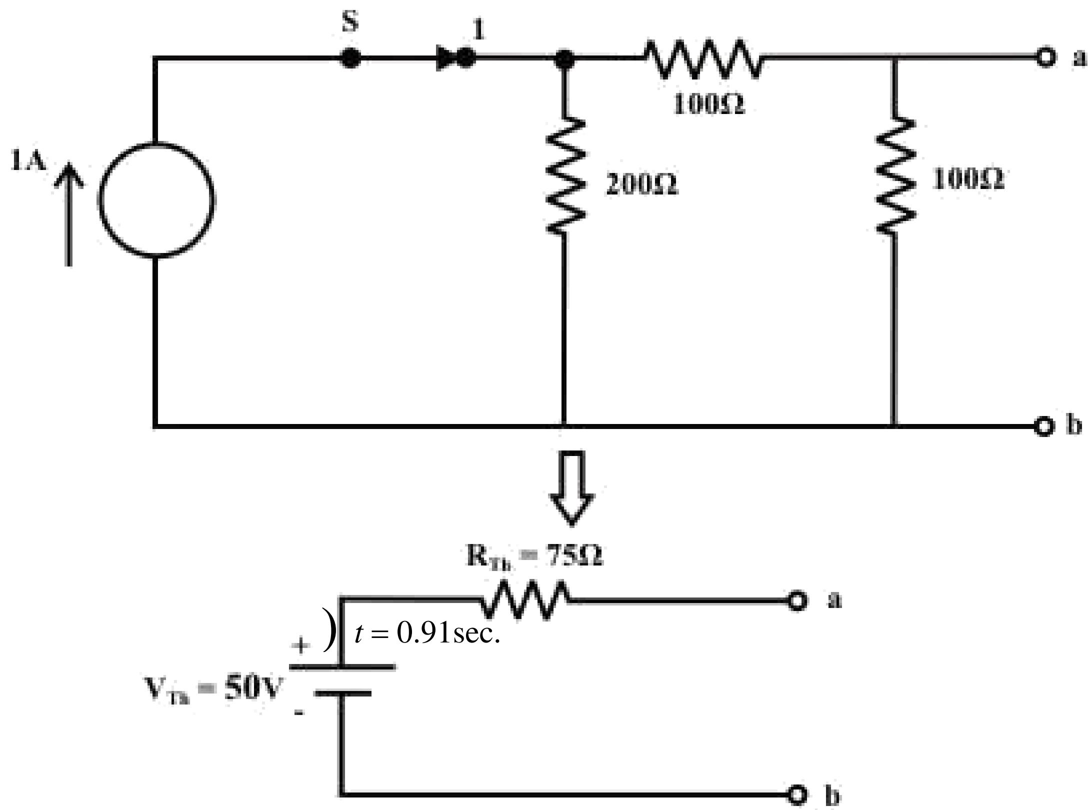


Fig. 10.24(b)

Fig.10.24(c) shows the capacitor C is connected across the Thevenin's voltage terminals ' a ' and ' b ' in series with Thevenin's resistance R_{Th} .

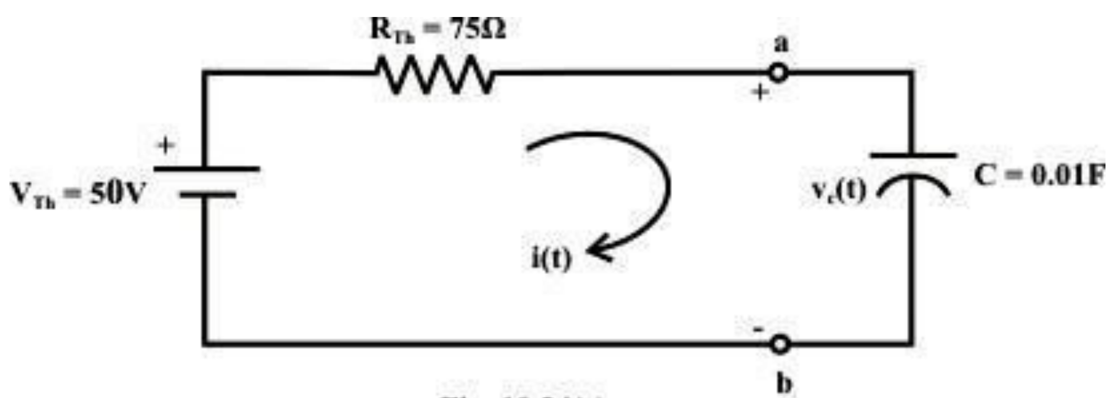


Fig. 10.24(c)

The parameters of Thevenin's voltage source are computed below:

$$V_{Th} = \frac{200}{200+100+100} \cdot 1 \cdot 100 = 50V \quad \text{and} \quad R_{Th} = \frac{100 \cdot 300}{100+300} = 75\Omega$$

Using KVL around the closed path, one can find the current through the capacitor and hence, the voltage across the capacitor.

$$50 = 75 \cdot i(t) + v_c(t) = 0.75\, \underline{dv_c(t)} + v_c(t) \qquad (10.58)$$

dt

The solution of the differential equation is given by

$$v_c(t) = A_1 e^{-\frac{1}{RC}t} + A \qquad (10.59)$$

Using the initial and boundary conditions of the circuit, we obtain the final expression of voltage across the capacitor $v_c(t)$ as

$$v_c(t) = 50\left(1 - e^{-1.33t}\right) \qquad (10.60)$$

Let ' t ' is the time required to reach the capacitor voltage 70% of its final (i.e., steady state) voltage.

$$50 \cdot 0.7 = 35 = 50\left(1 - e^{-1.33t}\right.$$

Example: L.10.9 The switch ' S of the circuit shown in fig.10.25(a) is closed at position '1' at $t = 0$.

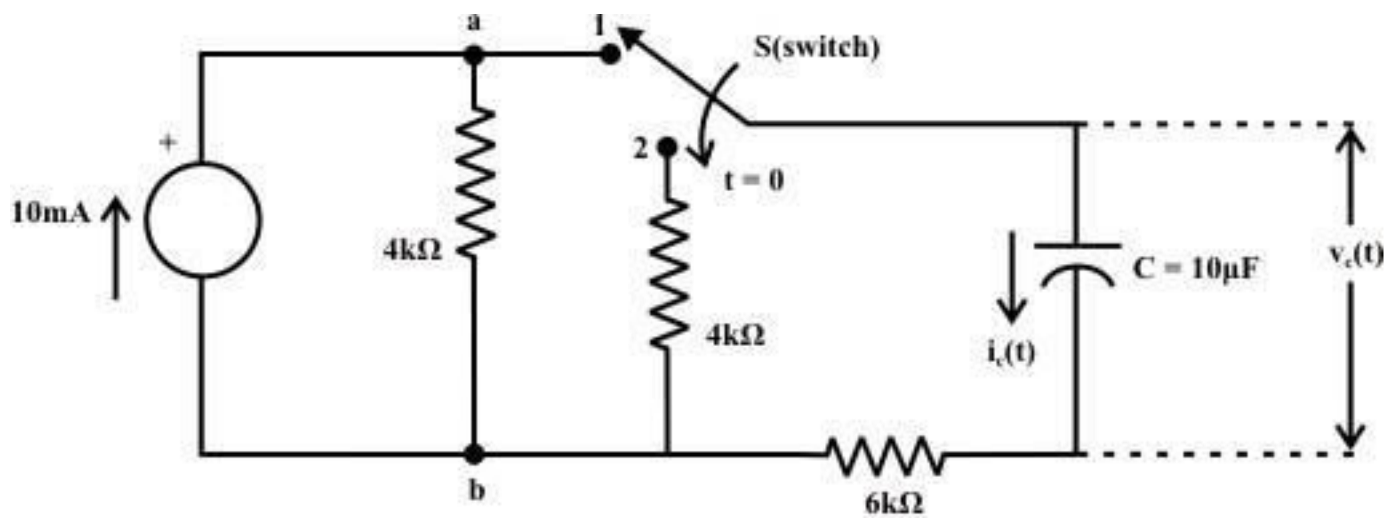


Fig. 10.25(a)

Find voltage $v_c(t)$ and current $i_c(t)$ expressions for $t \geq 0$. Assume that the capacitor is initially fully uncharged (i.e., . $v_c(0) = 0$).

(i) find the mathematical expressions for $v_c(t)$ and $i_c(t)$ if the switch ' S ' is thrown into position ' 2 ' at $t = \tau$ (sec.) of the charging phase.

(ii) plot the waveforms obtained in parts (i) to (ii) on the same time axis for the voltage $v_c(t)$ and the current $i_c(t)$ using the identified polarity of voltage and current directions.

Solution: (i) The current source is converted to an equivalent voltage source and it is redrawn in fig.10.25(b) when the switch 'S' is in position '1'.

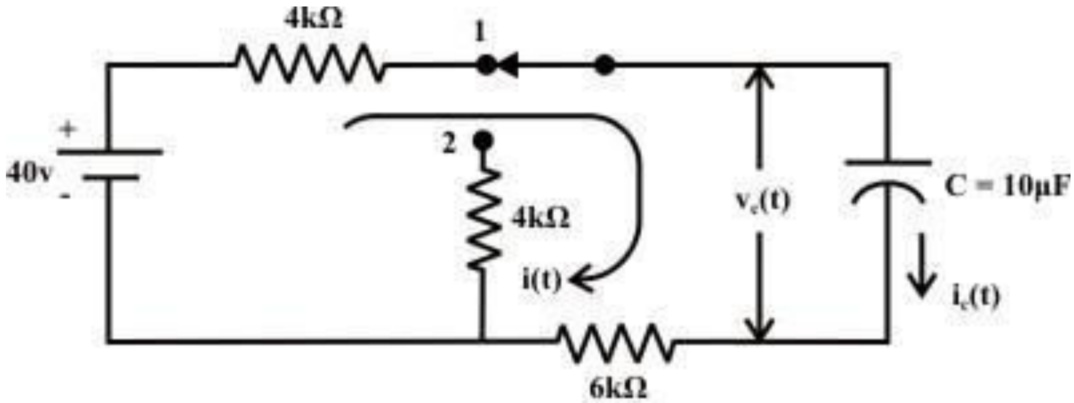


Fig. 10.25(b)

KVL around the closed path:

$$40 = 10 \cdot i(t) + v_c(t), \quad where \quad i(t) \text{ is in } mA.$$

$$40 = 10 \cdot C \frac{dv_c(t)}{dt} + v_c(t) \tag{10.61}$$

The voltage expression across the capacitor using the initial and boundary conditions of the circuit, one can write $v_c(t)$ as

$$v_c(t) = 40\left(1 - e^{-\frac{1}{RC}t}\right) = 40\left(1 - e^{-\frac{1}{10\cdot 10^{3}\cdot 10\cdot 10^{-6}}t}\right) = 40\left(1 - e^{-10t}\right) \tag{10.62}$$

$$i_c(t) = \frac{40 - v_c(t)}{10} = \frac{40e^{-10t}}{10} = 4 \cdot e^{-10t} \ (in\ mA) \tag{10.63}$$

Note that the time constant of the circuit in part (i) is $\tau = RC = 100$ $m$sec.

(ii) The switch 'S' is thrown into position '2' at $t = \tau = 0.1$sec. and the corresponding circuit diagram is shown in fig.10.25(c).

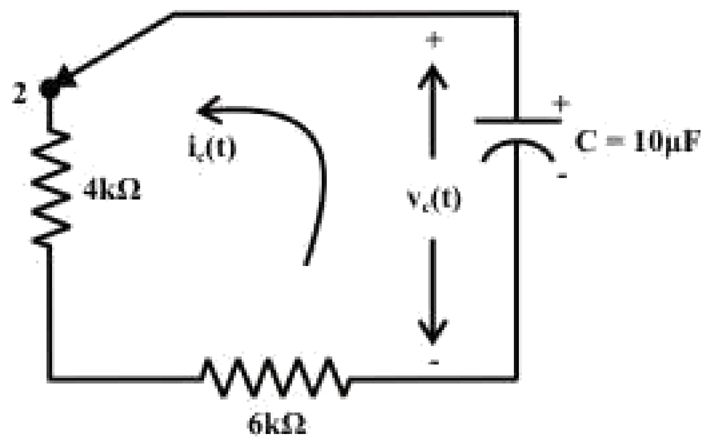


Fig. 10.25(c)

Note, at time $t = \tau = 0.1$sec., the capacitor is charged with a voltage $= v_c(\tau = 0.1) = 40\left(1 - e^{-10\cdot 0.1}\right) = 40 \cdot 0.632 = 25.28V$ and at the same time ($t = \tau = 0.1$sec.) the current in capacitor is $4\cdot e^{-10t} = 4 \cdot 0.368 = 1.472\ (in\ mA)$. Considering the fig.10.25(c), one can write KVL around the closed path

$$v_c(t) + C\frac{dv_c(t)}{dt}\cdot R_{eq} = 0 \qquad (10.64)$$

where $R_{eq} = 4 + 6 = 10\ k\Omega$ and the capacitor is now in discharging phase.
The solution of Eq.(10.64) can be found using the initial and final voltage of the capacitor (initial voltage $v_c(t = \tau = 0.1) = 25.28\ V$, $v_c(t - \tau = \infty) = 0\ V$) and it is given by

$$v_c(t) = v_c(\tau = 0.1)\cdot e^{-\frac{1}{R_{eq}C}(t-\tau)} = 25.28\cdot e^{-10(t-\tau)} \qquad (10.65)$$

Discharging current expression is given by (note, current direction is just opposite to the assigned direction and it is taken into account with a –ve sign)

$$i_c(t) = -\frac{v_c(t)}{R_{eq}} = -\frac{25.28\cdot e^{-10(t-\tau)}}{10} = -2.528\cdot e^{-10(t-\tau)}\ (in\ mA) \qquad (10.66)$$

(Note, the above two expressions are valid only for $t \geq \tau$)
The circuit responses for charging and discharging phases in (i) and (ii) are shown in fig.10.25 (d).

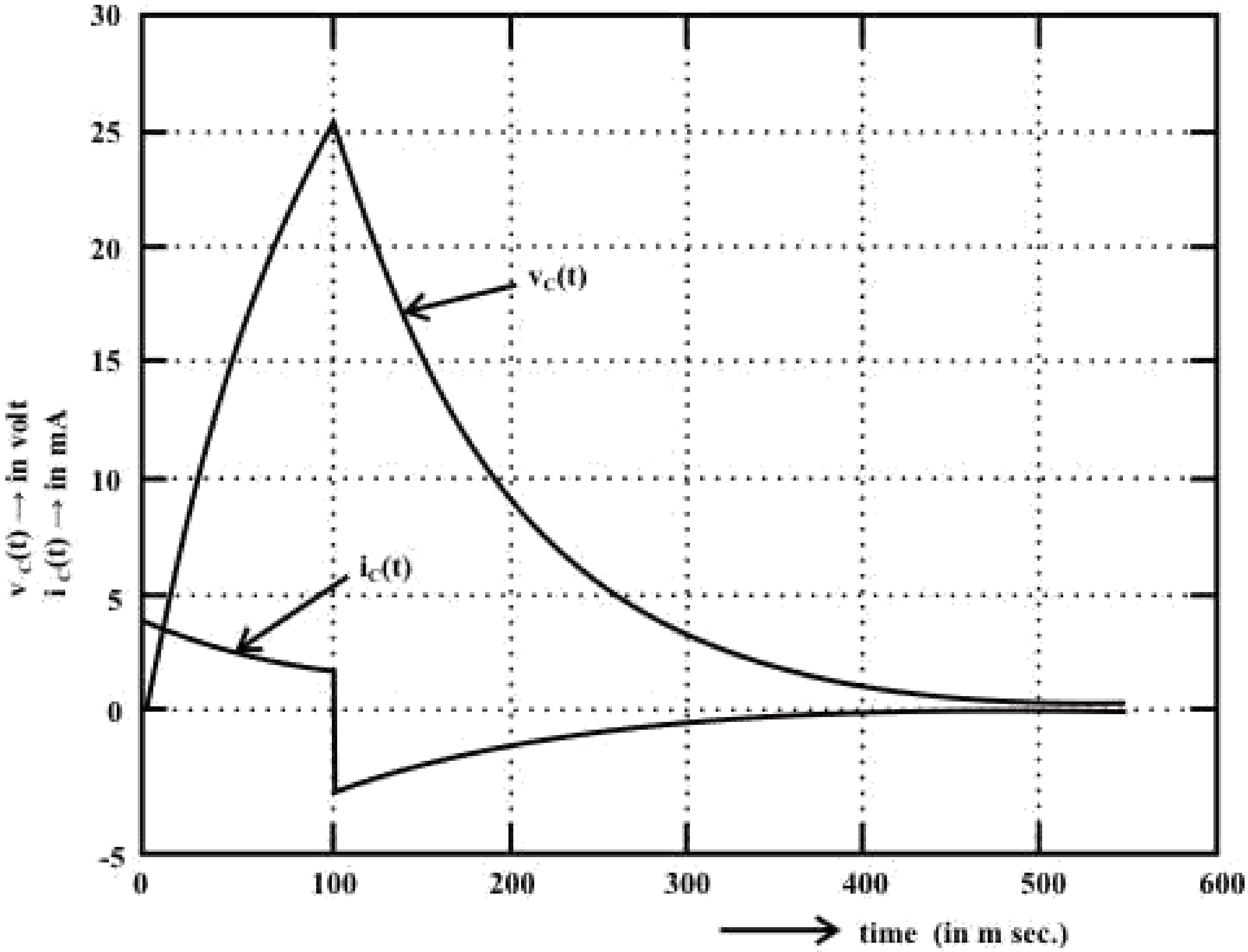


Fig. 10.25(d): Complete response of the circuit indicating charging and discharging phases.

Remark-6 Note that the current through the capacitor (see fig. 10.25(d)) can change instantaneously like the voltage across the inductor.

Appendix-A

L.10.A Solution of nth order linear time invariant differential equation excited by forcing function.

Let us consider a linear time invariant circuit having several energy source elements is described by the following dynamic equation.

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + a_{n-2} \frac{d^{n-2} x}{dt^{n-2}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = f(t) \qquad (10.A1)$$

where $a_1, a_2, a_3, \cdots a_{n-1}, a_n$ are constant coefficients associated in the differential equation and they are dependent on circuit parameters (like, R, L, *and* C for electric circuit) but independent of time, $f(t)$ is the forcing or driving function and $x(t)$ is the solution of differential equation or response of the system. We shall discussion the solution of differential equation restricted to second order differential, say $n = 2$ in equation (10.A1).

$$a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = f(t) \tag{10.A2}$$

The solution of this differential equation provides the response of circuit and it is given by

$$x(t) = x_n(t) + x_f(t) \tag{10.A3}$$

where $x_n(t)$ is the natural response of circuit, obtained by setting $f(t) = 0$, and $x_f(t)$ is the forced response that satisfies the original differential equation (10.A2).

By setting $f(t) = 0$ in equation (10.A2), as given in equation (10.A4), the force free equation is obtained.

$$a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0 \qquad \text{(Homogeneous equation)} \tag{10.A4}$$

The solution of such differential equation (or homogeneous equation) is known as natural solution or complementary solution or transient solution and it is denoted by $x_n(t)$. To get the natural solution $x_n(t)$ of equation (10.A4) the following steps are considered.

Let us use the following operators

$$\frac{d}{dt} = \alpha; \quad \frac{d^2}{dt^2} = \frac{d}{dt}\frac{d}{dt} = \alpha^2$$

in equation (10.A4) and results an equation given by

$$\left(a_2 \alpha^2 + a_1 \alpha + a_0\right) x = 0$$

Since $x \neq 0$, the above equation can be written as

$$\left(a_2 \alpha^2 + a_1 \alpha + a_0\right) = 0 \tag{10.A5}$$

which is known as characteristic equation for a circuit whose force free equation is Eq.(10.A4). The natural or transient solution of Eq.(10.A4) is expressed by the exponential terms as given below.

$$x_n(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} \tag{10.A6}$$

where α_1 *and* α_2 are the roots of characteristic equation (10.A5). The roots of second order characteristic equation with real coefficients is either real or complex occur in conjugate pairs. The constants A_1 *and* A_2 are evaluated from initial or boundary conditions of circuit. The principles of continuity of inductance current and capacitance voltage are used to establish the required boundary conditions.

If $x_n(t)$ is the natural or transient solution of unforced (or homogeneous) equation differential, it must satisfy its own differential equation

$$a_2 \frac{d^2 x_n}{dt^2} + a_1 \frac{dx_n}{dt} + a_0 x_n = 0 \tag{10.A7}$$

Further, if $x(t) = x_n(t) + x_f(t)$ is the complete solution of given differential Eq.(10.A2), it must satisfy its own equation

$$a_2 \frac{d^2 \left(x_n + x_f\right)}{dt^2} + a_1 \frac{d\left(x_n + x_f\right)}{dt} + a_0 \left(x_n + x_f\right) = f(t) \tag{10.A8}$$

Using the equation (10.A7) in Eq.(10.A8), we get

$$a_2 \frac{d^2 x_f}{dt^2} + a_1 \frac{dx_f}{dt} + a_0 x_f = f(t) \tag{10.A9}$$

The above equation implies that $x_f(t)$ is the forced solution or steady state solution of second order differential equation (10.A2). Steady state solution of some common forcing functions is listed in Table (assume $a_2 > 0$, $a_1 > 0$ *and* $a_0 > 0$).

Table: Steady state solution $x_f(t)$ for any order differential equation excited by some common forcing function.

Type of forcing function $f(t)$ (input)	Steady state solution $x_f(t)$ (output)
□ $f(t) = K$ (constant)	□ $x_f(t) = A$ (constant)
□ $f(t) = K\,t$	□ $x_f(t) = At + B$
□ $f(t) = K\,t^2$	□ $x_f(t) = At^2 + B\,t + C$
□ $f(t) = K\,e^{a\,t}$	□ $x_f(t) = Ae^{at}$
□ $f(t) = \sin b\,t$	□ $i_f(t) = A \sin b\,t + B \cos b\,t$
□ $f(t) = cos\,b\,t$	□ $i_f(t) = A \sin b\,t + B \cos b\,t$
□ $f(t) = e^{a\,t} \sin b\,t$	□ $i_f(t) = e^{at}\left(A \sin b\,t + B \cos b\,t\right)$
□ $f(t) = e^{a\,t} \cos b\,t$	□ $i_f(t) = e^{at}\left(A \sin b\,t + B \cos b\,t\right)$

Coefficients involve in the steady state solution can be found out by using the boundary conditions of the circuit.

Remark-7

(i) Eq. (10.A2) is the differential equation description of a linear circuit, superposition may be used to find the complete solution of a forcing function which is sum of natural and steady state responses. (ii) Eq.(10.A6) is the natural solution of force-free linear differential equation. Note that the constants α_1 *and* α_2 are the roots of the characteristic equation (10.A5) and they are entirely depending on the circuit parameters. The roots of the characteristic equation may be classified as

Case-1: Real or Complex but distinct

The natural solution of homogeneous equation (10.A4) is given as

$x_n(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$

Case-2: Roots are repeated (i.e. $\alpha_1 = \alpha_2 = \alpha$ or multiplicity of roots of order 2)
The natural solution of homogeneous equation (10.A4) is given as

$x_n(t) = \beta_0 e^{\alpha t} + \beta_1 t$

Using initial and final conditions of the circuit, β_0 *and* β_1 constants are computed. More discussions on these issues can be seen in Lesson-11.

L.10.6 Test your understanding **(Marks: 70)**

T.10.1 Inductor tends to block ----------- current but pass ---------- current.

T.10.2 The basic fundamental principle that explains the action of an inductor is known as ------------- law.

T.10.3 Exponential waveforms start ------ and finish -------------- .

T.10.4 A transient approximately always has a duration of -------- time constants.

T.10.5 After the first time constant, a transient goes through ---------- % of its steady state value.

T.10.6 -------- through inductor cannot change --------- but -------- across the inductor can --------- instantaneously at the switching phase.

T.10.7 A simple series $R - L$ circuit is excited with a constant voltage source, the speed of response depends on ---------- and ------- of the circuit.

T.10.8 The energy stored in an inductor in the form of -------------- .

T.10.9 In a first order circuit if the resistor value is doubled, the time constant is halved for an -------- circuit.

T.10.10 An inductor acts as ----------- for a ---------- current through it.

T.10.11 Once a capacitor has been charged up, it is able to act like a -------- .

T.10.12 If the spacing between the plates is doubled, the capacitance value is ----------.

T.10.13 After a capacitor is fully charged in a dc circuit, it --------- dc current.

T.10.14 The time rate of change of capacitor voltage is represented by the ------- tangent line to the $v_c(t)$ -versus- t curve.

T.10.15 Immediately after a switch has been thrown, a capacitor's --------- must maintain the same value that excited just before the switching instant.

T.10.16 At the instant of switching, current through the capacitor ------------- instantaneously.

T.10.17 At steady state condition in a dc circuit, the capacitor acts as an ----- circuit.

T.10.18 A first order circuit with a single resistor, if the resistor is doubled in value, the time constant is also ----- for an $R - C$ circuit.

T.10.19 Time constant of a first order system is the measure of ---------- response of the circuit.

T.10.20 The energy stored in a capacitor in the form of ------------- . [1· 20]

T.10.21 For the circuit of fig.10.26, find (i) $i_1(0^-), i_L(0^-)$ (ii) $i_1(0^+), i_L(0^+)$ (iii) $i_1(t=\infty), i_L(t=\infty)$ (iv) $v_{ab}(0^+), v_{ab}(t=\infty)$.

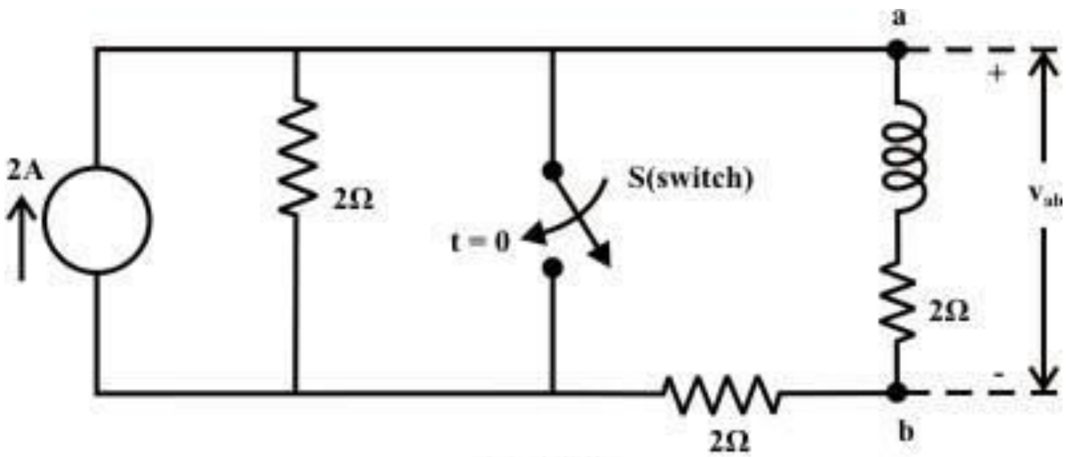


Fig. 10.26

(Ans. (i) 0 , 0.666 A (ii) 1.333 A, 0.666 A (iii) 2 A, 0 A (iv) −1.332 V, 0 V) [8]

T.10.22 For the circuit shown in Fig.10.12, the switch ‘ S ’ has been opened for a long time and then closes at t=0.

Find,
(i) $v_{ab}(0^-)$ (ii) $i_x(0^-)$ (iii) $i_x(0^+)$ (iv) $v_{ab}(0^+)$ (v) $i_x(t=\infty)$ (vi) $v_{ab}(t=\infty)$ (vii) $i_x(t)$ for $t > 0$

(Ans. [10]

T.10.23 In the circuit shown in fig.10.27, the switch was initially open and no current was flowing in inductor (L). The switch was closed at $t = 0$ and than re opened at $t = 2\tau$ sec. At $t = 0$, $\frac{d\, i_L(t)}{dt}$ was 50 A/s.

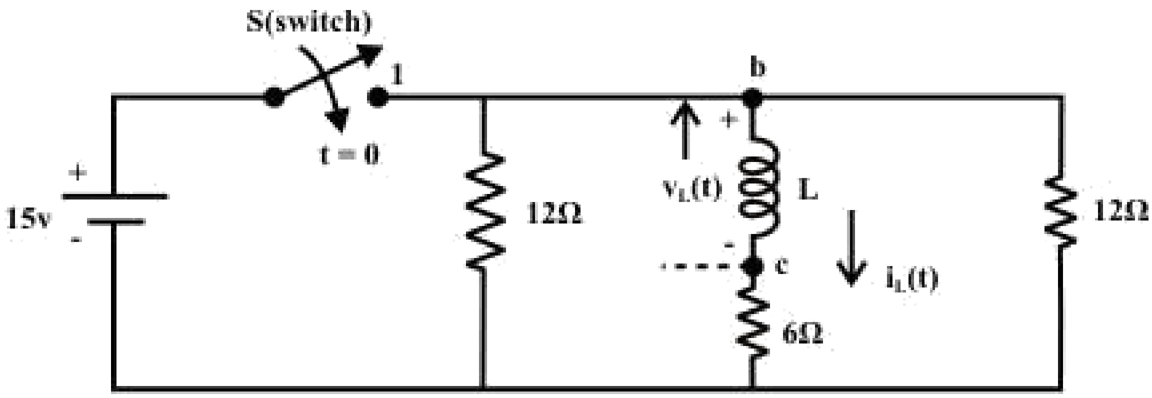


Fig. 10.27

Find,

(i) The value of L

(ii) Find the current $i_L(t)$ and voltage $v_{bc}(t)$ expressions for $t \geq 0$. Assume, no current was flowing through the inductor at $t = 0$ (i.e., . $i_L(0) = 0$).

(iii) Find the mathematical expressions for $i_L(t)$ and $v_{bc}(t)$ if the switch ' S ' is reopened at $t = 2\tau$ (sec.).

(iv) Plot the waveforms obtained in parts (ii) to (iii) on the same time axis (time → in ms.) for the current $i_L(t)$ and the voltage $v_{bc}(t)$ considering the indicated current directions and identified polarity of voltage across the $b - c$ terminals.

(Ans. (i) 0.3 H (ii) $i_L(t) = 1.25 \cdot \left(1 - e^{-40t}\right)$ *amp.*, $v_{bc}(t) = 15 \cdot e^{-40t}$
(iii) $i_L(t) = 1.081 \cdot e^{-40(t-\tau)}$, $v_{bc}(t) = 12.96 \cdot e^{-40(t-2\tau)}$, *for* $t \geq 2\tau$.) [10]

T.10.24 At steady state condition, find the values of I_1 , I_2 , I_3 , I_4 , I_5 , V_1 and V_2 for the circuit shown in fig.10.28.

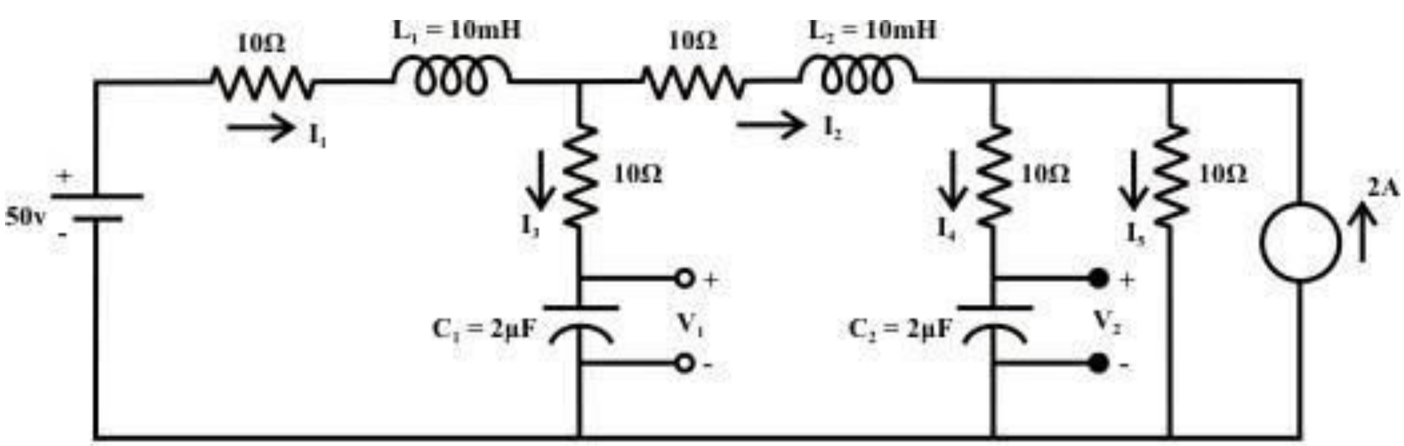


Fig. 10.28

(**Ans.** $I_1 = I_2 = I_5 = 1\ A, I_4 = I_3 = 0, V_1 = 40\ V \text{ and } V_2 = 30V$) [6]

T.10.25 Switch ' S ' shown in fig.10.29 is kept in position '1'for a long time.

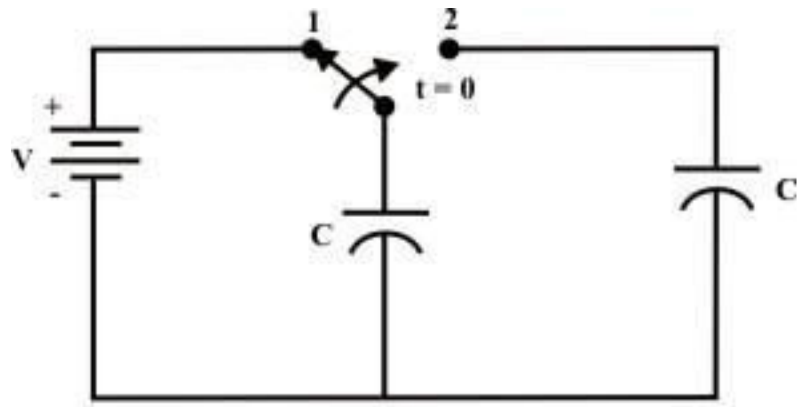


Fig. 10.29

When the switch is thrown in position ' 2 ', find at steady state condition
(i) the voltage across the each capacitor (ii) the charge across the each capacitor (iii) the energy stored by the each capacitor

(Ans. (i) $(i)\ \frac{V}{2}\quad (ii)\ C\frac{V}{2}\ (iii)\ C\frac{V^2}{8}$) [6]

T.10.26 For the circuit shown in fig.10.30, Switch ' S ' is kept in position '1 ' for a long time and then it is thrown in position ' 2 ' at time $t = 0$. Find (a) the current expression $i\,(t)$ for $t \geq 0$ (b) calculate the time constants of the circuit before and after the switching phases.

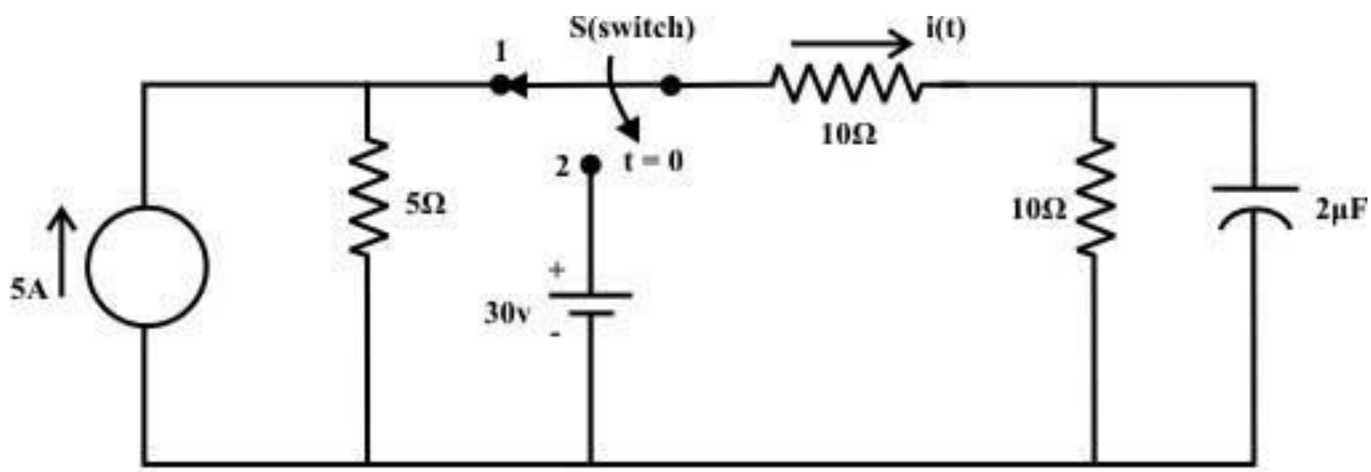


Fig. 10.30

(Ans. (a) $i\,(t) = 1.5 + 0.5e^{-10_5\,t}$ (b) 12 μs (before the switch is opened), (b) 10 μs (after the switch is opened, i.e., when the switch is in position ' 2 ')) [10]

Single-phase AC Circuits

Generation of Sinusoidal Voltage Waveform (AC) and Some Fundamental Concepts

In this lesson, firstly, how a sinusoidal waveform (ac) is generated, is described, and then the terms, such as average and effective (rms) values, related to periodic voltage or current waveforms, are explained. Lastly, some examples to find average and root mean square (rms) values of some periodic waveforms are presented.

Keywords: Sinusoidal waveforms, Generation, Average and RMS values of Waveforms.

After going through this lesson, the students will be able to answer the following questions:

1. What is an ac voltage waveform?
2. How a sinusoidal voltage waveform is generated, with some detail?
3. For periodic voltage or current waveforms, to compute or obtain the average and rms values, and also the time period.
4. To compare the different periodic waveforms, using above values.

Generation of Sinusoidal (AC) Voltage Waveform

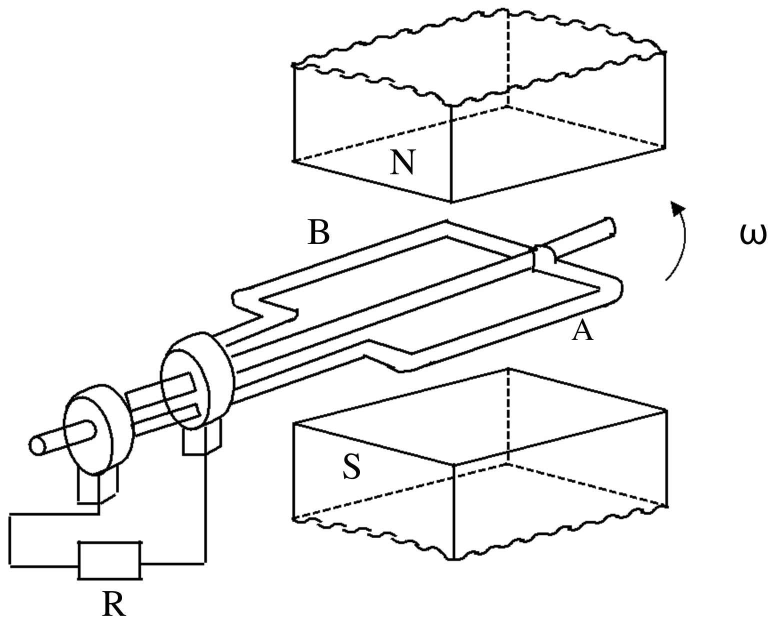


Fig. 12.1 Schematic diagram for single phase ac generation

A multi-turn coil is placed inside a magnet with an air gap as shown in Fig. 12.1. The flux lines are from North Pole to South Pole. The coil is rotated at an angular speed, $\omega = 2\pi n$ (rad/s).

$n = \frac{\omega}{2\pi}$ = speed of the coil (rev/sec, or rps)

$N = 60\,n$ = speed of the coil (rev/min, or rpm)
l = length of the coil (m)
b = width (diameter) of the coil (m)
T = No. of turns in the coil

B = flux density in the air gap (Wb / m^2)
$v = \pi b n$ = tangential velocity of the coil (m/sec)

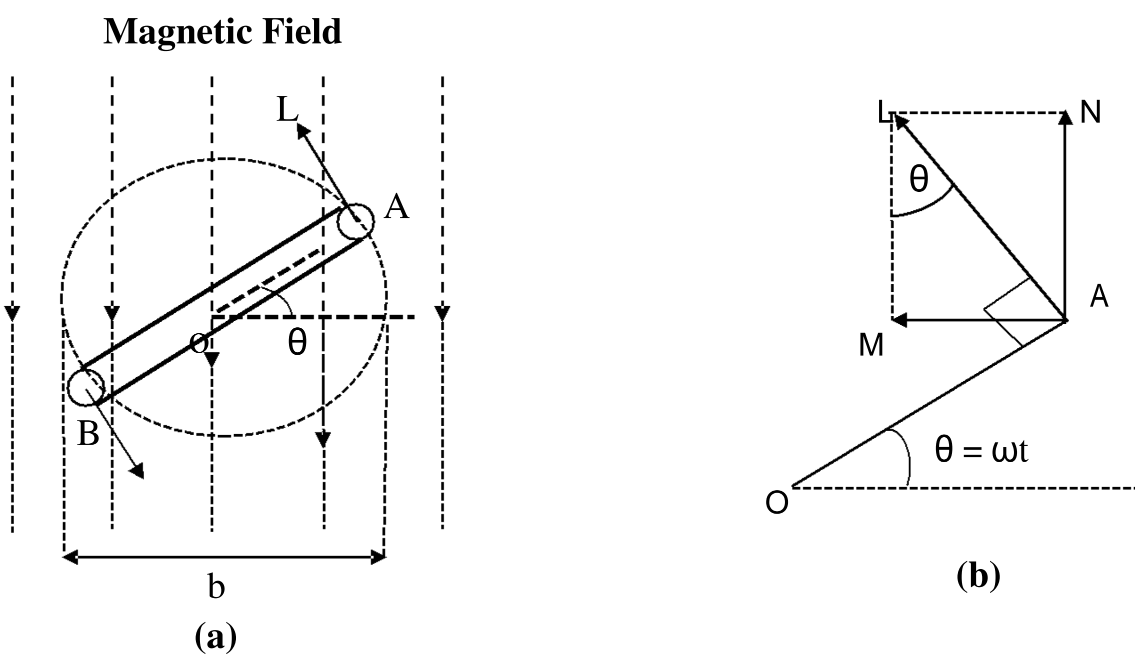


Fig. 12.2 (a) Coil position for Fig. 12.1, and (b) Details

At a certain instant t, the coil is an angle (rad), $\theta = \omega t$ with the horizontal (Fig. 12.2).
The emf (V) induced on one side of the coil (conductor) is $B\, l\, v \sin \theta$,
θ can also be termed as angular displacement.
The emf induced in the coil (single turn) is $2\, B\, l\, v \sin \theta = 2\, B\, l\, \pi\, b\, n \sin \theta$ The total emf induced or generated in the multi-turn coil is
$e(\theta) = T\, 2\, B\, l\, \pi\, b\, n \sin\theta = 2\pi B\, l\, b\, nT \sin\theta = E_m \sin\theta$

This emf as a function of time, can be expressed as, $e(t) = E_m \sin \omega t$. The graph of $e(t)$ or $e(\theta)$, which is a sinusoidal waveform, is shown in Fig. 12.4a

Area of the coil (m^2) $= a = l\, b$
Flux cut by the coil (Wb) = $\varphi = a\, B = l\, b\, B$
Flux linkage (Wb) = $\psi = T\, \varphi = T\, B\, l\, b$

It may be noted these values of flux φ and flux linkage ψ, are maximum, with the coil being at horizontal position, $\theta = 0$. These values change, as the coil moves from the horizontal position (Fig. 12.2). So, also is the value of induced emf as stated earlier.

The maximum value of the induced emf is,

$$E_m = 2\pi n\, B\, l\, bT = 2\pi n\, \varphi T = 2\pi n\, \psi = \omega\, \psi = \psi \frac{d\theta}{dt}$$

Determination of frequency (f) in the ac generator

In the above case, the frequency (Hz) of the emf generated is

$f = \omega/(2\pi) = n$, no. of poles being 2, i.e. having only one pole pair.

In the ac generator, no. of poles = p, and the speed (rps) = n, then the frequency in Hz or cycles/sec, is

f = no. of cycles/sec = no. of cycles per rev × no. of rev per sec = no. of pairs of poles × no. of rev per sec = $(p/2)\,n$

or, $f = \frac{pN}{120} = \frac{p}{2}\,\frac{\omega}{2\pi}$

Example

For a 4-pole ac generator to obtain a voltage having a frequency of 50 Hz,

the speed is, $n = \frac{2f}{p} = \frac{2 \cdot 50}{4} = 25$ rps = 25 60 = 1,500 rpm

For a 2-pole (p = 2) machine, the speed should be 3,000 rpm.

Similarly, the speed of the machine having different no. of poles, required to generate a frequency of 50 Hz can be computed.

Sinusoidal voltage waveform having frequency, f with time period (sec), $T = 1/f$

Periodic Voltage or Current Waveform

Average value

The current waveform shown in Fig. 12.3a, is periodic in nature, with time period, T. It is positive for first half cycle, while it is negative for second half cycle.

The average value of the waveform, $i(t)$ is defined as

$$I_{av} = \frac{Area\ over\ half\ cycle}{Time\ period\ of\ half\ cycle} = \frac{1}{T/2}\int_0^{T/2} i(t)\,dt = \frac{2}{T}\int_0^{T/2} i(t)\,dt$$

Please note that, in this case, only half cycle, or half of the time period, is to be used for computing the average value, as the average value of the waveform over full cycle is zero (0).

If the half time period (T/2) is divided into 6 equal time intervals (T),

$$I_{av} = \frac{(i_1 + i_2 + i_3 + \quad i_6)\ T}{6\ T} = \frac{(i_1 + i_2 + i_3 + \quad i_6)}{6} = \frac{Area\ over\ half\ cycle}{Time\ period\ of\ \ half\ cycle}$$

Please note that no. of time intervals is n = 6.

Root Mean Square (RMS) value

For this current in half time period subdivided into 6 time intervals as given above, in the resistance R, the average value of energy dissipated is given by

$$\frac{(i_1^2 + i_2^2 + i_3^2 + \quad i_6^2)}{6}\ R$$

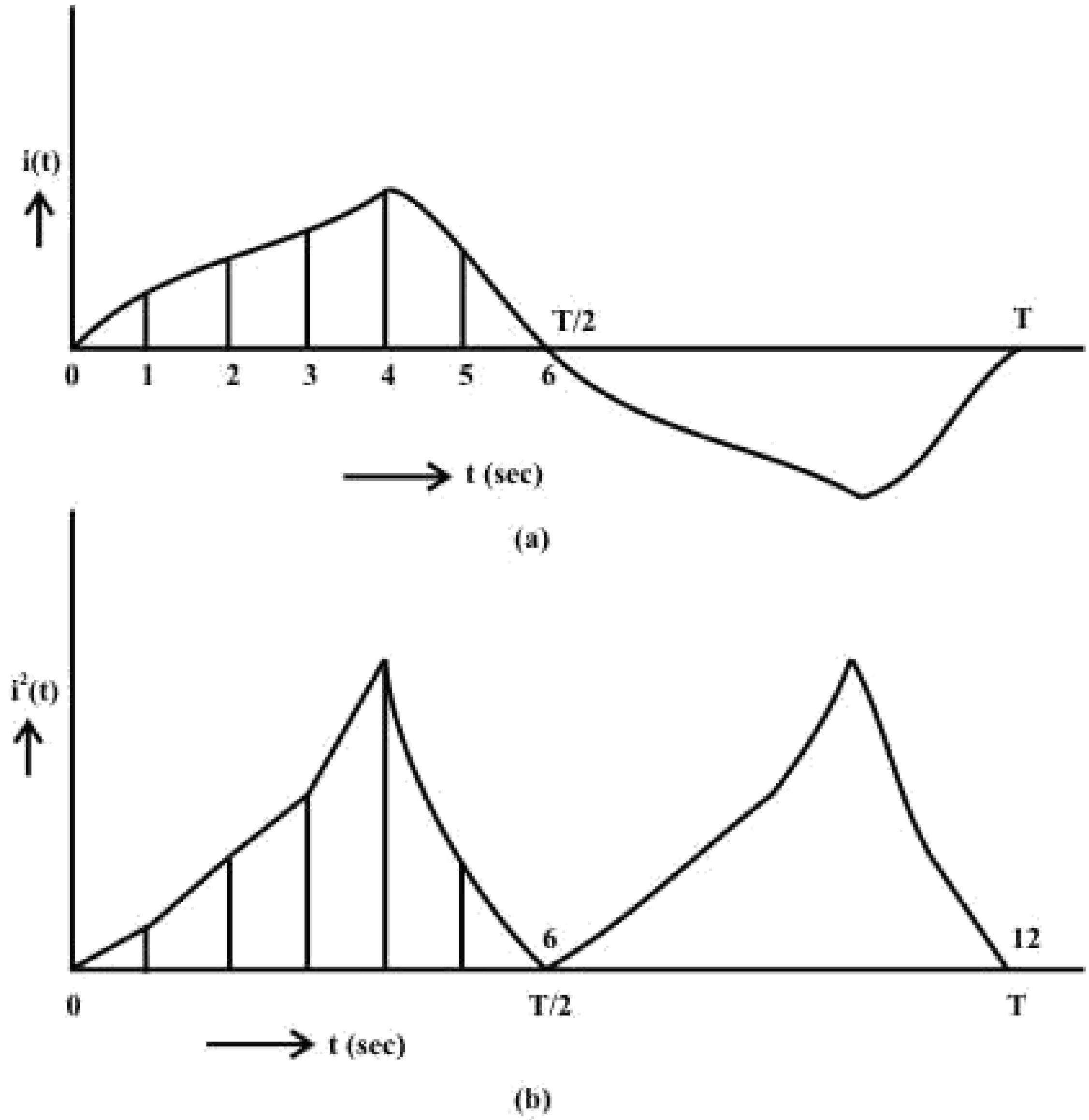


Fig. 12.3 Periodic current waveform
(a) Current (i), (b) Square of current (i^2)

The graph of the square of the current waveform, $i^2\,(t)$ is shown in Fig. 12.3b. Let I be the value of the direct current that produces the same energy dissipated in the resistance R, as produced by the periodic waveform with half time period subdivided into n time intervals,

$$I^2\, R = \frac{(i_1^2 + i_2^2 + i_3^2 + \quad i_n^2)\; T}{n \quad T}\; R$$

$$I = \sqrt{\frac{(i_1^2 + i_2^2 + i_3^2 \;+\; i_n^2)\; T}{n \quad T}} = \sqrt{\frac{Area\ of\ i^2\ \ curve\ over\ half\ cycle}{Time\ period\ of\ half\ cycle}}$$

$$= \sqrt{\frac{1}{T/2}\int_0^{T/2} i^2\,dt} = \sqrt{\frac{2}{T}\int_0^{T/2} i^2\,dt}$$

This value is termed as Root Mean Square (RMS) or effective one. Also to be noted that the same rms value of the current is obtained using the full cycle, or the time period.

Average and RMS Values of Sinusoidal Voltage Waveform

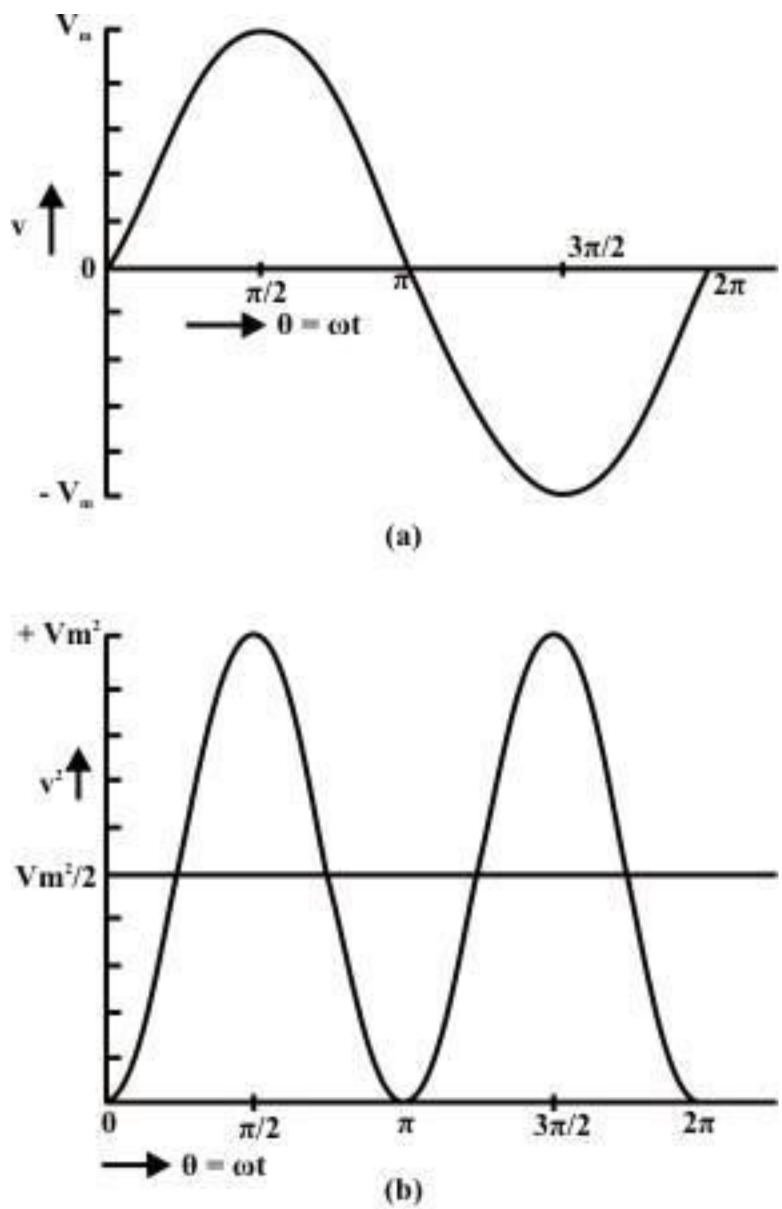


Fig. 12.4 Sinusoidal voltage waveform
(a) Voltage (v), (b) Square of voltage (v^2)

As shown earlier, normally the voltage generated, which is also transmitted and then distributed to the consumer, is the sinusoidal waveform with a frequency of 50 Hz in this country. The waveform of the voltage $v(t)$, and the square of waveform, $v^2(t)$, are shown in figures 12.4a and12.4b respectively.

Time period, $T = 1/f = (2\pi)/\omega$; in angle $(\omega T = 2\pi)$

Half time period, $T/2 = 1/(2f) = \pi/\omega$; in angle $(\omega T/2 = \pi)$

$v(\theta) = V_m \sin\theta \;\; for \;\; \pi \le \theta \le 0; \quad v(t) = V_m \sin \omega t \;\; for \; (\pi/\omega) \le t \le 0$

$$V_{av} = \frac{1}{\pi}\int_0^{\pi} v(\theta)\, d\theta = \frac{1}{\pi}\int_0^{\pi} V_m \sin\theta\, d\theta = \frac{V_m}{\pi}\cos\theta \Big|_{\pi}^{0} = \frac{2}{\pi} V_m = 0.637\, V_m$$

$$V = \left[\frac{1}{\pi}\int_0^{\pi} v^2\, d\theta\right]^{\frac{1}{2}} = \left[\frac{1}{\pi}\int_0^{\pi} V_m^2 \sin^2\theta\, d\theta\right]^{\frac{1}{2}} = \left[\frac{V_m^2}{\pi}\int_0^{\pi} \frac{1}{2}(1-\cos 2\theta)\, d\theta\right]^{\frac{1}{2}}$$

$$= \left[\frac{V_m^2}{2\pi}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi}\right]^{\frac{1}{2}} = \left[\frac{V_m^2}{2\pi}\,\pi\right]^{\frac{1}{2}} = \frac{V_m}{\sqrt{2}} = 0.707\, V_m$$

or, $V_m = \sqrt{2}\, V$

If time t, is used as a variable, instead of angle θ,

$$V_{av} = \frac{1}{\pi/\omega}\int_0^{\pi/\omega} v(t)\, dt = \frac{\omega}{\pi}\int_0^{\pi/\omega} V_m \sin \omega t\, dt = \frac{\omega V_m}{\pi\,\omega}\cos(\omega t)\Big|_{\pi}^{0} = \frac{2}{\pi} V_m = 0.637\, V_m$$

In the same way, the rms value, V can be determined.

If the average value of the above waveform is computed over total time period T, it comes out as zero, as the area of first (positive) half cycle is the same as that of second (negative) half cycle. However, the rms value remains same, if it is computed over total time period.

The different factors are defined as:

$$\text{Form factor} = \frac{\textit{RMS value}}{\textit{Average value}} = \frac{0.707\, V_m}{0.637\, V_m} = 1.11$$

$$\text{Peak factor} = \frac{\textit{Maximum value}}{\textit{Average value}} = \frac{V_m}{0.707\, V_m} = 1.414$$

Note: The rms value is always greater than the average value, except for a rectangular waveform, in which case the heating effect remains constant, so that the average and the rms values are same.

Example

The examples of the two waveforms given are periodic in nature.

1. Triangular current waveform (Fig. 12.5)

Time period = T

$$i(t) = I_m \frac{t}{T} \quad for\; T \le t \le 0$$

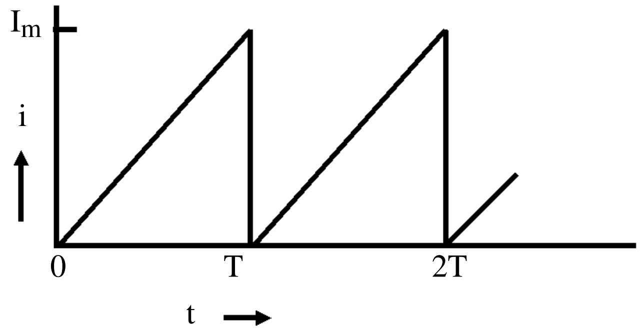


Fig. 12.5 Triangular current waveform

$$I_{av} = \frac{1}{T}\int_0^T i(t)\,dt = \frac{1}{T}\int_0^T I_m \frac{t}{T}\,dt = \frac{I_m}{T^2}\left.\frac{t^2}{2}\right|_0^T = \frac{I_m}{T^2}\,\frac{T^2}{2} = \frac{I_m}{2} = 0.5\,I_m$$

$$I = \left[\frac{1}{T}\int_0^T i^2\,dt\right]^{\frac{1}{2}} = \left[\frac{1}{T}\int_0^T I_m^2\,\frac{t^2}{T^2}\,dt\right]^{\frac{1}{2}} = \left[\frac{I_m^2}{T^3}\left.\frac{t^3}{3}\right|_0^T\right]^{\frac{1}{2}} = \left[\frac{I_m^2}{T^3}\,\frac{T^3}{3}\right]^{\frac{1}{2}} = \left[\frac{I_m^2}{3}\right]^{\frac{1}{2}}$$

$$= \frac{I_m}{\sqrt{3}} = 0.57735\,I_m$$

Two factors of the waveform are:

Form factor = $\dfrac{RMS\ value}{Average\ value} = \dfrac{0.57735\,I_m}{0.5\,I_m} = 1.1547$

Peak factor = $\dfrac{Maximum\ value}{Average\ value} = \dfrac{I_m}{0.5\,I_m} = 2.0$

To note that the form factor is slightly higher than that for the sinusoidal waveform, while the peak factor is much higher.

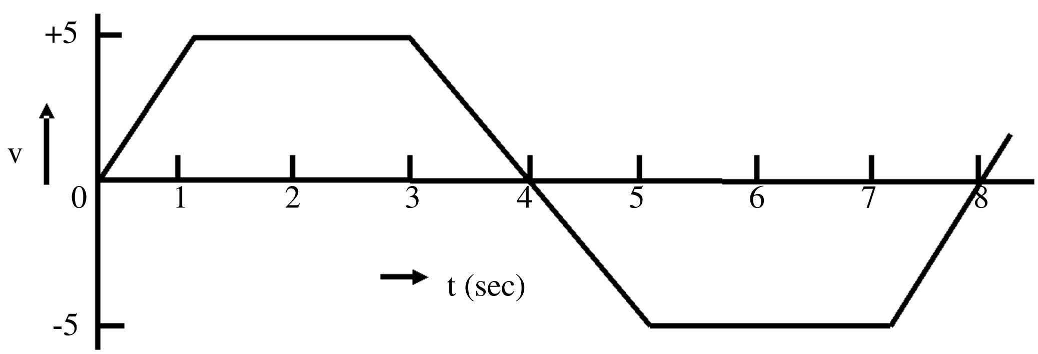


Fig. 12.6 Trapezoidal voltage waveform

2. Trapezoidal voltage waveform (Fig. 12.6)

   Time period (T) = 8 ms

   Half time period $(T/2) = 8/2 = 4$

$v(t) = m\,t = (5/1)\,t = 5t$ *for* $1 \le t \le 0$; $v(t) = 5$ *for* $3 \le t \le 1$; $v(t) = 5(4 - t)$ *for* $4 \le t \le 3$

Please note that time, t is in ms, and slope, m is in V/ms. Also to be noted that, as in the case of sinusoidal waveform, only half time period is taken here for the computation of the average and rms values.

$$V_{av} = \frac{1}{T/2}\int_0^{T/2} v(t)\,dt = \frac{1}{4}\left[\int_0^1 5t\,dt + \int_1^3 5\,dt + \int_3^4 5\,(4-t)\,dt\right] = \frac{1}{4}\left[\frac{5}{2}t^2\Big|_0^1 + 5t\Big|_1^3 + \frac{5}{2}(4-t)^2\Big|_4^3\right]$$

$$= \frac{1}{4}\left[\frac{5}{2} + 5(3-1) + \frac{5}{2}\right] = \frac{15}{4} = 3.75\ V$$

$$V = \left[\frac{1}{T/2}\int_0^{T/2} v^2\,dt\right]^{\frac{1}{2}} = \left[\frac{1}{4}\left\{\int_0^1 (5t)^2\,dt + \int_1^3 (5)^2\,dt + \int_3^4 5^2\,(4-t)^2\,dt\right\}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{4}\left\{\frac{25}{3}t^3\Big|_0^1 + 25t\Big|_1^3 + \frac{25}{3}(4-t)^3\Big|_4^3\right\}\right]^{\frac{1}{2}} = \left[\frac{1}{4}\left\{\frac{25}{3} + 25\,(3-1) + \frac{25}{3}\right\}\right]^{\frac{1}{2}}$$

$$= \sqrt{\frac{50}{3}} \cong \sqrt{16.67} = 4.0825\ V$$

Two factors of the waveform are:

Form factor $= \dfrac{RMS\ value}{Average\ value} = \dfrac{4.0825}{3.75} = 1.0887$

Peak factor $= \dfrac{Maximum\ value}{Average\ value} = \dfrac{5.0}{3.75} = 1.3333$

To note that the both the above factors are slightly lower than those for the sinusoidal waveform.

Similarly, the average and rms or effective values of periodic voltage or current waveforms can be computed.

In this lesson, starting with the generation of single phase ac voltage, the terms, such as average and rms values, related to periodic voltage and current waveforms are explained with examples. In the next lesson, the background material required – the representation of sinusoidal voltage/current as phasors, the rectangular and polar forms of the phasors, as complex quantity, and the mathematical operations – addition/subtraction and multiplication/division, using phasors as complex quantity, are discussed in detail with numerical examples. In the following lessons, the study of circuits fed from single phase ac supply, is presented.

Problems

12.1 What is the speed in rpm of an ac generator with 4 poles, to produce a voltage with a frequency of 50 Hz

(a) 3000 (b) 1500 (c) 1000 (d) 750

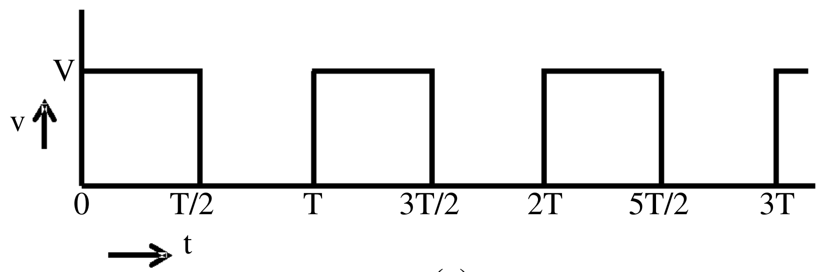
12.2 Determine the No. of poles required in an ac generator running at 1,000 rpm, to produce a voltage with a frequency of 50 Hz.

(a) 2 (b) 4 (c) 6 (d) 8

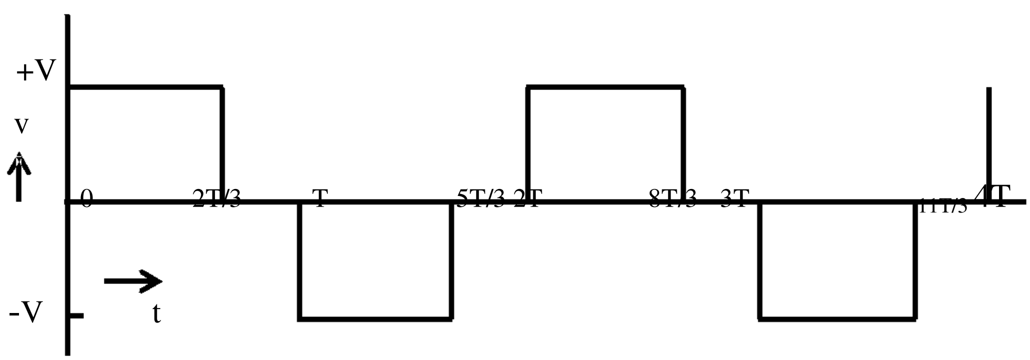
12.3 Calculate the speed in rpm of an ac generator with 24 poles, to produce a voltage with a frequency of 50 Hz.

(a) 300 (b) 250 (c) 200 (d) 150

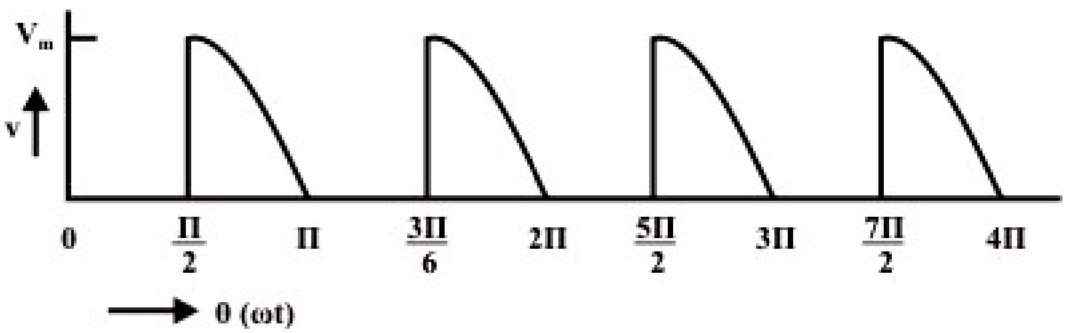
12.4 Determine the average and root mean square (rms) values of the following waveforms.



(a)

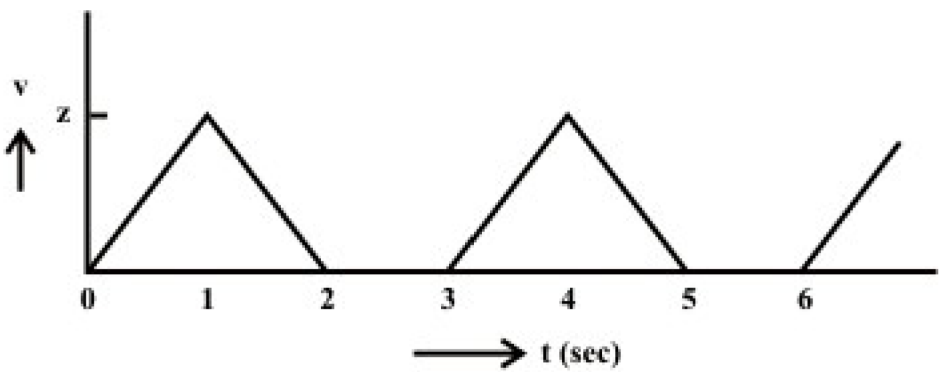


(b)



V_m
v
0
$\frac{\Pi}{2}$
Π
$\frac{3\Pi}{6}$
2Π
$\frac{5\Pi}{2}$
3Π
$\frac{7\Pi}{2}$
4Π
θ (ωt)

(c)



v
z
0
1
2
3
4
5
6
t (sec)

(d)

Fig. 12.7

UNIT-3

Transformer

Ideal Transformer

Contents

23.1 Goals of the lesson

In this lesson, we shall study two winding *ideal transformer*, its properties and working principle under no load condition as well as under load condition. Induced voltages in primary and secondary are obtained, clearly identifying the factors on which they depend upon. The ratio between the primary and secondary voltages are shown to depend on ratio of turns of the two windings. At the end, how to draw phasor diagram under no load and load conditions, are explained. Importance of studying such a transformer will be highlighted. At the end, several objective type and numerical problems have been given for solving.
Key Words: Magnetising current, HV & LV windings, no load phasor diagram, reflected current, equivalent circuit.

After going through this section students will be able to understand the following.

1. necessity of transformers in power system.
2. properties of an ideal transformer.
3. meaning of load and no load operation.
4. basic working principle of operation under no load condition.
5. no load operation and phasor diagram under no load.
6. the factors on which the primary and secondary induced voltages depend.
7. fundamental relations between primary and secondary voltages.
8. the factors on which peak flux in the core depend.
9. the factors which decides the magnitude of the *magnetizing current*.
10. What does loading of a transformer means?
11. What is reflected current and when does it flow in the primary?
12. Why does VA (or kVA) remain same on both the sides?
13. What impedance does the supply see when a given impedance Z_2 is connected across the secondary?
14. Equivalent circuit of ideal transformer referred to different sides.

23.2 Introduction

Transformers are one of the most important components of any power system. It basically changes the level of voltages from one value to the other at constant frequency. Being a static machine the efficiency of a transformer could be as high as 99%.

Big generating stations are located at hundreds or more km away from the load center (where the power will be actually consumed). Long transmission lines carry the power to the load centre from the generating stations. Generator is a rotating machines and the level of voltage at which it generates power is limited to several kilo volts only –

a typical value is 11 kV. To transmit large amount of power (several thousands of mega watts) at this voltage level means large amount of current has to flow through the transmission lines. The cross sectional area of the conductor of the lines accordingly should be large. Hence cost involved in transmitting a given amount of power rises many folds. Not only that, the transmission lines has their own resistances. This huge amount of current will cause tremendous amount of *power loss* or I^2r loss in the lines. This loss will simply heat the lines and becomes a *wasteful energy*. In other words, *efficiency of transmission* becomes poor and cost involved is high.

The above problems may addressed if we could transmit power at a very high voltage say, at 200 kV or 400 kV or even higher at 800 kV. But as pointed out earlier, a generator is incapable of generating voltage at these level due to its own practical limitation. The solution to this problem is to use an appropriate *step-up transformer* at the generating station to bring the transmission voltage level at the desired value as depicted in figure 23.1 where for simplicity single phase system is shown to understand the basic idea. Obviously when power reaches the load centre, one has to step down the voltage to suitable and safe values by using transformers. Thus transformers are an integral part in any modern power system. Transformers are located in places called *substations*. In cities or towns you must have noticed transformers are installed on poles – these are called *pole mounted distribution transformers*. These type of transformers change voltage level typically from 3-phase, 6 kV to 3-phase 440 V line to line.

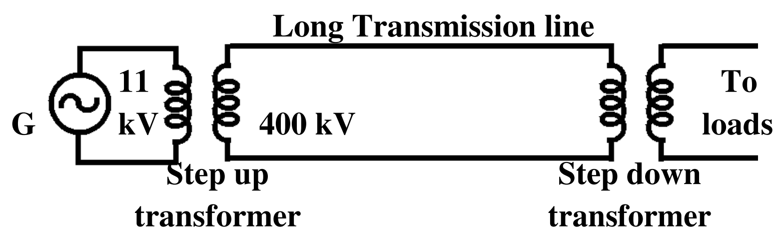


Figure 23.1: A simple single phase power system.

In this and the following lessons we shall study the basic principle of operation and performance evaluation based on equivalent circuit.

23.2.1 Principle of operation

A transformer in its simplest form will consist of a rectangular *laminated magnetic structure* on which two coils of different number of turns are wound as shown in Figure 23.2.

The winding to which a.c voltage is impressed is called the *primary* of the transformer and the winding across which the load is connected is called the *secondary* of the transformer.

23.3 Ideal Transformer

To understand the working of a transformer it is always instructive, to begin with the concept of an *ideal* transformer with the following properties.

1. Primary and secondary windings has no resistance.

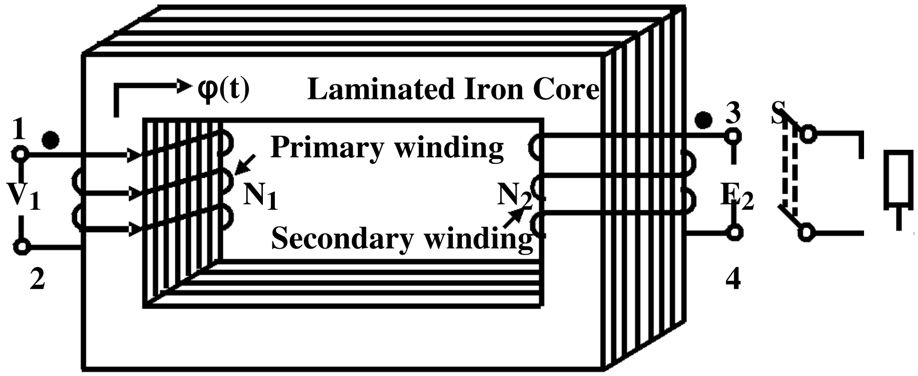


Figure 23.2: A typical transformer.

2. All the flux produced by the primary links the secondary winding i,e., there is no leakage flux.
3. Permeability μ_r of the core is infinitely large. In other words, to establish flux in the core vanishingly small (or zero) current is required.
4. Core loss comprising of *eddy current* and *hysteresis* losses are neglected.

23.3.1 Core flux gets fixed by voltage & frequency

The flux level B_{max} in the core of a given magnetic circuit gets fixed by the magnitude of the supply voltage and frequency. This important point has been discussed in the previous lecture 20. It was shown that:

$$B_{\max} = \frac{V}{\sqrt{2}\pi fAN} = \frac{1}{4.44\,AN}\frac{V}{f}$$

where, V is the applied voltage at frequency f, N is the number of turns of the coil and A is the cross sectional area of the core. For a given magnetic circuit A and N are constants, so B_{max} developed in core is decided by the ratio $\frac{V}{f}$. The peak value of the coil current I_{max}, drawn from the supply now gets decided by the B-H characteristics of the core material.

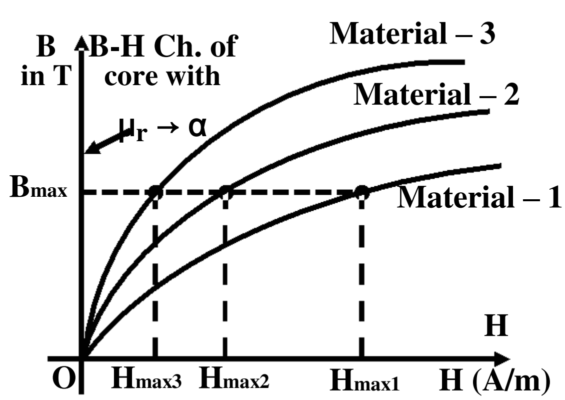


Figure 23.3: Estimating current drawn for different core materials.

To elaborate this, let us consider a magnetic circuit with N number of turns and core section area A with mean length l. Let material-3 be used to construct the core whose B-H characteristic shown in figure 23.3. Now the question is: if we apply a voltage V at frequency f, how much current will be drawn by the coil? We follow the following steps to arrive at the answer.

1. First calculate maximum flux density using $B_{max} = \frac{1}{4.44\,AN}\frac{V}{f}$. Note that value of B_{max} is independent of the core material property.
2. Corresponding to this B_{max}, obtain the value of H_{max3} from the B-H characteristic of the material-3 (figure 23.3).
3. Now calculate the required value of the current using the relation $I_{max\,3} = \frac{H_{max\,3}\,l}{N}$.
4. The rms value of the exciting current with material-3 as the core, will be $I_3 = I_{max\,3}\sqrt{2}$.

By following the above steps, one could also estimate the exciting currents (I_2 or I_3) drawn by the coil if the core material were replaced by material-2 or by material-3 with other things remaining same. Obviously current needed, to establish a flux of B_{max} is lowest for material-3. Finally note that if the core material is such that $\mu_r \rightarrow \infty$, the B-H characteristic of this ideal core material will be the B axis itself as shown by the thick line in figure 23.3 which means that for such an ideal core material current needed is practically zero to establish any B_{max} in the core.

23.3.2 Analysis of ideal transformer

Let us assume a sinusoidally varying voltage is impressed across the primary with secondary winding open circuited. Although the current drawn I_m will be practically zero, but its position will be 90° lagging with respect to the supply voltage. The flux produced will obviously be in phase with I_m. In other words the supply voltage will lead the flux phasor by 90°. Since flux is common for both the primary and secondary coils, it is customary to take flux phasor as the reference.

$$\text{Let, } \varphi(t) = \varphi_{max} \sin \omega t$$

$$\text{then, } v_1 = v_{max} \sin\left(\omega t + \frac{\pi}{2}\right) \qquad (23.1)$$

The time varying flux $\varphi(t)$ will link both the primary and secondary turns inducing in voltages e_1 and e_2 respectively

Instantaneous induced voltage in primary $= -N_1 \frac{d\varphi}{dt} = \omega N_1 \varphi_{max} \sin\left(\omega t - \frac{\pi}{2}\right)$

$$= 2\pi f N_1 \varphi_{max} \sin\left(\omega t - \frac{\pi}{2}\right) \quad (23.2)$$

Instantaneous induced voltage in secondary $= -N_2 \frac{d\varphi}{dt} = \omega N_2 \varphi_{max} \sin\left(\omega t - \frac{\pi}{2}\right)$

$$= 2\pi f N_2 \varphi_{max} \sin\left(\omega t - \frac{\pi}{2}\right) \quad (23.3)$$

Magnitudes of the rms induced voltages will therefore be

$$E_1 = \sqrt{2}\pi f N_1 \varphi_{max} = 4.44 f N_1 \varphi_{max} \quad (23.4)$$

$$E_2 = \sqrt{2}\pi f N_2 \varphi_{max} = 4.44 f N_2 \varphi_{max} \quad (23.5)$$

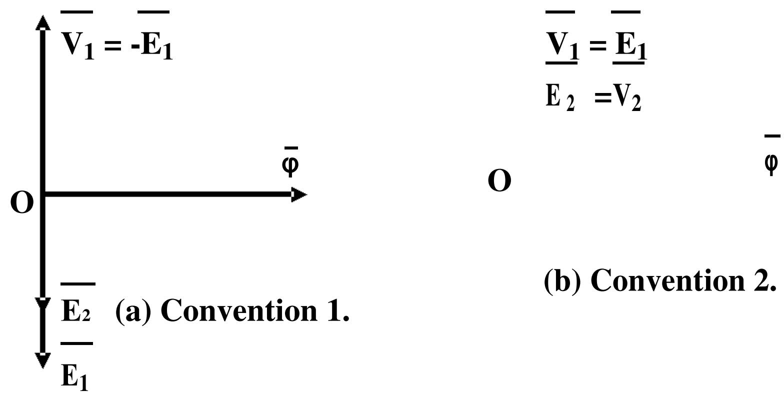
The time phase relationship between the applied voltage v_1 and e_1 and e_2 will be same. The 180° phase relationship obtained in the mathematical expressions of the two merely indicates that the induced voltage opposes the applied voltage as per *Lenz's law*. In other words if e_1 were allowed to act alone it would have delivered power in a direction opposite to that of v_1. By applying Kirchoff's law in the primary one can easily say that $V_1 = E_1$ as there is no other drop existing in this ideal transformer. Thus udder no load condition,

$$\frac{V_2}{V_1} = \frac{E_2}{E_1} = \frac{N_2}{N_1}$$

Where, V_1, V_2 are the terminal voltages and E_1, E_2 are the rms induced voltages. In convention 1, phasors $\bar{E}_1$ and $\bar{E}_2$ are drawn 180° out of phase with respect to $\bar{V}_1$ in order to convey the respective power flow directions of these two are opposite. The second convention results from the fact that the quantities $v_1(t)$, $e_1(t)$ and $e_2(t)$ vary in unison, then why not show them as co-phasal and keep remember the power flow business in one's mind.

23.3.3 No load phasor diagram

A transformer is said to be under no load condition when no load is connected across the secondary i.e., the switch S in figure 23.2 is kept opened and no current is carried by the secondary windings. The phasor diagram under no load condition can be drawn starting with $\bar{\varphi}$ as the reference phasor as shown in figure 23.4.



(a) Convention 1. **(b) Convention 2.**

Figure 23.4: No load Phasor Diagram following two conventions.

In convention 1, phsors $\overline{E_1}$ and $\overline{E_2}$ are drawn 180° out of phase with respect to $\overline{V_1}$ in order to convey that the respective power flow directions of these two are opposite. The second convention results from the fact that the quantities v_1 (t), $e_1(t)$ and $e_2(t)$ vary in unison then why not show them as co-phasal and keep remember the power flow business in one's mind. Also remember vanishingly small magnetizing current is drawn from the supply creating the flux and in time phase with the flux.

23.4 Transformer under loaded condition

In this lesson we shall study the behavior of the transformer when loaded. A transformer gets loaded when we try to draw power from the secondary. In practice loading can be imposed on a transformer by connecting impedance across its secondary coil. It will be explained how the primary reacts when the secondary is loaded. It will be shown that any attempt to draw current/power from the secondary, is immediately responded by the primary winding by drawing extra current/power from the source. We shall also see that mmf balance will be maintained whenever both the windings carry currents. Together with the *mmf balance equation* and *voltage ratio* equation, invariance of Volt-Ampere (VA or KVA) irrespective of the sides will be established.

We have seen in the preceding section that the secondary winding becomes a seat of emf and ready to deliver power to a load if connected across it when primary is energized. Under no load condition power drawn is zero as current drawn is zero for ideal transformer. However when loaded, the secondary will deliver power to the load and same amount of power must be sucked in by the primary from the source in order to maintain *power balance*. We expect the primary current to flow now. Here we shall examine in somewhat detail the mechanism of drawing extra current by the primary when the secondary is loaded. For a fruitful discussion on it let us quickly review the *dot* convention in mutually coupled coils.

23.4.1 Dot convention

The primary of the transformer shown in figure 23.2 is energized from a.c source and potential of terminal 1 with respect to terminal 2 is $v_{12} = V_{max} sin \omega t$. Naturally polarity of 1 is sometimes +ve and some other time it is –ve. The *dot* convention helps us to determine the polarity of the induced voltage in the secondary coil marked with terminals 3 and 4. Suppose at some time *t* we find that terminal 1 is +ve and it is increasing with respect to terminal 2. At that time what should be the status of the induced voltage polarity in the secondary – whether terminal 3 is +ve or –ve? If possible let us assume terminal 3 is –ve and terminal 4 is positive. If that be current the secondary will try to deliver current to a load such that current comes out from terminal 4 and enters terminal 3. Secondary winding therefore, produces flux in the core in the same direction as that of the flux produced by the primary. So core flux gets strengthened in inducing more voltage. This is contrary to the dictate of Lenz's law which says that the polarity of the induced voltage in a coil should be such that it will try to oppose the cause for which it is due. Hence terminal 3 can not be –ve.

If terminal 3 is +ve then we find that secondary will drive current through the load leaving from terminal 3 and entering through terminal 4. Therefore flux produced by the secondary clearly opposes the primary flux fulfilling the condition set by Lenz's law. Thus when terminal 1 is +ve terminal 3 of the secondary too has to be positive. In mutually coupled coils dots are put at the appropriate terminals of the primary and secondary merely to indicative the status of polarities of the voltages. Dot terminals will have at any point of time identical polarities. In the transformer of figure 23.2 it is appropriate to put dot markings on terminal 1 of primary and terminal 3 of secondary. It is to be noted that if the *sense* of the windings are known (as in figure 23.2), then one can ascertain with confidence where to place the dot markings without doing any testing whatsoever. In practice however, only a pair of primary terminals and a pair of secondary terminals are available to the user and the sense of the winding can not be ascertained at all. In such cases the dots can be found out by doing some simple tests such as *polarity test* or *d.c kick test*.

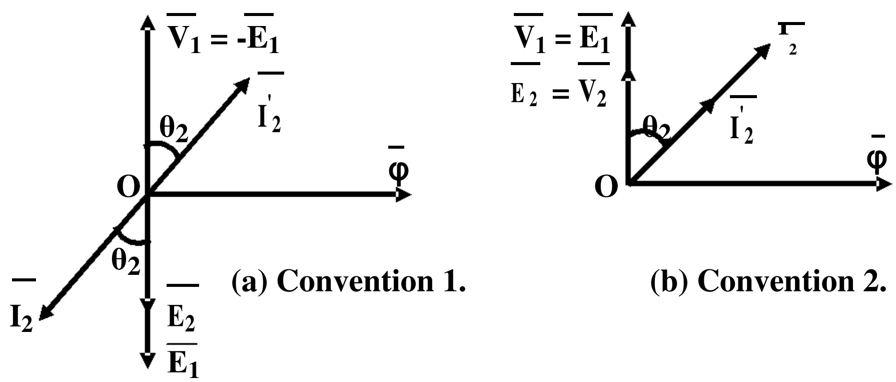
If the transformer is loaded by closing the switch *S*, current will be delivered to the load from terminal 3 and back to 4. Since the secondary winding carries current it produces flux in the anti clock wise direction in the core and tries to reduce the original flux. However, KVL in the primary demands that core flux should remain constant no matter whether the transformer is loaded or not. Such a requirement can only be met if the primary draws a definite amount of extra current in order to nullify the effect of the mmf produced by the secondary. Let it be clearly understood that net mmf acting in the core is given by: mmf due to vanishingly small magnetizing current + mmf due to secondary current + mmf due to additional primary current. But the last two terms must add to zero in order to keep the flux constant and net mmf eventually be once again be due to vanishingly small magnetizing current. If I_2 is the magnitude of the secondary current and $I_2^{'}$ is the additional current drawn by the primary then following relation must hold good:

$$N_1 I_2' = N_2 I_2$$

$$\text{or } I_2' = \frac{N_2}{N_1} I_2 = \frac{I_2}{a}$$

$$\text{where, } a = \frac{N_1}{N_2} = \text{turns ratio} \tag{23.6}$$

To draw the phasor diagram under load condition, let us assume the power factor angle of the load to be θ_2, lagging. Therefore the load current phasor $\overline{I_2}$, can be drawn lagging the secondary terminal voltage $\overline{E_2}$ by θ_2 as shown in the figure 23.5.



(a) Convention 1. (b) Convention 2.

Figure 23.5: Phasor Diagram when transformer is loaded.

The reflected current magnitude can be calculated from the relation $I_2' = \frac{I_2}{a}$ and is shown directed 180° out of phase with respect to $\overline{I_2}$ in convention 1 or in phase with $\overline{I_2}$ as per the convention 2. At this stage let it be suggested to follow one convention only and we select convention 2 for that purpose. Now,

$$\text{Volt-Ampere delivered to the load} = V_2 I_2$$
$$= E_2 I_2$$
$$= aE_1 \frac{I}{a}$$
$$= E_1 I_1 = V_1 I_1 = \text{Volt-Ampere drawn from the supply.}$$

Thus we note that for an ideal transformer the output VA is same as the input VA and also the power is drawn at the same power factor as that of the load.

23.4.2 Equivalent circuit of an ideal transformer

The equivalent circuit of a transformer can be drawn (i) showing both the sides along with parameters, (ii) referred to the primary side and (iii) referred to the secondary side.

In which ever way the equivalent circuit is drawn, it must represent the operation of the transformer correctly both under no load and load condition. Figure 23.6 shows the equivalent circuits of the transformer.

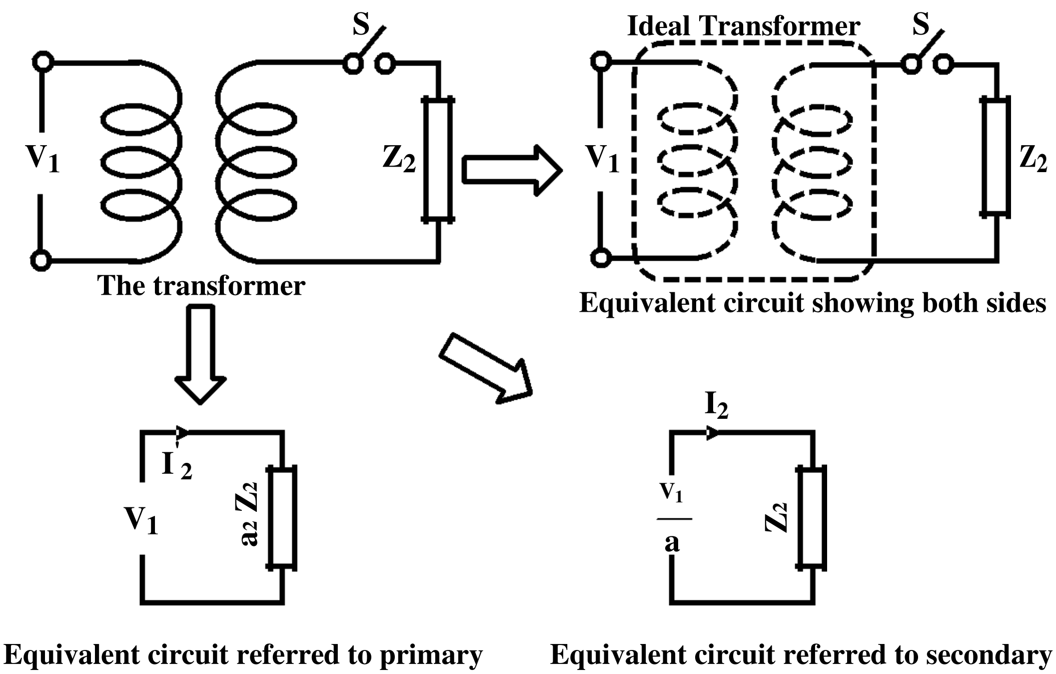


Figure 23.6: Equivalent circuits of an ideal transformer.

Think in terms of the supply. It supplies some current at some power factor when a load is connected in the secondary. If instead of the transformer, an impedance of value $a^2 Z_2$ is connected across the supply, supply will behave identically. This corresponds to the equivalent circuit referred to the primary. Similarly from the load point of view, forgetting about the transformer, we may be interested to know what voltage source should be impressed across Z_2 such that same current is supplied to the load when the transformer was present. This corresponds to the equivalent circuit referred to the secondary of the transformer. When both the windings are shown in the equivalent circuit, they are shown with chain lines instead of continuous line. Why? This is because, when primary is energized and secondary is opened no current is drawn, however current is drawn when a load is present on the secondary side. Although supply two terminals are physically joined by the primary winding, the current drawn depends upon the load on the secondary side.

23.5 Tick the correct answer

1. An ideal transformer has two secondary coils with number of turns 100 and 150 respectively. The primary coil has 125 turns and supplied from 400 V, 50 Hz, single phase source. If the two secondary coils are connected in series, the possible voltages across the series combination will be:

(A) 833.5 V or 166.5 V (B) 833.5 V or 320 V
(C) 320 V or 800 V (D) 800 V or 166.5 V

2. A single phase, ideal transformer of voltage rating 200 V / 400 V, 50 Hz produces a flux density of 1.3 T when its LV side is energized from a 200 V, 50 Hz source. If the LV side is energized from a 180 V, 40 Hz source, the flux density in the core will become:

   (A) 0.68 T (B) 1.44 T (C) 1.62 T (D) 1.46 T

3. In the coil arrangement shown in Figure 23.7, A dot (□) marking is shown in the first coil. Where would be the corresponding dot (□) markings be placed on coils 2 and 3?
   (A) At terminal *P* of coil 2 and at terminal *R* of coil 3
   (B) At terminal *P* of coil 2 and at terminal *S* of coil 3
   (C) At terminal *Q* of coil 2 and at terminal *R* of coil 3
   (D) At terminal *Q* of coil 2 and at terminal *S* of coil 3

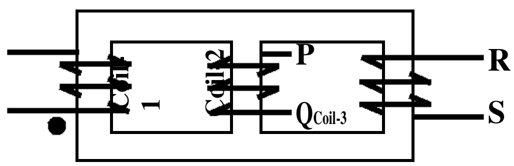
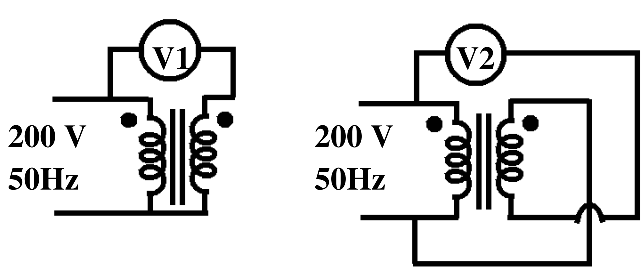


Figure 23.7:

4. A single phase ideal transformer is having a voltage rating 200 V / 100 V, 50 Hz. The HV and LV sides of the transformer are connected in two different ways with the help of voltmeters as depicted in figure 23.8 (a) and (b). If the HV side is energized with 200 V, 50 Hz source in both the cases, the readings of voltmeters V1 and V2 respectively will be:

   (A) 100 V and 300 V (B) 300 V and 100 V
   (C) 100 V or 0 V (D) 0 V or 300 V



Connection (a) **Connection (b)**

Figure 23.8:

5. Across the HV side of a single phase 200 V / 400 V, 50 Hz transformer, an impedance of $32 + j24\Omega$ is connected, with LV side supplied with rated voltage & frequency. The supply current and the impedance seen by the supply are respectively:

   (A) 20 A & $128 + j96\Omega$ (B) 20 A & $8 + j6\Omega$
   (C) 5 A & $8 + j6\Omega$ (D) 20 A & $16 + j12\Omega$

6. The rating of the primary winding a transformer, having 60 number of turns, is 250 V, 50 Hz. If the core cross section area is 144 cm^2 then the flux density in the core is:

   (A) 1 T (B) 1.6 T (C) 1.4 T (D) 1.5 T

23.6 Solve the following

1. In Figure 23.9, the *ideal transformer* has turns ratio 2:1. Draw the equivalent circuits referred to primary and referred to secondary. Calculate primary and secondary currents and the input power factor and the load power factor.

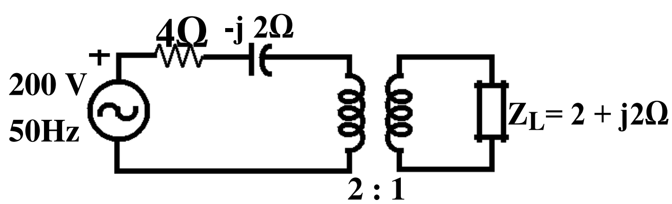


Figure 23.9: Basic scheme of protection.

2. In the Figure 23.10, a 4-winding transformer is shown along with number of turns of the windings. The first winding is energized with 200 V, 50 Hz supply. Across the 2nd winding a pure inductive reactance $X_L = 20\ \Omega$ is connected. Across the 3rd winding a pure resistance $R = 15\ \Omega$ and across the 4th winding a capacitive reactance of $X_C = 10\ \Omega$ are connected. Calculate the input current and the power factor at which it is drawn.

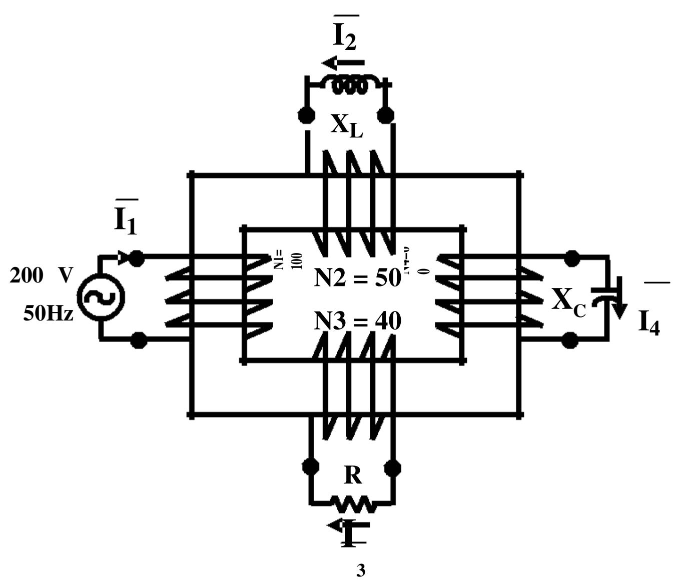


Figure 23.10:

3. In the circuit shown in Figure 23.11, T1, T2 and T3 are *ideal* transformers.

   a) Neglecting the impedance of the transmission lines, calculate the currents in primary and secondary windings of all the transformers. Reduce the circuit refer to the primary side of T1.
   b) For this part, assume the transmission line impedance in the section AB to be $\overline{Z}_{AB} = 1 + j3\Omega$. In this case calculate, what should be $\overline{V}_s$ for maintaining 450 V across the load $\overline{Z}_L = 60 + j80\Omega$. Also calculate the net impedance seen by $\overline{V}_s$.

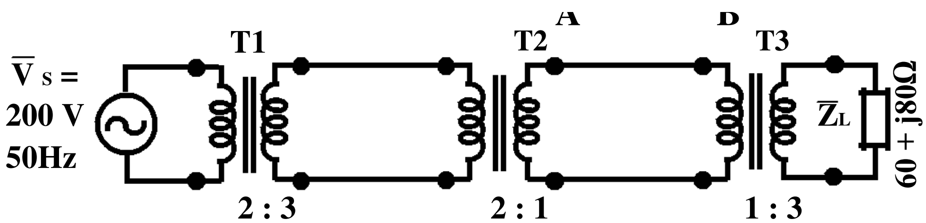


Figure 23.11:

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Transformer

Efficiency & Regulation

Contents

25.1 Goals of the lesson

In the previous lesson we have seen how to draw equivalent circuit showing magnetizing reactance (X_m), resistance (R_{cl}), representing core loss, equivalent winding resistance (r_e) and equivalent leakage reactance (x_e). The equivalent circuit will be of little help to us unless we know the parameter values. In this lesson we first describe two basic simple laboratory tests namely (i) open circuit test and (ii) short circuit test from which the values of the equivalent circuit parameters can be computed. Once the equivalent circuit is completely known, we can predict the performance of the transformer at various loadings. *Efficiency* and *regulation* are two important quantities which are next defined and expressions for them derived and importance highlighted. A number of objective type questions and problems are given at the end of the lesson which when solved will make the understanding of the lesson clearer.

Key Words: O.C. test, S.C test, efficiency, regulation.

After going through this section students will be able to answer the following questions.

- Which parameters are obtained from O.C test?
- Which parameters are obtained from S.C test?
- What percentage of rated voltage is needed to be applied to carry out O.C test?
- What percentage of rated voltage is needed to be applied to carry out S.C test?
- From which side of a large transformer, would you like to carry out O.C test?
- From which side of a large transformer, would you like to carry out S.C test?
- How to calculate efficiency of the transformer at a given load and power factor?
- Under what condition does the transformer operate at maximum efficiency?
- What is regulation and its importance?
- How to estimate regulation at a given load and power factor?
- What is the difference between *efficiency* and *all day efficiency*?

25.2 Determination of equivalent circuit parameters

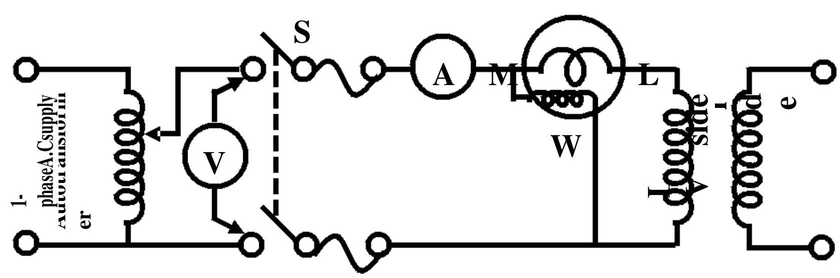
After developing the equivalent circuit representation, it is natural to ask, how to know equivalent circuit the parameter values. Theoretically from the detailed design data it is possible to estimate various parameters shown in the equivalent circuit. In practice, two basic tests namely the *open circuit test* and the *short circuit test* are performed to determine the equivalent circuit parameters.

25.2.1 Qualifying parameters with suffixes LV & HV

For a given transformer of rating say, 10 kVA, 200 V / 100 V, 50 Hz, one should not be under the impression that 200 V (HV) side will always be the primary (as because this value appears first in order in the voltage specification) and 100 V (LV) side will always be secondary. Thus, for a given transformer either of the HV and LV sides may be used as primary or secondary as decided by the user to suit his/her goals in practice. Usually suffixes 1 and 2 are used for expressing quantities in terms of primary and secondary respectively – there is nothing wrong in it so long one keeps track clearly which side is being used as primary. However, there are situation, such as carrying out O.C & S.C tests (discussed in the next section), where naming parameters with suffixes HV and LV become imperative to avoid mix up or confusion. Thus, it will be useful to qualify the parameter values using the suffixes HV and LV (such as $r_{e\,HV}$, $r_{e\,LV}$ etc. instead of r_{e1}, r_{e2}). Therefore, it is recommended to use suffixes as LV, HV instead of 1 and 2 while describing quantities (like voltage V_{HV}, V_{LV} and currents I_{HV}, I_{LV}) or parameters (resistances r_{HV}, r_{LV} and reactances x_{HV}, x_{LV}) in such cases.

25.2.2 Open Circuit Test

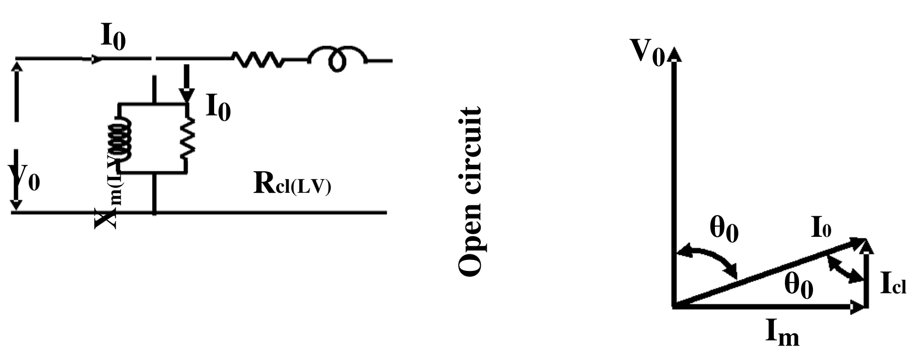
To carry out open circuit test it is the LV side of the transformer where rated voltage at rated frequency is applied and HV side is left opened as shown in the circuit diagram 25.1. The voltmeter, ammeter and the wattmeter readings are taken and suppose they are V_0, I_0 and W_0 respectively. During this test, rated flux is produced in the core and the current drawn is the no-load current which is quite small about 2 to 5% of the rated current. Therefore low range ammeter and wattmeter current coil should be selected. Strictly speaking the wattmeter will record the core loss as well as the LV winding copper loss. But the winding copper loss is very small compared to the core loss as the flux in the core is rated. In fact this approximation is built-in in the approximate equivalent circuit of the transformer referred to primary side which is LV side in this case.



Open Circuit Test

Figure 25.1: Circuit diagram for O.C test

The approximate equivalent circuit and the corresponding phasor diagrams are shown in figures 25.2 (a) and (b) under no load condition.



(a) Equivalent circuit under O.C test (b) Corresponding phasor diagram

Below we shall show how from the readings of the meters the parallel branch impedance namely $R_{cl(LV)}$ and $X_{m(LV)}$ can be calculated.

Calculate no load power factor $\cos\theta_0 = \frac{W}{V_0 I_0}$

Hence θ_0 is known, calculate $\sin\theta_0$

Calculate magnetizing current $I_m = I_0 \sin\theta_0$

Calculate core loss component of current $I_{cl} = I_0 \cos\theta_0$

Magnetising branch reactance $X_{m(LV)} = \frac{V_0}{I_m}$

Resistance representing core loss $R_{cl(LV)} = \frac{V_0}{I_{cl}}$

We can also calculate $X_{m(HV)}$ and $R_{cl(HV)}$ as follows:

$$X_{m(HV)} = \frac{X_{m(LV)}}{a^2}$$

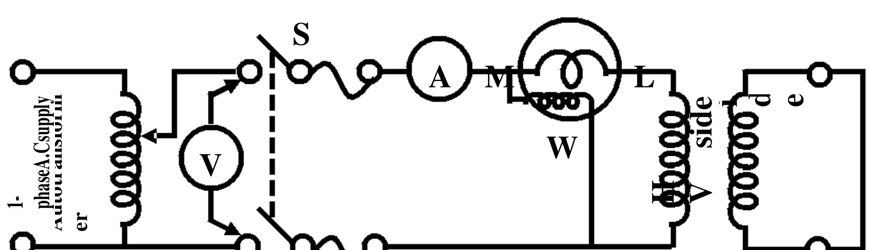
$$R_{cl(HV)} = \frac{R_{cl(LV)}}{a^2}$$

Where, $a = \frac{N_{LV}}{N_{HV}}$ the turns ratio

If we want to draw the equivalent circuit referred to LV side then $R_{cl(LV)}$ and $X_{m(LV)}$ are to be used. On the other hand if we are interested for the equivalent circuit referred to HV side, $R_{cl(HV)}$ and $X_{m(HV)}$ are to be used.

25.2.3 Short circuit test

Short circuit test is generally carried out by energizing the HV side with LV side shorted. Voltage applied is such that the *rated* current flows in the windings. The circuit diagram is shown in the figure 25.3. Here also voltmeter, ammeter and the wattmeter readings are noted corresponding to the rated current of the windings.

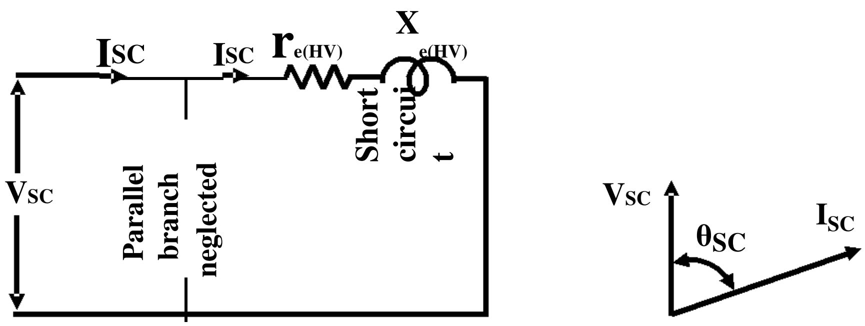


Short Circuit Test

Figure 25.3: Circuit diagram during S.C test

Suppose the readings are V_{sc}, I_{sc} and W_{sc}. It should be noted that voltage required to be applied for rated short circuit current is quite small (typically about 5%). Therefore flux level in

the core of the transformer will be also very small. Hence core loss is negligibly small compared to the winding copper losses as rated current now flows in the windings. Magnetizing current too, will be pretty small. In other words, under the condition of the experiment, the parallel branch impedance comprising of $R_{cl(HV)}$ and $X_{m(LV)}$ can be considered to be absent. The equivalent circuit and the corresponding phasor diagram during circuit test are shown in figures 25.4 (a) and
(b).



(a) Equivalent circuit under S.C test **(b) Corresponding phasor diagram**

Figure 25.4: Equivalent circuit & phasor diagram during S.C test

Therefore from the test data series equivalent impedance namely $r_{e(HV)}$ and $x_{e(HV)}$ can easily be computed as follows:

$$\text{Equivalent resistance ref. to HV side } r_{e(HV)} = \frac{W_{sc}}{I_{sc}^2}$$

$$\text{Equivalent impedance ref. to HV side } z_{e(HV)} = \frac{V_{sc}}{I_{sc}}$$

$$\text{Equivalent leakage reactance ref. to HV side } x_{e(HV)} = \sqrt{z_{e(HV)}^2 - r_{e(HV)}^2}$$

We can also calculate $r_{e(LV)}$ and $x_{e(LV)}$ as follows:

$$r_{e(LV)} = a^2 r_{e(HV)}$$

$$x_{e(LV)} = a^2 x_{e(HV)}$$

$$\text{where, } a = \frac{N_{LV}}{N_{HV}} \text{ the turns ratio}$$

Once again, remember if you are drawing equivalent circuit referred to LV side, use parameter values with suffixes LV, while for equivalent circuit referred to HV side parameter values with suffixes HV must be used.

25.3 Efficiency of transformer

In a practical transformer we have seen mainly two types of major losses namely core and copper losses occur. These losses are wasted as heat and temperature of the transformer rises. Therefore output power of the transformer will be always less than the input power drawn by the primary from the source and efficiency is defined as

$$\eta = \frac{\text{Output power in KW}}{\text{Output power in Kw + Losses}}$$

$$= \frac{\text{Output power in KW}}{\text{Output power in Kw + Core loss + Copper loss}} \qquad (25.1)$$

We have seen that from no load to the full load condition the core loss, P_{core} remains practically constant since the level of flux remains practically same. On the other hand we know that the winding currents depend upon the degree of loading and copper loss directly depends upon the square of the current and not a constant from no load to full load condition. We shall write a general expression for efficiency for the transformer operating at *x* per unit loading and delivering power to a known power factor load. Let,

KVA rating of the transformer be $= S$

Per unit degree of loading be $= x$

Transformer is delivering $= x\,S$ KVA

Power factor of the load be $= \cos\theta$

Output power in KW $= xS\cos\theta$

Let copper loss at full load (i.e., x = 1) $= P_{cu}$

Therefore copper loss at x per unit loading $= x^2 P_{cu}$

Constant core loss $= P_{core}$ (25.2)

(25.3)

Therefore efficiency of the transformer for general loading will become:

$$\eta = \frac{xS\cos\theta}{xS\cos\theta + P_{core} + x^2 P_{cu}}$$

If the power factor of the load (i.e., cos θ) is kept constant and degree of loading of the transformer is varied we get the *efficiency* Vs *degree of loading* curve as shown in the figure 25.5. For a given load power factor, transformer can operate at maximum efficiency at some unique value of loading i.e., *x*. To find out the condition for *maximum efficiency*, the above equation for η can be differentiated with respect to *x* and the result is set to 0. Alternatively, the right hand side of the above equation can be simplified to, by dividing the numerator and the denominator by *x*. the expression for η then becomes:

$$\eta = \frac{S\cos\theta}{S\cos\theta + \frac{P_{core}}{x} + x\,P_{cu}}$$

For efficiency to be maximum, $\frac{d}{dx}$ (Denominator) is set to zero and we get,

$$\text{or } \frac{d}{dx}\left(S\cos\theta + \frac{P_{core}}{x} + x\,P_{cu}\right) = 0$$

$$\text{or} -\frac{P_{core}}{x^2} + P_{cu} = 0$$

$$\text{or } x^2 P_{cu} = P_{core}$$

$$\text{The loading for maximum efficiency, } x = \sqrt{\frac{P_{core}}{P_{cu}}}$$

Thus we see that for a given power factor, transformer will operate at maximum efficiency when it is loaded to $\sqrt{\frac{P_{core}}{P_{cu}}}\, S$ KVA. For transformers intended to be used continuously at its rated KVA, should be designed such that maximum efficiency occurs at $x = 1$. Power transformers fall under this category. However for transformers whose load widely varies over time, it is not desirable to have maximum efficiency at $x = 1$. Distribution transformers fall under this category and the typical value of x for maximum efficiency for such transformers may between 0.75 to 0.8. Figure 25.5 show a family of efficiency Vs. degree of loading curves with power factor as parameter. It can be seen that for any given power factor, maximum efficiency occurs at a loading of $x = \sqrt{\frac{P_{core}}{P_{cu}}}$. Efficiencies η_{max1}, η_{max2} and η_{max3} are respectively the maximum efficiencies corresponding to power factors of unity, 0.8 and 0.7 respectively. It can easily be shown that for a given load (i.e., fixed x), if power factor is allowed to vary then maximum efficiency occurs at unity power factor. Combining the above observations we can say that the efficiency is obtained when the loading of the transformer is $x = \sqrt{\frac{P_{core}}{P_{cu}}}$ and load power factor is unity. Transformer being a static device its efficiency is very high of the order of 98% to even 99%.

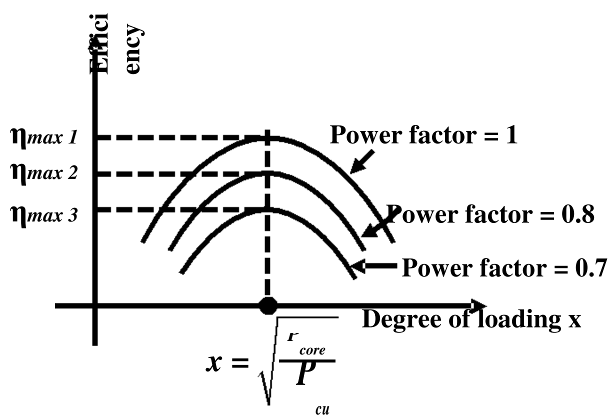


Figure 25.5: Efficiency VS degree of loading curves.

25.3.1 All day efficiency

In the earlier section we have seen that the efficiency of the transformer is dependent upon the degree of loading and the load power factor. The knowledge of this efficiency is useful provided the load and its power factor remains constant throughout.

For example take the case of a *distribution transformer*. The transformers which are used to supply LT consumers (residential, office complex etc.) are called distribution transformers. For obvious reasons, the load on such transformers vary widely over a day or 24 hours. Some times the transformer may be practically under no load condition (say at mid night) or may be over loaded during peak evening hours. Therefore it is not fare to judge efficiency of the transformer calculated at a particular load with a fixed power factor. *All day efficiency*, alternatively called *energy efficiency* is calculated for such transformers to judge how efficient are they. To estimate the efficiency the whole day (24 hours) is broken up into several time blocks over which the load remains constant. The idea is to calculate total amount of energy output in KWH and total amount of energy input in KWH over a complete day and then take the ratio of these two to get the energy efficiency or all day efficiency of the transformer. Energy or All day efficiency of a transformer is defined as:

$$\eta_{\text{all day}} = \frac{\text{Energy output in KWH in 24 hours}}{\text{Energy input in KWH in 24 hours}}$$

$$= \frac{\text{Energy output in KWH in 24 hours}}{\text{Output in KWH in 24 hours + Energy loss in 24 hours}}$$

$$= \frac{\text{Output in KWH in 24 hours}}{\text{Output in KWH in 24 hours + Loss in core in 24 hours + Loss in the Winding in 24 hours}}$$

$$= \frac{\text{Energy output in KWH in 24 hours}}{\text{Energy output in KWH in 24 hours} + 24\,P_{\text{core}} + \text{Energy loss (cu) in the winding in 24 hours}}$$

With primary energized all the time, constant P_{core} loss will always be present no matter what is the degree of loading. However copper loss will have different values for different time blocks as it depends upon the degree of loadings. As pointed out earlier, if P_{cu} is the full load copper loss corresponding to $x = 1$, copper loss at any arbitrary loading x will be $x^2\, P_{cu}$. It is better to make the following table and then calculate $\eta_{all\ day}$.

Time blocks	KVA Loading	Degree of loading x	P.F of load	KWH output	KWH cu loss
T_1 hours	S_1	$x_1 = S_1/S$	$\cos\theta_1$	$S_1 \cos\theta_1 T_1$	$x_1^2\, P_{cu}\, T_1$
T_2 hours	S_2	$x_2 = S_2/S$	$\cos\theta_2$	$S_2 \cos\theta_2 T_2$	$x_2^2\, P_{cu}\, T_2$
…	…	…	…	…	…
T_n hours	S_n	$x_n = S_n/S$	$\cos\theta_n$	$S_n \cos\theta_n T_n$	$x_n^2\, P_{cu}\, T_n$

$$\text{Note that } \sum_{i=1}^{n} T_i = 24$$

$$\text{Energy output in 24 hours} = \sum_{i=1}^{n} S_i \cos\theta_i T_i$$

$$\text{Total energy loss} = 24\,P_{core} + \sum_{i=1}^{n} x_i^2\, P_{cu}\, T_i$$

$$\eta_{allday} = \frac{\sum_{i=1}^{n} S_i \cos\theta_i\, T_i}{\sum_{i=1}^{n} S_i \cos\theta_i T_i + \sum_{i=1}^{n} x_i^2\, P_{cu} T_i + 24\,P_{core}}$$

25.4 Regulation

The output voltage in a transformer will not be maintained constant from no load to the full load condition, for a fixed input voltage in the primary. This is because there will be internal voltage drop in the series leakage impedance of the transformer the magnitude of which will depend upon the degree of loading as well as on the power factor of the load. The knowledge of *regulation* gives us idea about change in the magnitude of the secondary voltage from no load to full load condition at a given power factor. This can be determined experimentally by direct loading of the transformer. To do this, primary is energized with rated voltage and the secondary terminal voltage is recorded in absence of any load and also in presence of full load. Suppose the readings of the voltmeters are respectively V_{20} and V_2. Therefore change in the magnitudes of the secondary voltage is $V_{20} - V_2$. This change is expressed as a percentage of the no load secondary voltage to express regulation. Lower value of regulation will ensure lesser fluctuation of the voltage across the loads. If the transformer were ideal regulation would have been zero.

$$\text{Percentage Regulation, } \%R = \frac{(V_{20} - V_2)}{V_{20}} \cdot 100$$

For a well designed transformer at full load and 0.8 power factor load percentage regulation may lie in the range of 2 to 5%. However, it is often not possible to fully load a large transformer in the laboratory in order to know the value of regulation. Theoretically one can estimate approximately, regulation from the equivalent circuit. For this purpose let us draw the equivalent circuit of the transformer referred to the secondary side and neglect the effect of no load current as shown in the figure 25.6. The corresponding phasor diagram referred to the secondary side is shown in figure 25.7.

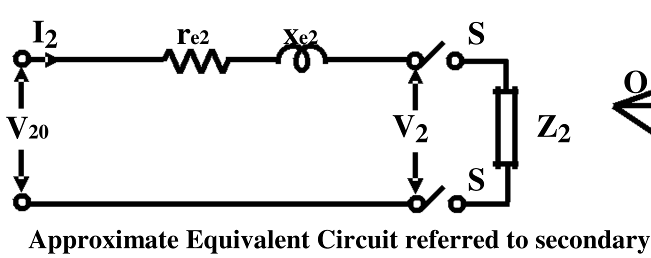


Figure 25.6: Equivalent circuit ref. to secondary.

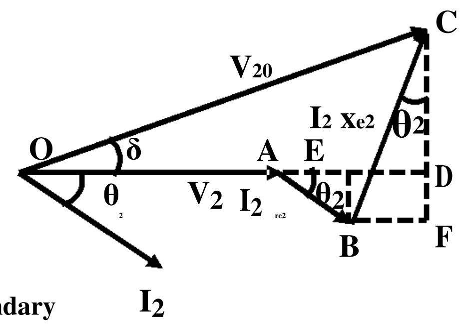


Figure 25.7: Phasor diagram ref. to secondary.

It may be noted that when the transformer is under no load condition (i.e., S is opened), the terminal voltage V_2 is same as V_{20}. However, this two will be different when the switch is closed due to drops in $I_2\, r_{e2}$ and $I_2\, x_{e2}$. For a loaded transformer the phasor diagram is drawn taking terminal voltage V_2 on reference. In the usual fashion I_2 is drawn lagging V_2, by the power factor angle of the load θ_2 and the drops in the equivalent resistance and leakage reactances are added to get the phasor V_{20}. Generally, the resistive drop $I_2\, r_{e2}$ is much smaller than the reactive drop $I_2\, x_{e2}$. It is because of this the angle between OC and OA (δ) is quite small. Therefore as per the definition we can say regulation is

$$R = \left(\frac{V_{20} - V_2}{V_{20}}\right) = \frac{OC - OA}{OC}$$

An approximate expression for regulation can now be easily derived geometrically from the phasor diagram shown in figure 25.7.

$$\begin{aligned}
OC &= OD \text{ since, } \delta \text{ is small} \\
\text{Therefore, } OC - OA &= OD - OA \\
&= AD \\
&= AE + ED \\
&= I_2 r_{e2} \cos\theta_2 + I_2 x_{e2} \sin\theta_2 \\
\text{So per unit regulation, } R &= \frac{OC - OA}{OC} \\
&= \frac{I_2 r_{e2} \cos\theta_2 + I_2 x_{e2} \sin\theta_2}{V_{20}} \\
\text{or, } R &= \frac{I_2 r_{e2}}{V_{20}} \cos\theta_2 + \frac{I_2 x_{e2}}{V_{20}} \sin\theta_2
\end{aligned}$$

It is interesting to note that the above regulation formula was obtained in terms of quantities of secondary side. It is also possible to express regulation in terms of primary quantities as shown below:

$$\text{We know, } R = \frac{I_2 r_{e2}}{V_{20}} \cos\theta_2 + \frac{I_2 x_{e2}}{V_{20}} \sin\theta_2$$

Now multiplying the numerator and denominator of the RHS by a the turns ratio, and further manipulating a bit with a in numerator we get:

$$R = \frac{(I_2/a) a^2 r_{e2}}{aV_{20}} \cos\theta_2 + \frac{(I_2/a) a^2 x_{e2}}{aV_{20}} \sin\theta_2$$

Now remembering, that $(I_2/a) = I_2'$, $a^2 r_{e2} = r_{e1}$, $a^2 x_{e2} = x_{e1}$ and $aV_{20} = V_{20}' = V_1$; we get regulation formula in terms of primary quantity as:

$$\begin{aligned}
R &= \frac{I_2' r_{e1}}{V_{20}'} \cos\theta_2 + \frac{I_2' x_{e1}}{V_{20}'} \sin\theta_2 \\
\text{Or, } R &= \frac{I_2' r_{e1}}{V_1} \cos\theta_2 + \frac{I_2' x_{e1}}{V_1} \sin\theta_2 \\
\text{Neglecting no load current: } R &\approx \frac{I_1 r_{e1}}{V_1} \cos\theta_2 + \frac{I_1 x_{e1}}{V_1} \sin\theta_2
\end{aligned}$$

Thus regulation can be calculated using either primary side quantities or secondary side quantities, since:

$$R = \frac{I_2 r_{e2}}{V_{20}} \cos\theta_2 + \frac{I_2 x_{e2}}{V_{20}} \sin\theta_2 = \frac{I_1 r_{e1}}{V_1} \cos\theta_2 + \frac{I_1 x_{e1}}{V_1} \sin\theta_2$$

Now the quantity $\frac{I_2 r_{e2}}{V_{20}}$, represents what fraction of the secondary no load voltage is dropped in the equivalent winding resistance of the transformer. Similarly the quantity $\frac{I_2 x_{e2}}{V_{20}}$ represents what fraction of the secondary no load voltage is dropped in the equivalent leakage reactance of the transformer. If I_2 is rated curerent, then these quantities are called the per unit resistance and per unit leakage reactance of the transformer and denoted by ${}_r$ and ${}_x$ respectively. The terms ${}_r = \frac{I_{2\,rated}\, r_{e2}}{V_{20}}$ and ${}_x = \frac{I_{2\,rated}\, x_{e2}}{V_{20}}$ are called the per unit resistance and per unit leakage reactance respectively. Similarly, per unit leakage impedance ${}_z$, can be defined.

It can be easily shown that the per unit values can also be calculated in terms of primary quantities as well and the relations are summarised below.

$${}_r = \frac{I_{2\,rated}\, r_{e2}}{V_{20}} = \frac{I_{1rated}\, r_{e1}}{V_1}$$

$${}_x = \frac{I_{2\,rated}\, x_{e2}}{V_{20}} = \frac{I_{1rated}\, x_{e1}}{V_1}$$

$${}_z = \frac{I_{2\,rated}\, z_{e2}}{V_{20}} = \frac{I_{1rated}\, z_{e1}}{V_1}$$

where, $z_{e2} = \sqrt{r_{e2}^2 + x_{e2}^2}$ and $z_{e1} = \sqrt{r_{e1}^2 + x_{e1}^2}$.

It may be noted that the per unit values of resistance and leakage reactance come out to be same irrespective of the sides from which they are calculated. So regulation can now be expressed in a simple form in terms of per unit resistance and leakage reactance as follows.

$$\text{per unit regulation, } R = {}_r \cos\theta_2 + {}_x \sin\theta_2$$

$$\text{and \% regulation } R = \left({}_r \cos\theta_2 + {}_x \sin\theta_2\right) \cdot 100$$

For leading power factor load, regulation may be negative indicating that secondary terminal voltage is more than the no load voltage. A typical plot of *regulation versus power factor* for rated current is shown in figure 25.8.

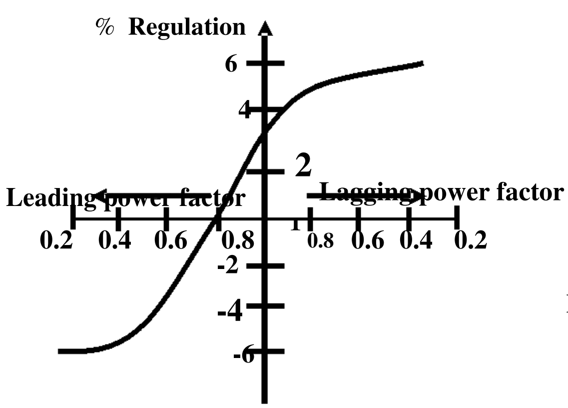


Figure 25.8: Regulation VS power factor curve.

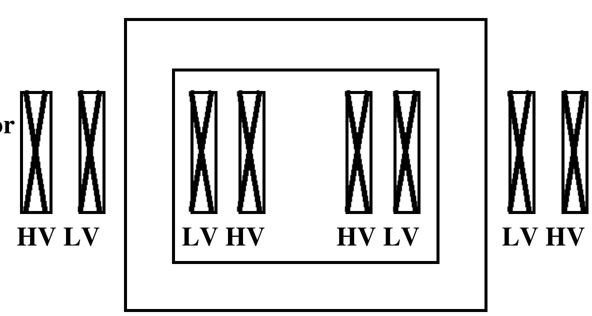


Figure 25.9: LV and HV windings in both the limbs.

To keep the regulation to a prescribed limit of low value, good material (such as copper) should be used to reduce resistance and the primary and secondary windings should be distributed in the limbs in order to reduce leakage flux, hence leakage reactance. The hole LV winding is divided into two equal parts and placed in the two limbs. Similar is the case with the HV windings as shown in figure 25.9.

25.5 Tick the correct answer

1. While carrying out OC test for a 10 kVA, 110 / 220 V, 50 Hz, single phase transformer from LV side at rated voltage, the watt meter reading is found to be 100 W. If the same test is carried out from the HV side at rated voltage, the watt meter reading will be

   (A) 100 W (B) 50 W (C) 200 W (D) 25 W

2. A 20 kVA, 220 V / 110 V, 50 Hz single phase transformer has full load copper loss = 200 W and core loss = 112.5 W. At what kVA and load power factor the transformer should be operated for maximum efficiency?

   (A) 20 kVA & 0.8 power factor (B) 15 kVA & unity power factor

   (C) 20 kVA & unity power factor (D) 15 kVA & 0.8 power factor.

3. A transformer has negligible resistance and has an equivalent *per unit* reactance 0.1. Its voltage *regulation* on full load at 30° leading load power factor angle is:

   (A) +5 %   (B) -5 %   (C) + 10 %   (D) -10 %

4. A transformer operates most efficiently at $\frac{3}{4}$th full load. The ratio of its *iron loss* and *full load copper loss* is given by:

   (A) 16:9   (B) 4:3   (C) 3:4   (D) 9:16

5. Two identical 100 kVA transformer have 150 W iron loss and 150 W of copper loss at rated output. Transformer-1 supplies a constant load of 80 kW at 0.8 power factor lagging throughout 24 hours; while transformer-2 supplies 80 kW at unity power factor for 12 hours and 120 kW at unity power factor for the remaining 12 hours of the day. The *all day efficiency*:

   (A) of transformer-1 will be higher.   (B) of transformer-2 will be higher.

   (B) will be same for both transformers. (D) none of the choices.

6. The current drawn on no load by a single phase transformer is $i_0 = 3\sin(314t - 60°)$ A, when a voltage $v_1 = 300\sin(314t)V$ is applied across the primary. The values of magnetizing current and the *core loss* component current are respectively:

(A) 1.2 A & 1.8 A   (B) 2.6 A & 1.5 A   (C) 1.8 A & 1.2 A   (D) 1.5 A & 2.6 A

7. A 4 kVA, 400 / 200 V single phase transformer has 2 % equivalent resistance. The equivalent resistance referred to the HV side in ohms will be:

   (A) 0.2   (B) 0.8   (C) 1.0   (D) 0.25

8. The % resistance and the % leakage reactance of a 5 kVA, 220 V / 440 V, 50 Hz, single phase transformer are respectively 3 % and 4 %. The voltage to be applied to the HV side, to carry out S.C test at rated current is:

   (A) 11 V   (B) 15.4 V   (C) 22 V   (D) 30.8 V

25.6 Solve the Problems

1. A 30KVA, 6000/230V, 50Hz single phase transformer has HV and LV winding resistances of 10.2Ω and 0.0016Ω respectively. The *equivalent* leakage reactance as referred to HV side is 34Ω. Find the voltage to be applied to the HV side in order to circulate the full load current with LV side short circuited. Also estimate the full load % *regulation* of the transformer at 0.8 lagging power factor.

2. A single phase transformer on open circuit condition gave the following test results:

Applied voltage	Frequency	Power drawn
192 V	40 Hz	39.2 W
288 V	60 Hz	73.2 W

Assuming Steinmetz exponent n = 1.6, find out the *hysteresis* and *eddy current* loss separately if the transformer is supplied with 240 V, 50 Hz.

3. Following are the test results on a 4KVA, 200V/400V, 50Hz single phase transformer. While no load test is carried out from the LV side, the short circuit test is carried out from the HV side.

No load test:	200 V	0.7 A	60 W
Short Circuit Test:	9 V	6 A	21.6 W

Draw the equivalent circuits (i) referred to LV side and then (ii) referred to HV side and insert all the parameter values.

4. The following data were obtained from testing a 48 kVA, 4800/240V, 50 Hz transformer.

O.C test (from LV side):	240 V	2 A	120 W
S.C test (from HV side):	150 V	10 A	600 W

(i) Draw the equivalent circuit referred to the HV side and insert all the parameter values.

(ii) At what kVA the transformer should be operated for *maximum efficiency*? Also calculate the value of maximum efficiency for 0.8 lagging factor load.

Transformer

Problem solving on Transformers

Contents

28.1 Introduction

In this lesson some typical problems on transformer are solved with emphasis on logical steps involved. For a practical two winding transformer, the knowledge of approximate equivalent circuit is of utmost importance in order to predict its performance. Equivalent circuit parameters are either supplied directly or indirectly in terms of O.C and S.C test data. The first problem enumerates in detail how to get the equivalent circuit parameters from test data. The importance of the side (LV or HV) in which calculations are carried out is highlighted. The second problem, in fact, is an extension of the first problem. Calculation of regulation, efficiency and maximum efficiency are dealt with in these problems.

Next few problems highlight the basic calculation steps involved in *ideal* 3-phase transformer and *ideal* auto transformer since the equivalent circuit of these transformers are outside the scope of first year electrical technology course.

28.2 Problems on 2 winding single phase transformers

1. The O.C and S.C test data are given below for a single phase, 5 kVA, 200V/400V, 50Hz transformer.

O.C test from LV side :	200V	1.25A	150W
S.C test from HV side :	20VV	12.5A	175W

Draw the equivalent circuit of the transformer (i) referred to LV side and (ii) referred to HV side inserting all the parameter values.

Solution

Let us represent LV side parameters with suffix 1 and HV side parameters with suffix 2.

Computation with O.C test data

Let us show the test data in the approximate equivalent circuit (Figure 28.1) of the transformer as given below.

Due to the fact that the HV side is open circuited, there will be no current in the branch $r_{e1} + jx_{e1}$.

So entire power of 150W is practically dissipated in R_{cl1}. The no load current $I_{01} = 1.25\ A$ is divided into: magnetizing component I_{m1} and core loss component I_{cl1} as depicted in the phasor diagram figure 28.1.

$$\text{No load (or O.C) power factor } \cos\theta_o = \frac{150}{200 \cdot 1.25} = 0.6$$

$$\theta_o = \cos^{-1} 0.6 = 53.13^\circ$$

$$\text{Hence, } \sin\theta_o = 0.8$$

After knowing the value of $\cos\theta_o$ and $\sin\theta_o$ and referring to the no load phasor diagram, I_{m1} and I_{cl1} can be easily calculated as follows.

$$\begin{aligned}
\text{Magnetizing component } I_{m1} &= I_{01}\sin\theta_o \\
&= 1.25 \cdot 0.8 \\
I_{m1} &= 1A \\
\text{core loss component, } I_{cl1} &= I_{01}\cos\theta_o \\
&= 1.25 \cdot 0.6 \\
I_{cl1} &= 0.75A
\end{aligned}$$

Thus the parallel branch parameters X_{m1} and R_{cl1} can be calculated.

$$\begin{aligned}
\text{Magnetizing reactance } X_{m1} &= \frac{V_1}{I_{m1}} \\
&= \frac{200}{1} \\
X_{m1} &= 200\Omega \\
\text{Resistance representing core loss } R_{cl1} &= \frac{V_1}{I_{cl1}} \\
&= \frac{200}{0.75} \\
,\ R_{cl1} &= 266.67\Omega
\end{aligned}$$

It may be noted that from the O.C test data we can get the parallel branch impedances namely the *magnetizing reactance* and the *resistance representing the core loss* referred to the side where measurements have been taken.

Computation with S.C test data

Since the test has been carried out from the HV side with LV side shorted, we draw the equivalent circuit referred to the HV side as shown in figure 28.2. Parameter values are denoted by using suffix 2. Important point to note here is the absence of the parallel branch. The reason

being, the voltage applied during S.C test is quite low causing a low flux level. Hence magnetizing and core loss component of currents will be pretty small compared to the rated current flowing through $r_{e2} + jx_{e2}$. In this case, power drawn from the supply gets practically dissipated in winding resistances i.e., r_{e2}.

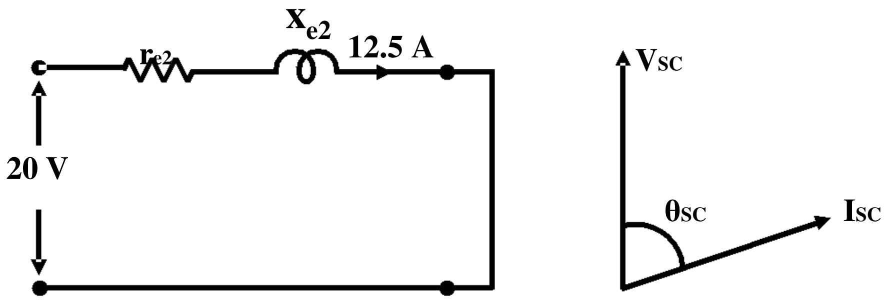


Figure 28.2: O.C equivalent circuit and phasor diagram.

Calculation of series parameters is rather simple and as follows.

$$\text{Power drawn } W_{sc} = I_{sc}^2 r_{e2}$$

$$\text{or, } r_{e2} = \frac{W_{sc}}{I_{sc}^2}$$

$$= \frac{175}{12.5^2}$$

$$r_{e2} = 1.12\Omega$$

$$\text{Now S.C impedance } z_{sc} = \frac{V_{sc}}{I_{sc}}$$

$$= 20/12.5$$

$$z_{sc} = 1.6\Omega = \sqrt{r_{e2}^2 + x_{e2}^2}$$

$$\text{Thus, } x_{e2} = \sqrt{z_{sc}^2 - r_{e2}^2}$$

$$= \sqrt{1.6^2 - 1.12^2}$$

$$x_{e2} = 1.14\Omega$$

Although calculation of parameters from the test results are over, it is *very important* to note that parallel branch parameters have been obtained referred to LV side and series branch parameters have been obtained referred to HV side. However to draw a meaningful equivalent circuit referred to a particular side, *all the parameters are to be represented/calculated referred to that side.*

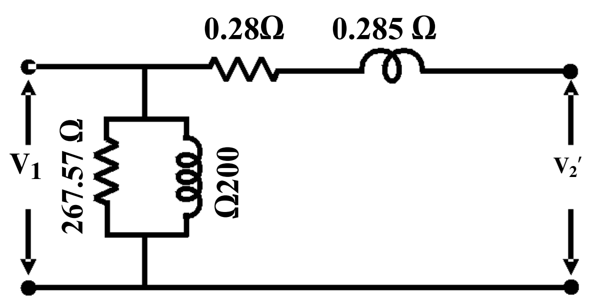
Equivalent circuit referred LV side

The parallel branch parameters R_{cl1} = 266.67Ω and X_{m1} = 200Ω have already been calculated wrt LV side. Naturally no further transformations are necessary. However, series parameters r_{e2} and x_{e2} have been calculated above from test data. So we need to calculate r_{e1} and x_{e1} in order to correctly represent the equivalent circuit referred to primary side.

$$
\begin{aligned}
\text{Turns ratio, } a &= 200/400 = 0.5 \\
\text{but we know, } r_{e1} &= a^2 r_{e2} \\
\text{and } x_{e1} &= a^2 x_{e2} \\
\text{Thus } r_{e1} &= 0.5^2 \cdot 1.12 = 0.28\Omega \\
\text{and } x_{e1} &= 0.5^2 \cdot 1.14 = 0.285\Omega
\end{aligned}
$$

So the equivalent circuit referred to LV side can now be drawn showing all the parameter values as shown below in figure 28.3.

Figure 28.3: Equivalent circuit referred to LV side.



Equivalent circuit referred HV side

Here we note that series parameters referred to HV side are already known to be r_{e2} = 1.12Ω and x_{e2} = 1.14Ω. However, the parallel branch parameters are to be transformed as follows.

$$\begin{aligned} \text{Turns ratio, } a &= 0.5 \\ \text{but we know, } R_{cl2} &= R_{cl1}/a^2 \\ \text{and } X_{m2} &= X_{m1}/a^2 \\ \text{Thus, } R_{cl2} &= 266.67/0.5^2 = 1066.68\Omega \\ \text{and } X_{m2} &= 200/0.5^2 = 800\Omega \end{aligned}$$

We are now in a position to draw the equivalent circuit of the same transformer referred to the HV side as shown in figure 28.4.

After getting the equivalent circuit, regulation, efficiency of the transformer can be predicted under various loading conditions. Solution of the next problem shows how equivalent circuit can be used to predict the performance,

2. For the same transformer (single phase, 5 kVA, 200V/400V, 50 Hz) of problem 1, the equivalent circuit of which is known, calculate the following:

i. the efficiency of the transformer at 75% loading with load power factor = 0.7

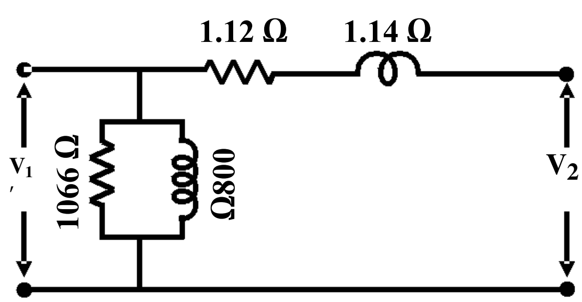


Figure 28.4: Equivalent circuit referred to HV side.

ii. At what load or kVA the transformer is to be operated for maximum efficiency? Also calculate the value of maximum efficiency.

iii. The regulation of the transformer at full load 0.8 power factor lag.

iv. What should be the applied voltage to the LV side when the transformer delivers rated current at 0.7 power factor lagging, at a terminal voltage of 400 V?

Solution

i. From the test data of the previous problem, we have:

$$\begin{aligned} \text{Full load kVA rating, } S &= 5kVA \\ \text{Core loss at rated voltage \& frequency, } P_{core} &= 150\text{W} \\ \text{Full load copper loss, } P_{cu} &= 175\text{W} \end{aligned}$$

$$\text{We know, efficiency, } \eta = \frac{x\,S\cos\theta}{x\,S\cos\theta + P_{core} + x^2 P_{cu}}$$

75% loading means, $x = 0.75$

load power factor, $\cos\theta = 0.7$

$$\eta = \frac{0.75\times5000\times0.7}{0.75\times5000\times0.7 + 150 + 0.75^2\times175}$$

$$= 2625/2873.44$$

% efficiency, η = 91.35%

ii. We know maximum efficiency occurs when $x^2 P_{cu} = P_{core}$, where P_{cu} is the full load copper loss and P_{core} is the iron loss. Now P_{cu} = 175 W and P_{core} = 120 W.

Per unit value of loading for η_{max} is $x = \sqrt{P_{core}/P_{cu}}$

$$= \sqrt{120/175}$$

$$x = 0.83$$

$$\text{Thus the load for } \eta_{max} = x\,S$$

$$= 0.83 \cdot 5\text{kVA}$$

$$\text{the required load for } \eta_{max} = 4.15\text{kVA}$$

iii. To calculate the regulation of the transformer at load current I_2 and load power factor $\cos\theta$, we use the following formula in terms of HV side parameters.

$$\text{Per unit regulation, } R = \frac{I_2 r_{e2} \cos\theta + I_2 x_{e2} \sin\theta}{V_{20}}$$

$$\text{Putting the values, } R = \frac{12.5\times 1.12\times 0.7 + 12.5\times 1.14\times 0.71}{400}$$

$$\%\ \text{regulation, } R = 4.9\%$$

iv. It is interesting to note that the difference between the reflected primary supply voltage magnitude V_1' and the secondary load terminal voltage magnitude V_2 is the numerator of the regulation formula used above.

$$V_1' - V_2 = I_2\, r_{e2} \cos\theta + I_2\, x_{e2} \sin\theta$$

$$\text{or, } V_1' = V_2 + I_2\, r_{e2} \cos\theta + I_2\, x_{e2} \sin\theta$$

$$= 12.5 \cdot 1.12 \cdot 0.7 + 12.5 \cdot 1.14 \cdot 0.71$$

$$= 400 + 19.92\text{V}$$

$$\text{so, } V_1' = 419.92\text{V}$$

Remember V_1' represents the applied voltage to LV calculated in terms of HV side. So the magnitude of the actual voltage to be applied across the primary is:

$$V_1 = a\, V_1'$$

$$= 0.5 \cdot 419.92$$

$$V_1 = 210\ \text{V}$$

28.3 Problems on 3-phase ideal transformer

It may be recalled that one can make a 3-phase transformer by using a bank of three numbers of identical single phase transformers or a single unit of a 3-phase transformers.

1. Three single phase *ideal* transformers, each of rating 5kVA, 200V/400V, 50 Hz are available.

   a) The LV sides are connected in star and HV sides are connected in delta. What line to line 3-phase voltage should be applied and what will be the corresponding HV side line to line voltage will be? Also calculate and show the line and phase current magnitudes in both LV & HV sides corresponding to rated condition.

   b) The LV sides are connected in delta and HV sides are connected in delta. What line to line 3-phase voltage should be applied and what will be the corresponding HV side line to line voltage will be? Also calculate and show the line and phase current magnitudes in both LV & HV sides corresponding to rated condition.

Solution

Here the idea is not to exceed the voltage and current rating of HV and LV coils of each single phase transformer. Now for each transformer having rating 5 kVA, 200V/100V, 50 Hz we have:

$$
\begin{aligned}
\text{Rated voltage of each HV coil is} &= 200\text{V} \\
\text{Rated voltage of each LV coil is} &= 100\text{V} \\
\text{Phase turns ratio is } a_{ph} &= 200/100 = 2 \\
\text{Rated current of each HV coil is} &= 5000/200 = 25\text{A} \\
\text{Rated current of each LV coil is} &= 5000/100 = 50\text{A}
\end{aligned}
$$

Solution of (a)

In this case HV sides are connected in star and LV sides are connected in delta as shown in figure 28.5. Thus line to line voltage to be applied to HV side must not exceed $200\sqrt{3} = 346.4V$. This will ensure that rated voltage is applied across each of the HV coil and rated voltage of 100 V is induced in each of the LV coil. Obviously the available line to line voltage on the LV side will be 100 V since the coils on this side are connected in delta.

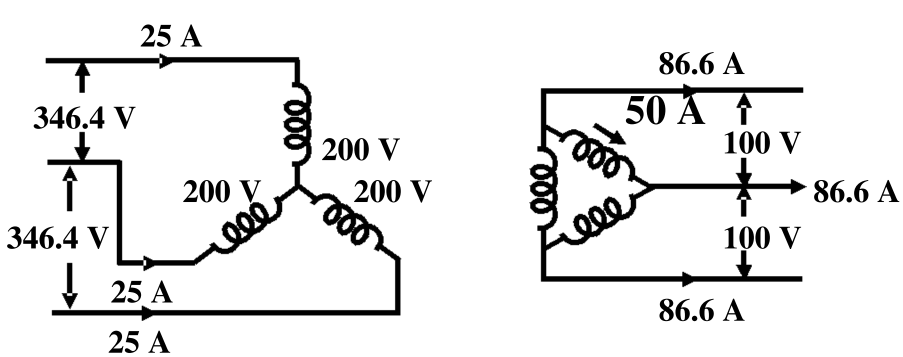


Figure 28.5: Connection of transformers for part (a).

Now the question is how much line current should be allowed to be supplied by the LV side when balanced 3-phase load is connected across it? The constrain is that we should not allow overloading of any of the coils in terms of current as well. Since rated current of each LV side coil is 50 A and the coils are connected in delta, so the corresponding allowed line current in the LV side will be is $50\sqrt{3} = 86.6\ A$ (Note: line current = $\sqrt{3}$ phase current in delta connection).

But we know for any individual ideal transformer if LV coil carries a 50 A current, the corresponding HV coil must carry a current of $50/a_{ph}$ = 25 A as shown in fig 28.5. Thus HV side line current drawn from the supply must be also 25 A as these coils are connected in star (Note: line current = phase current in star connection).

Now we are in a position to calculate the total kVA handled by the bank of 3-phase transformer. Referring to the LV side the transformers supplies 86.6 A line current at a line to line voltage of 100 V. Therefore, total kVA supplied is equal to $\sqrt{3}\ V_{LL}\ I_L = \sqrt{}$[redacted]$\times 100 \times 86.6$ VA = 15 kVA[redacted]. Similarly total kVA drawn from the supply is calculated as $\sqrt{3} \times 346.4 \times 25$ *mbox* VA = 15 kVA . Thus we see the total kVA becomes 3 times the individual kVA rating of the transformers. Since the transformers are assumed to be ideal *Total kVA input = Total kVA output*.

Solution of (b)

In this case HV sides are connected in delta and LV sides are connected in star as shown in figure 28.6. Thus line to line voltage to be applied to HV side must not exceed 200V. This will ensure that rated voltage is applied across each of the HV coil and rated voltage of 100 V is induced in each of the LV coils. The available line to line voltage on the LV side will be $100\sqrt{3} =$ 173.2 V since coils on this side are connected in star.

Since LV coils are connected in star allowed line current to be delivered is 50 A. So total kVA output is $\sqrt{3} \times 173.2 \times 50$ VA = 15 kVA . In each HV coil current has to be 25 A and the corresponding supply line current is $\sqrt{3} \cdot 25 = 43.3A$. Total input kVA is $\sqrt{3} \times 200 \times 43.3$ VA = 15 kVA . Distribution of phase and line currents in LV and HV sides are shown in figure 28.6.

2. Three identical single phase transformers each of rating 5 kVA, 200V/100V, 50Hz are connected in delta-delta. Calculate what line to line voltage to be applied to the HV side? Also find out corresponding LV side line to line voltage. Find out the kVA rating of the bank such that none of the transformers get over loaded.

Solution

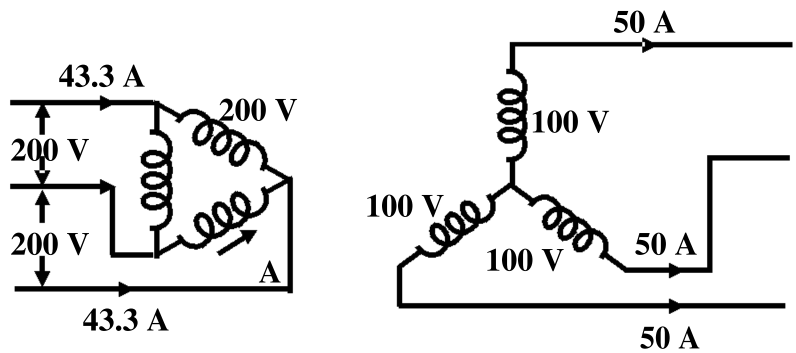


Figure 28.6: Connection of transformers for part (b).

The connection diagram of the delta-delta arrangement is shown in figure 28.7

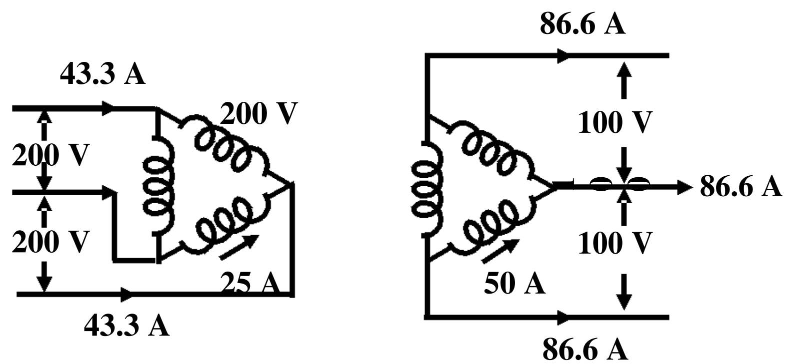


Figure 28.7: Connection of transformers for delta-delta.

As explained in the first two problems, line to line voltage to be applied to the HV side is 200 V because of delta connection. Induced voltage in each coil has to be 100 V in the LV side. Since the LV coils are also connected in delta the line to line voltage on the LV side is 100 V.
Since coil current has to be rated values, line currents on HV and LV sides are obtained as 43.3 A and 86.6 A. Total kVA that can be handled by the bank is $\sqrt{3} \times 200 \times 43.3$ VA $= \sqrt{3} \times 100 \times 86.6$ VA $= 15$ kVA .

3. Two identical transformers each of rating 5 kVA, 200V/100V, 50 Hz transformers are connected in *open delta*. Calculate the kVA rating of the open delta bank when HV side is used as primary.

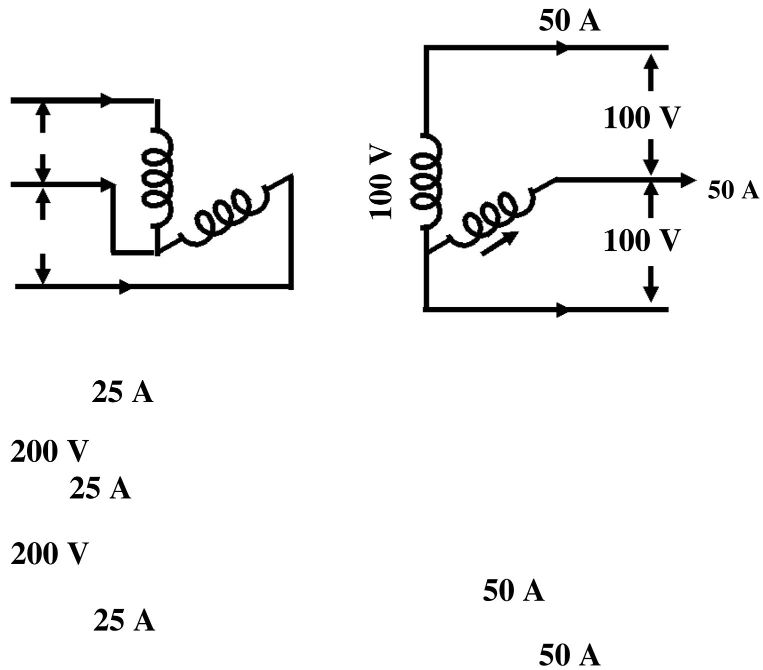


Figure 28.8: Connection of transformers for open delta.

In open delta connection each coil is connected across the lines; therefore, the line to line voltage to be applied to the HV side is 200 V. Induced voltage in the LV coils will be 100 V. Hence line to line voltage in the LV side is 100 V.

A careful look at the circuit in fig 28.8 shows that both HV and LV coils are in series with the lines. Thus if we want the transformers not to be over loaded, line currents on the LV side must be 50 A which automatically fixes the HV side line current to be 25 A.

Let us use $\sqrt{3}\, V_{LL} I_L$ to calculate the kVA handled by the bank of two single phase transformers i.e;

Total kVA = $\sqrt{3} \times 100 \times 50 = \sqrt{3} \times 200 \times 25$ VA = 8.66 kVA

It is interesting to note that in other types of 3-phase connection of transformers such as star-star, star-delta, delta-delta, total kVA handled without overloading any of the transformers is 3 times the individual rating of the transformers. This we learned while solving previous problems where we got the total kVA as 15 kVA $(= 3\times 5 \text{ kVA})$. But in open delta connection where two single phase identical transformers each of rating 5 kVA has been employed we note the total kVA handled is **not** 10 kVA $(= 2\times 5)$kVA but 8.66 kVA only. Thus total kVA available as open delta is only $\frac{8.66}{10} \times 100 = 86.6\%$ of the installed capacity.

4. A 3-phase, 500 kVA, 6000V/400V, 50Hz, delta-star connected transformer is delivering 300 kW, at 0.8 pf lagging to a balanced 3-phase load connected to the LV side with HV side supplied from 6000 V, 3- phase supply. Calculate the line and winding currents in both the sides. Assume the transformer to be ideal.

Solution

First note that it is not a bank of single phase transformers. In fact it is a single unit of 3-phase transformer with the name plate rating as 500 kVA, 6000 V/400 V, 50Hz, delta-star connected 3-phase transformer. 500 kVA represents the *total* kVA and voltages specified are always line to line. Similarly unless otherwise specified, kW rating of a 3-phase load is the total kW absorbed by the load. The connection diagram is shown in figure 28.9.

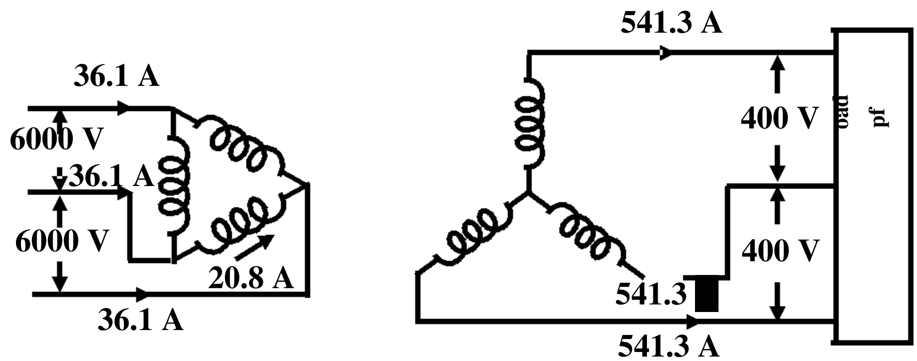


Figure 28.9: Connection diagram with 3-phase load.

Noting the relation kVA, $S = P / \cos\theta$ and $I = S / \sqrt{3}\, V_{LL}$ let us start out calculation.

$$
\begin{aligned}
\text{Load kVA} &= 300/0.8 = 375 \text{ kVA} = \text{input kVA} \\
\text{Line current drawn by the load, } I_{2L} &= 375000/\sqrt{3} \cdot 400 \\
I_{2L} &= 541.3\ A \\
\text{Because of star connection, LV coil current} &= 541.3\ A \\
\text{since input kVA} &= 375 \text{ kVA} \\
\text{HV side line current, } I_{1L} &= \frac{375000}{\sqrt{3} \cdot 6000} \\
I_{1L} &= 36.1\ A
\end{aligned}
$$

Actual phase winding currents can also be calculated as:

$$
\begin{aligned}
\text{LV side phase coil current} &= \text{LV side line current} \\
\text{or, } I_{2ph} &= I_{2L} \\
I_{2ph} &= 541.3 \text{ A due to star connection.} \\
\text{HV side phase coil current} &= \text{LV side line current} / \sqrt{3} \\
\text{or, } I_{1ph} &= I_{1L}/\sqrt{3} \\
I_{1ph} &= 36.1/\sqrt{3} = 20.8\ A \text{ due to delta connection.}
\end{aligned}
$$

28.4 Problems on ideal auto transformers

Recall that an auto transformer essentially is essentially a single winding transformer with a portion of the winding common to both supply and the load side. In contrast to a two winding transformer it can not provide isolation between HV and LV side. Here VA is transferred from one side to the other not only by magnetic coupling but also by electrical conduction. Autotransformer becomes cheaper than a similarly rated two winding transformer when the voltage transformation ratio is close to unity. A single phase two winding transformer can be suitably connected to perform like an auto transformer.

1. A 5kVA, 200 V/ 100 V, 50 Hz, single phase ideal two winding transformer is to used to step up a voltage of 200 V to 300 V by connecting it like an auto transformer. Show the connection diagram to achieve this. Calculate the maximum kVA that can be handled by the autotransformer (without over loading any of the HV and LV coil). How much of this kVA is transferred magnetically and how much is transferred by electrical conduction.

Solution

Two connect a two winding transformer as an auto transformer, it is essential to know the dot markings on the two coils. The coils are to be now series connected appropriately so as to identify clearly between which two terminals to give supply and between which two to connect the load. Since the input voltage here is 200 V, supply must be connected across the HV terminals. The induced voltage in the LV side in turn gets fixed to 100 V. But we require 300 V as output, so LV coil is to be connected in additive series with the HV coil. This is what has been shown in figure 28.10.

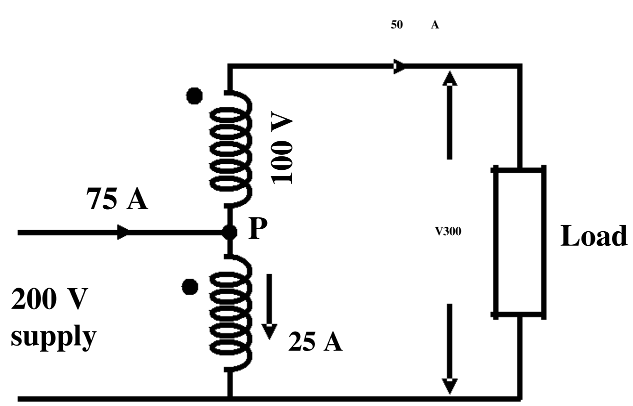


Figure 28.10: Two winding transformer as an autotransformer.

Here the idea is not to exceed the voltage and current rating of HV and LV coils of the two winding transformer. Now for the transformer having rating 5 kVA, 200 V/ 100 V, 50 Hz we have:

$$\begin{aligned}
\text{Rated voltage of HV coil is} &= 200\text{ V} \\
\text{Rated voltage of LV coil is} &= 100\text{ V} \\
\text{Phase turns ratio is } a &= 200/100 = 2 \\
\text{Rated current of each HV coil is} &= 5000/200 = 25\text{ A} \\
\text{Rated current of each LV coil is} &= 5000/100 = 50\text{ A}
\end{aligned}$$

Since the load is in series with LV coil, so load current is same as the current flowing through the LV coil. Thus a maximum of 50 A can be drawn by the load otherwise overloading of the coils take place.

$$\text{Output kVA} = 300 \cdot 50 \text{ VA} = 15 \text{ kVA}$$

$$\text{input kVA} = \text{Output kVA} = 15 \text{ kVA}$$

$\because$ transformer is ideal

$$\text{Current drawn form the supply} = 15000/200 = 75 \text{ A}$$

Now the question is now much current is flowing in the HV coil and in which direction? However, this is quite easy since supply and load currents are already known along with their directions as shown in figure 28.10. Applying KCl at the junction P, we get:

$$\text{Current through HV coil } I_{HV} = 75 - 50 = 25 \text{ A}$$

The direction of I_{HV} is obviously from top to bottom. No matter whether a two winding transformer is used as a two winding transformer or as an autotransformer, mmf must be balanced in the coils. If current comes out through the dot terminal in the LV coil, current must flow in through the dot of the HV coil.

It is important to note that as a two winding transformer, kVA handling capacity is 5 kVA, the rating of the transformer. However, the same transformer when connected as auto transformer, kVA handling capacity becomes 15 kVA without overloading any of the coils.

$$\begin{aligned} \text{kVA transferred magnetically} &= \text{kVA of either HV or LV coil} \\ &= 200 \cdot 25 \text{ VA} = 100 \cdot 50 \text{ VA} = 5 \text{ kVA} \end{aligned}$$

$$\text{kVA transferred magnetically} = 5kVA$$

$$\begin{aligned} \text{kVA transferred electrically} &= \text{total kVA transferred} - \text{kVA transferred magnetically} \\ &= 15 - 5 = 10 \text{ kVA} \end{aligned}$$

2. An autotransformer has a coil with total number of turns $N_{CD} = 200$ between terminals C and D. It has got one tapping at A such that $N_{AC} = 100$ and another tapping at B such that $N_{BA} = 50$.
   Calculate currents in various parts of the circuit and show their directions when 400 V supply is connected across AC and two resistive loads of 60Ω & 40Ω are connected across BC and DC respectively.

Solution

Let us first draw the circuit diagram (shown in figure 28.11) as per data given in the problem. First let us calculate the voltages applied across the loads remembering the fact that voltage per turn in a transformer remains constant.

$$\begin{aligned} \text{Supply voltage across AC, } V_{AC} &= 400\text{V} \\ \text{Number of turns between A \& C } N_{AC} &= 100 \\ \text{Voltage per turn} &= 400/100 = 4\text{V} \\ \text{Voltage across the } 40\Omega \text{ load} &= N_{DC} \cdot \text{Voltage per turn} \\ &= 200 \cdot 4 = 800\text{V} \\ \text{So, current through } 40\Omega &= 800/40 = 20\text{A} \\ \text{Voltage across the } 60\Omega\ load &= N_{BC} \cdot \text{Voltage per turn} \end{aligned}$$

$$= \quad 150 \cdot 4 = 600 \text{ V}$$

$$\text{So, current through } 60\Omega \quad = \quad 600/60 = 10\text{A} \tag{28.1}$$

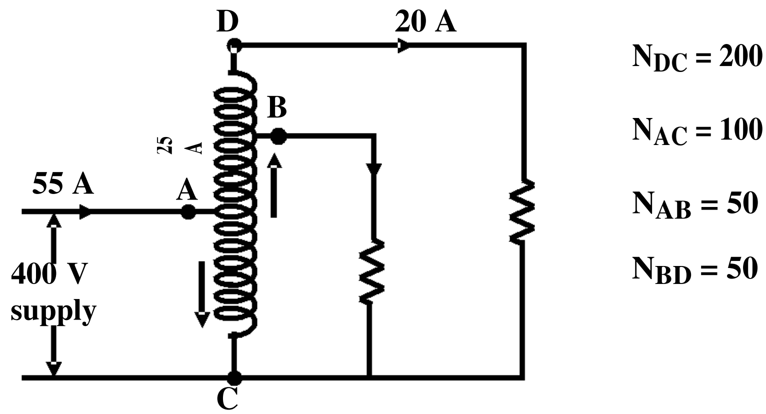


Figure 28.11: Circuit arrangement.

Total output kVA will be the simple addition of the kVAs supplied to the loads i,e.,

$$(600 \cdot 10 + 800 \cdot 20) \text{ VA} = 22000 \text{ VA} = 22 \text{ kVA}$$

Assuming the autotransformer to be ideal, input kVA must also be 22 kVA. We are therefore in a position to calculate the current drawn from the supply.

$$\text{Current drawn from the supply} = 22000/400 = 55 \text{ A}$$

Now we know all the load currents and the current drawn from the supply. Current calculations in different parts of the transformer winding becomes pretty simple-one has to apply KCL at the tap points B and A.

$$\begin{aligned}
\text{Current in DB part of the winding } I_{BD} &= 20 \text{ A} \\
\text{Applying KCL at B, current in AB part } I_{AB} &= 20 + 10 = 30 \text{ A} \\
\text{Applying KCL at A, current in AC part } I_{AC} &= 55 - 30 = 25 \text{ A}
\end{aligned}$$

It is suggested to repeat the problem if 40Ω resistor is replaced by an impedance (30 + j40) Ω other things remaining unchanged.

Magnetic Circuits and Core Losses

Eddy Current & Hysteresis Loss

Contents

Chapter 22

Eddy Current & Hysteresis Losses (Lesson 22)

22.1 Lesson goals

In this lesson we shall show that (i) a time varying field will cause *eddy currents* to be induced in the core causing power loss and (ii) *hysteresis* effect of the material also causes additional power loss called *hysteresis loss*. The effect of both the losses will make the core hotter. We must see that these two losses, (together called *core loss*) are kept to a minimum in order to increase efficiency of the apparatus such as transformers & rotating machines, where the core of the magnetic circuit is subjected to time varying field. If we want to minimize something we must know the origin and factors on which that something depends. In the following sections we first discuss *eddy current phenomenon* and then the phenomenon of *hysteresis*.

Finally expressions for (i) inductance, (ii) stored energy density in a magnetic field and (iii) force between parallel faces across the air gap of a magnetic circuit are derived.
Key Words: Hysteresis loss; hysteresis loop; eddy current loss; Faraday's laws;

After going through this section students will be able to answer the following questions. After going through this lesson, students are expected to have clear ideas of the following:

1. Reasons for core losses.
2. That core loss is sum of hysteresis and eddy current losses.
3. Factors on which hysteresis loss depends.
4. Factors on which eddy current loss depends.

5. Effects of these losses on the performance of magnetic circuit.
6. How to reduce these losses?
7. Energy storing capability in a magnetic circuit.
8. Force acting between the parallel faces of iron separated by air gap.
9. Iron cored inductance and the factors on which its value depends.

22.2 Introduction

While discussing magnetic circuit in the previous lesson (no. 21) we assumed the exciting current to be constant d.c. We also came to know how to calculate flux (φ) or flux density (B) in the core for a constant exciting current. When the exciting current is a function of time, it is expected that flux (φ) or flux density (B) will be functions of time too, since φ produced depends on i. In addition if the current is also alternating in nature then both the magnitude of the flux and its direction will change in time. The magnetic material is now therefore subjected to a time varying field instead of steady constant field with d.c excitation. Let:

$$\text{The exciting current } i(t) = I_{max} \sin \omega t$$

$$\text{Assuming linearity, flux density } B(t) = \mu_0 \mu_r H(t)$$

$$= \mu_0 \mu_r \frac{Ni}{l}$$

$$= \mu_0 \mu_r \frac{N I_{max} \sin \omega t}{l}$$

$$\therefore \quad B(t) = B_{max} \sin \omega t$$

22.2.1 Voltage induced in a stationary coil placed in a time varying field

If normal to the area of a coil, a time varying field $\varphi(t)$ exists as in figure 22.1, then an emf is induced in the coil. This emf will appear across the free ends 1 & 2 of the coil. Whenever we talk about some voltage or emf, two things are important, namely the magnitude of the voltage and its polarity. Faraday's law tells us about the both. Mathematically it is written as $e(t) = -N\frac{d\varphi}{dt}$

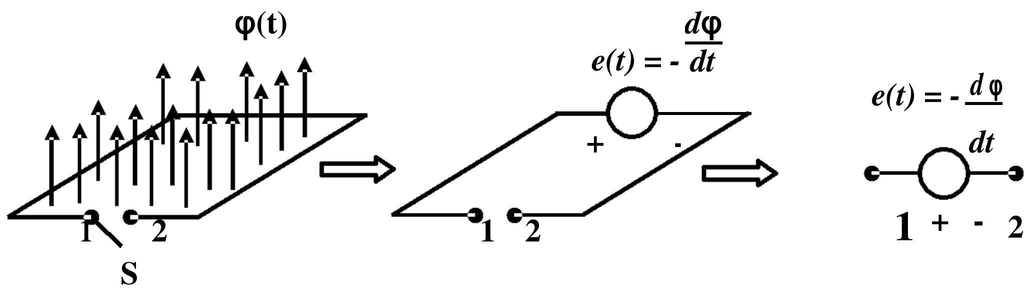


Figure 22.1:

Let us try to understand the implication of this equation a bit deeply. $\varphi(t)$ is to be taken normal to the surface of the coil. But a surface has two *normals*; one in the upward direction and the other in downward direction for the coil shown in the figure. Which one to take? The choice is entirely ours. In this case we have chosen the normal along the upward direction. This direction is obtained if you start your journey from the terminal-2 and reach the terminal-1 in the anticlockwise direction along the contour of the coil. Once the direction of the normal is chosen what we have to do is to express $\varphi(t)$ along the same direction. Then calculate $N\frac{d\varphi}{dt}$ and put a – ve sign before it. The result obtained will give you e_{12} i.e., potential of terminal -1 wrt terminal-2. In other words, the whole coil can be considered to be a source of emf wrt terminals 1 & 2 with polarity as indicated. If at any time flux is increasing with time in the upward direction, $\frac{d\varphi}{dt}$ is + ve and e_{12} will come out to be – ve as well at that time. On the other hand, at any time flux is decreasing with time in the upward direction, $\frac{d\varphi}{dt}$ is – ve and e_{12} will come out to be + ve as well at that time. Mathematically let:

$$
\begin{aligned}
\text{Flux density } B(t) &= \quad \omega \\
\text{Area of the coil} &= A \\
\text{Flux crossing the area } \varphi(t) &= B(t)\,A \\
&= \quad \omega \\
&= \varphi_{max} \sin \omega t \\
\text{Induced voltage in the coil } e_{12} &= -N\frac{d\varphi}{dt} \\
&= -1\times\frac{d\varphi}{dt} \quad \because N = 1 \text{ here} \\
&= \varphi_{max}\omega \cos \omega t \\
\therefore e_{12} &= E_{max} \cos \omega t \\
\text{RMS value of } e_{12}\ E &= \frac{\varphi_{max}\omega}{\sqrt{2}} \\
\therefore E &= \sqrt{2}\pi f\, \varphi_{max} \quad \text{putting } \omega = 2\pi f
\end{aligned}
$$

If the switch S is closed, this voltage will drive a circulating current i_c in the coil the direction of which will be such so as to oppose the cause for which it is due. Correct instantaneous polarity of the induced voltage and the direction of the current in the coil are shown in figure 22.2, for different time intervals with the switch S closed. In the interval $0 < \omega t < \frac{\pi}{2}$, $\frac{d\varphi}{dt}$ is + ve

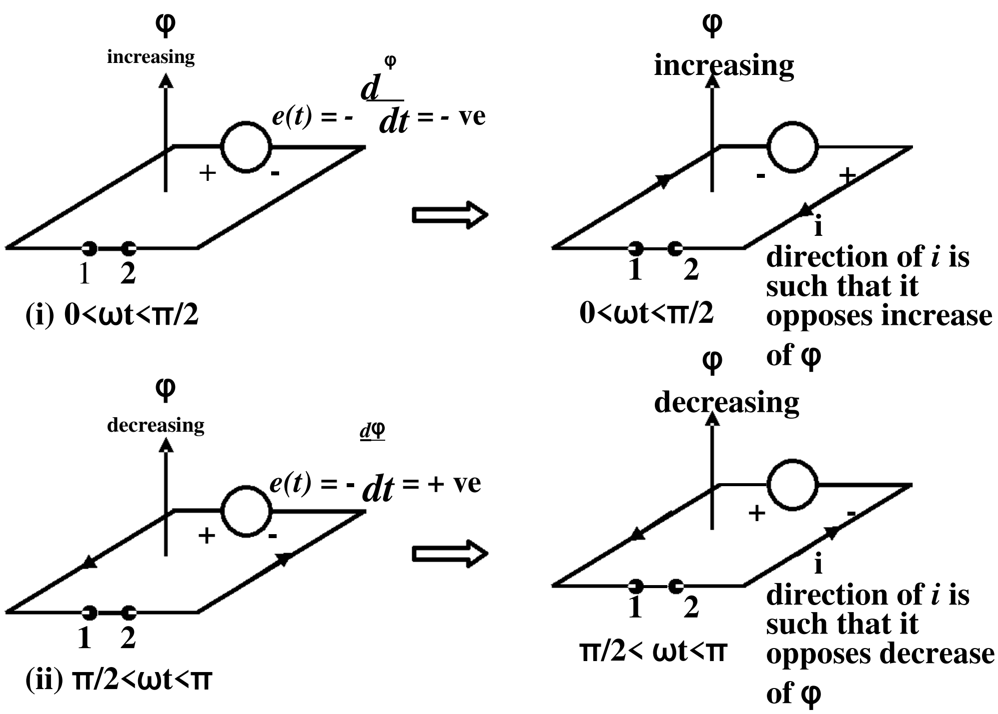


Figure 22.2: Direction of induced current.

22.2.2 Eddy current

Look at the Figure 22.3 where a rectangular core of magnetic material is shown along with the exciting coil wrapped around it. Without any loss of generality, one may consider this to be a part of a magnetic circuit. If the coil is excited from a sinusoidal source, exciting current flowing will be sinusoidal too. Now put your attention to any of the cross section of the core and imagine any arbitrary rectangular closed path *abcd*. An emf will be induced in the path *abcd* following Faraday's law. Here of course we don't require a switch S to close the path because the path is closed by itself by the conducting magnetic material (say iron). Therefore a circulating current i_{eddy} will result. The direction of i_{eddy} is shown at the instant when $B(t)$ is increasing with time. It is important to note here that to calculate induced voltage in the path, the value of flux to be taken is the flux enclosed by the path i.e., $\varphi_{max} = B_{max} \times$ area of the loop abcd. The magnitude of the eddy current will be limited by the *path resistance* , R_{path} neglecting reactance effect. Eddy current will therefore cause power loss in R_{path} and heating of the core. To calculate the total eddy current loss in the material we have to add all the power losses of different eddy paths covering the whole cross section.

22.2.3 Use of *thin* plates or *laminations* for core

We must see that the power loss due to eddy current is minimized so that heating of the core is reduced and efficiency of the machine or the apparatus is increased. It is obvious if the cross sectional area of the eddy path is reduced then eddy voltage induced too will be reduced ($E_{eddy} \propto$ area), hence eddy loss will be less. This can be achieved by using several thin electrically insulated plates (called laminations) stacked together to form the core instead a solid block of iron. The idea is depicted in the Figure 22.4 where the plates have been shown for clarity, rather separated from each other. While assembling the core the *laminations* are kept closely pact. Conclusion is that solid block of iron should not be

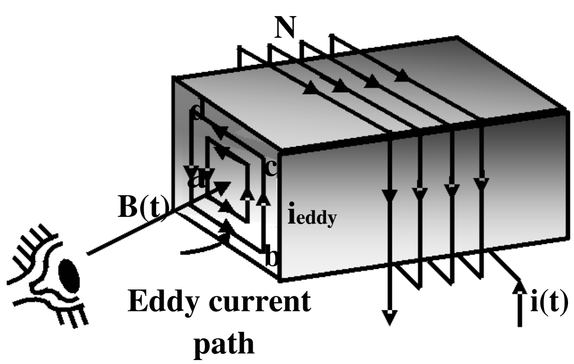


Figure 22.3: Eddy current paths

used to construct the core when exciting current will be ac. However, if exciting current is dc, the core need not be laminated.

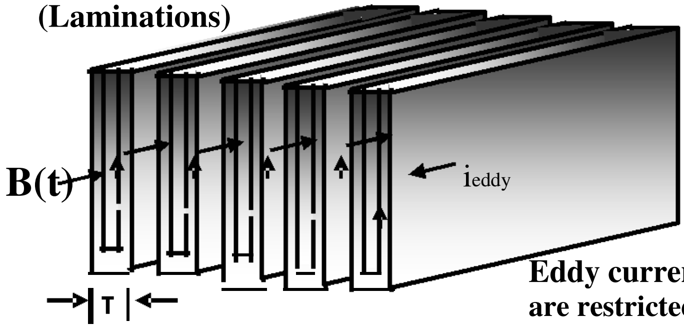


Figure 22.4: Laminated core to reduce eddy loss.

22.2.4 Derivation of an expression for eddy current loss in a thin plate

From physical consideration we have seen that *thin* plates each of thickness τ, are to be used to reduce eddy loss. With this in mind we shall try to derive an approximate expression for eddy loss in the following section for a thin plate and try to identify the factors on which it will depend. Section of a thin plate τ << *L* and *h* is shown in the plane of the screen in Figure 22.5.

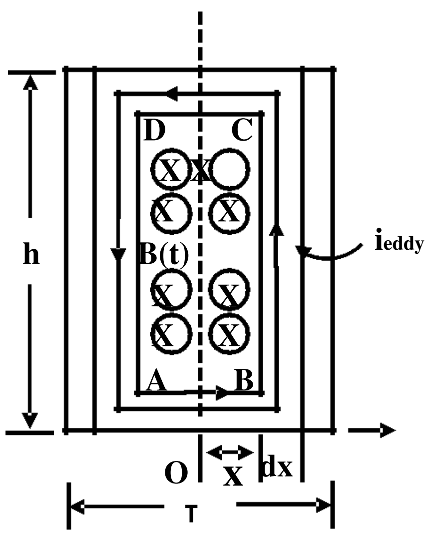


Figure 22.5: Elemental eddy current path.

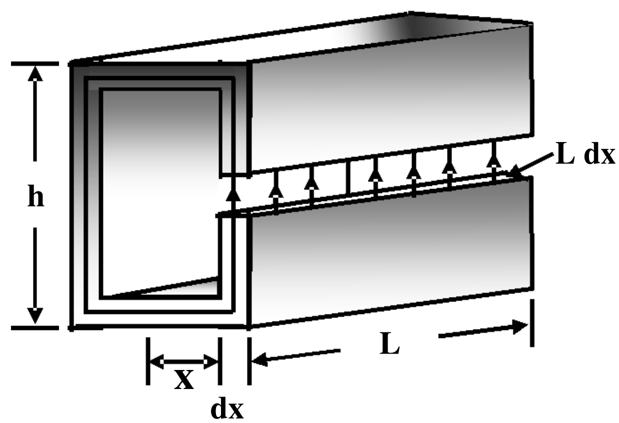


Figure 22.6: Section of the elemental eddy current path.

Eddy current loss is essentially I^2R loss occurring inside the core. The current is caused by the induced voltage in any conceivable closed path due to the time varying field as shown in the diagram 22.5.

Let us consider a thin magnetic plate of length L, height h and thickness τ such that τ is very small compared to both L and h. Also let us assume a sinusoidally time varying field $b = B_{max} \sin\omega t$ exists perpendicular to the rectangular area formed by τ and h as shown in figure 22.5.

Let us consider a small elemental rectangular closed path ABCDA of thickness dx and at a distance x from the origin. The loop may be considered to be a single coil through which time varying flux is crossing. So there will be induced voltage in it, in similar manner as voltage is induced in a coil of single turn shown in the previous section. Now,

$$\text{Area of the loop ABCD} = 2hx$$

$$\text{Flux crossing the loop} = B_{max}\, 2hx \sin \omega t$$

$$\text{RMS voltage induced in the loop, } E = \sqrt{2}\pi f B_{max}\, 2hx$$

$$\text{Resistance of the path through which eddy current flows, } R_{path} = \frac{\rho\left(2h + 4x\right)}{L\, dx}$$

To derive an expression for the eddy current loss in the plate, we shall first calculator the power loss in the elemental strip and then integrate suitably to for total loss. Power loss in the loop dP is given by:

$$
\begin{aligned}
dP &= \frac{E^2}{R_{path}} \\
&= \frac{E^2 L\, dx}{\rho\left(2h+4x\right)} \\
&= \frac{E^2 L\, dx}{\rho 2h} \quad since\ \tau << h \\
\text{Total eddy current loss, } P_{eddy} &= \frac{4\pi^2 B_{max}^2 f^2 hL}{\rho} \int_{x=0}^{\frac{\tau}{2}} x^2\, dx \\
&= \frac{\pi^2 f^2 B_{max}^2 \tau^2}{6\rho}\left(hL\tau\right) \\
\text{Volume of the thin plate} &= hL\tau \\
\text{Eddy loss per unit volume, boldmath } P_{eddy} &= \frac{\pi^2 f^2 B_{max}^2 \tau^2}{6\rho} \\
\text{or, } Peddy &= k_e f^2 B_{max}^2 \tau^2
\end{aligned}
$$

Thus we find eddy current loss per unit volume of the material directly depends upon the *square* of the *frequency, flux density and thickness* of the plate. Also it is inversely proportional to the *resistivity* of the material. The core of the material is constructed using thin plates called laminations. Each plate is given a *varnish coating* for providing necessary insulation between the plates. *Cold Rolled Grain Oriented*, in short CRGO sheets are used to make transformer core.

22.3 Hysteresis Loss

22.3.1 Unidirectional time varying exciting current

Consider a magnetic circuit with constant (d.c) excitation current I_0. Flux established will have fixed value with a fixed direction. Suppose this final current I_0 has been attained from zero current slowly by energizing the coil from a potential divider arrangement as depicted in Figure 22.7. Let us also assume that initially the core was not magnetized. The exciting current therefore becomes a function of time till it reached the desired current I and we stopped further increasing it. The flux too naturally will be function of time and cause induced voltage e_{12} in the coil with a polarity to oppose the increase of inflow of current as shown. The coil becomes a source of emf with terminal-1, +ve and terminal-2, -ve. Recall that a source in which current enters through its +ve terminal absorbs power or energy while it delivers power or energy when current comes out of the +ve terminal. Therefore during the interval when $i(t)$ is increasing the coil absorbs energy. Is it possible to know how much energy does the coil absorb when current is increased from 0 to I_0? This is possible if we have the B-H curve of the material with us.

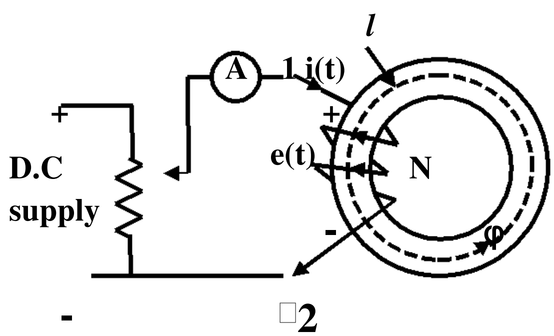


Figure 22.7:

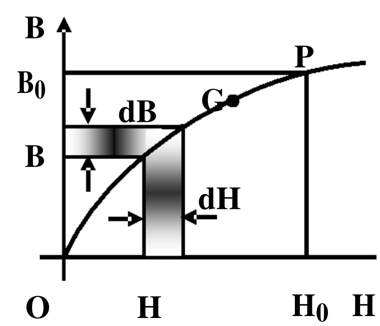


Figure 22.8:

22.3.2 Energy stored, energy returned & energy density

Let:

i = current at time t
H = field intensity corresponding to i at time t
B = flux density corresponding to i at time t

(22.1)

Let an infinitely small time dt elapses so that new values become:

$$i + di = \text{Current at time } t + dt$$
$$H + dH = \text{Field intensity corresponding to } i + di \text{ at time } t + dt$$
$$B + dB = \text{Flux density corresponding to } i + di \text{ at time } t + dt$$

Voltage induced in the coil $e_{12} = N\dfrac{d\varphi}{dt}$

$$= NA \frac{dB}{dt}$$

Power absorbed at t $= e_{12} i$

$$= NA \frac{dB}{dt} i$$

$$= A\, l\, H \frac{dB}{dt} \quad \text{noting, } H = \frac{Ni}{l}$$

Energy absorbed in time dt, dW $= A\, l\, H \frac{dB}{dt} i \cdot dt$

$$= A\, l\, H\, dB$$

total energy absorbed per unit volume, *W* $= \boldsymbol{\int_0^{B_0} H\, dB}$

Graphically therefore, the closed area $OKPB_0BO$ is a measure of the energy stored by the field in the core when current is increased from 0 to I_0. What happens if now current is gradually reduced back to 0 from I_0? The operating point on B- H curve does not trace back the same path when current was increasing from 0 to I_0. In fact, B- H curve (PHT) remains above during decreasing current with respect the B-H curve (OGP) during increasing current as shown in figure 22.9. This lack of retracing the same path of the curve is called *hysteresis*. The portion OGP should be used for increasing current while the portion (PHT) should be used for decreasing current. When the current is brought back to zero external applied field H becomes zero and the material is left magnetized with a residual field OT. Now the question is when the exciting current is decreasing, does the coil absorb or return the energy back to supply. In this case $\frac{dB}{dt}$ being –ve, the induced voltage reverses its polarity although direction of i remains same. In other words, current leaves from the +ve terminal of the induced voltage thereby returning power back to the supply. Proceeding in the same fashion as adopted for increasing current, it can be shown that the area PMTRP represents amount of energy returned per unit volume. Obviously energy absorbed during rising current from 0 to I_0 is more than the energy returned during lowering of current from I_0 to 0. The balance of the energy then must have been lost as heat in the core.

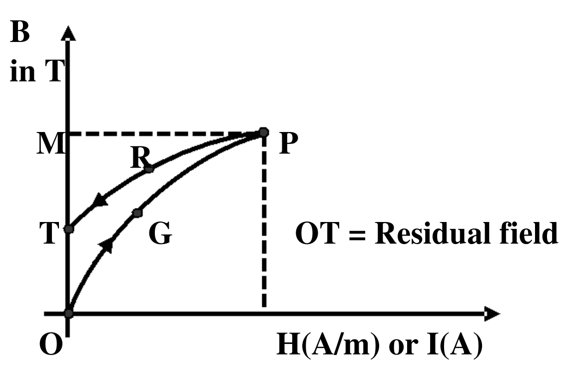


Figure 22.9:

22.4 Hysteresis loop with alternating exciting current

In the light of the above discussion, let us see how the operating point is traced out if the exciting current is $i = I_{max} \sin \omega t$. The nature of the current variation in a complete cycle can be enumerated as follows:

$$\text{In the interval } 0 \leq \omega t \leq \frac{\pi}{2} \quad : i \text{ is +ve and } \frac{di}{dt} \text{ is +ve.}$$

$$\text{In the interval } \frac{\pi}{2} \leq \omega t \leq \pi \quad : i \text{ is +ve and } \frac{di}{dt} \text{ is –ve.}$$

$$\text{In the interval } \pi \leq \omega t \leq \frac{3\pi}{2} \quad : i \text{ is –ve and } \frac{di}{dt} \text{ is –ve.}$$

$$\text{In the interval } \frac{3\pi}{2} \leq \omega t \leq 2\pi \quad : i \text{ is –ve and } \frac{di}{dt} \text{ is +ve.}$$

Let the core had no residual field when the coil is excited by $i = I_{max} \sin \omega t$. In the interval $0 < \omega t < \frac{\pi}{2}$, B will rise along the path OGP. Operating point at P corresponds to $+I_{max}$ or $+H_{max}$. For the interval $\frac{\pi}{2} < \omega t < \pi$ operating moves along the path PRT. At point T, current is zero. However, due to sinusoidal current, i starts increasing in the –ve direction as shown in the Figure 22.10 and operating point moves along TSEQ. It may be noted that a –ve H of value OS is necessary to bring the residual field to zero at S. OS is called the *coercivity* of the material. At the end of the interval $\pi < \omega t < \frac{3\pi}{2}$, current reaches $-I_{max}$ or field $-H_{max}$. In the next internal, $\frac{3\pi}{2} < \omega t < 2\pi$, current changes from $-I_{max}$ to zero and operating point moves from M to N along the path MN. After this a new cycle of current variation begins and the operating point now never enters into the path OGP. The movement of the operating point can be described by two paths namely: (i) QFMNKP for increasing current from $-I_{max}$ to $+I_{max}$ and (ii) from $+I_{max}$ to $-I_{max}$ along PRTSEQ.

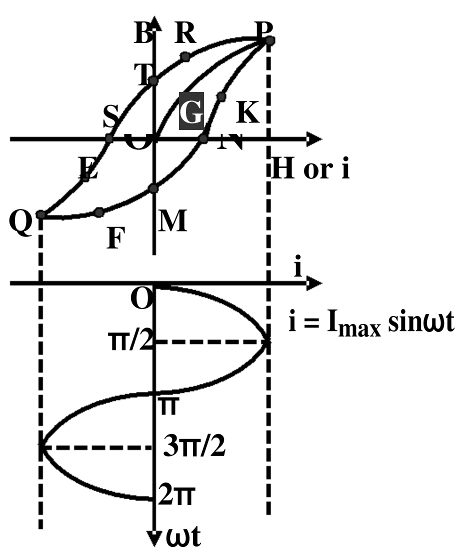


Figure 22.10: B-H loop with sinusoidal current.

22.4.1 Hysteresis loss & loop area

In other words the operating point trace the *perimeter* of the closed area QFMNKPRTSEQ. This area is called the B-H loop of the material. We will now show that the area enclosed by the loop is the *hysteresis loss* per unit volume per cycle variation of the current. In the interval $0 \le \omega t \le \pi/2$, i is +ve and $\frac{di}{dt}$ is also +ve, moving the operating point from M to P along the path MNKP. Energy *absorbed* during this interval is given by the shaded area MNKPLTM shown in Figure 22.11 (i).

In the interval $\pi/2 \le \omega t \le \pi$, i is +ve but $\frac{di}{dt}$ is –ve, moving the operating point from P to T along the path PRT. Energy *returned* during this interval is given by the shaded area PLTRP shown in Figure 22.11 (ii). Thus during the +ve half cycle of current variation net amount of energy absorbed is given by the shaded area MNKPRTM which is nothing but half the area of the loop.

In the interval $\pi \le \omega t \le 3\pi/2$, i is –ve and $\frac{di}{dt}$ is also –ve, moving the operating point from T to Q along the path TSEQ. Energy *absorbed* during this interval is given by the shaded area QJMTSEQ shown in Figure 22.11 (iii).

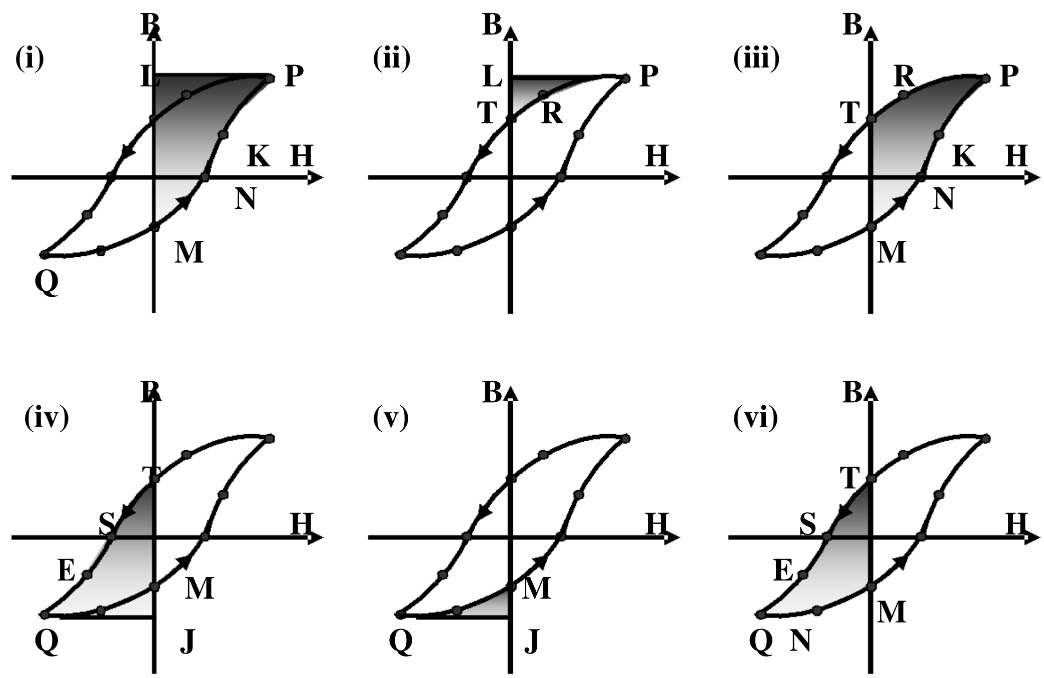


Figure 22.11: B-H loop with sinusoidal current.

In the interval $3\pi/2 \le \omega t \le 2\pi$, i is –ve but $\frac{di}{dt}$ is + ve, moving the operating point from Q to M along the path QEM. Energy *returned* during this interval is given by the shaded area QJMFQ shown in Figure 22.11 (iv).

Thus during the –ve half cycle of current variation net amount of energy absorbed is given by the shaded area QFMTSEQ which is nothing but the other half the loop area.

Therefore total area enclosed by the B-H loop is the measure of the hysteresis loss per unit volume per unit cycle. To reduce hysteresis loss one has to use a core material for which area enclosed will be as small as possible.

Steinmetz's empirical formula for hysteresis loss

Based on results obtained by experiments with different ferromagnetic materials with sinusoidal currents, Charles Steimetz *proposed* the empirical formula for calculating hysteresis loss analytically.

$$\text{Hysteresis loss per unit volume, } P_h = k_h \, f B_{\max}{}^n$$

Where, the coefficient k_h depends on the material and n, known as Steinmetz exponent, may vary from 1.5 to 2.5. For iron it may be taken as 1.6.

22.5 Seperation of core loss

The sum of hyteresis and eddy current losses is called core loss as both the losses occur within the core (magnetic material). For a given magnetic circuit with a core of ferromagnetic material, volume and thickness of the plates are constant and the total core loss can be expressed as follows.

$$\text{Core loss} = \text{Hysteresis loss} + \text{Eddy current loss}$$
$$P_{core} = K_h \, f B_{\max}{}^n + K_e \, f^2 \, B_{\max}{}^2$$

It is rather easier to measure the core loss with the help of a wattmeter (W) by energizing the N turn coil from a sinusoidal voltage of known frequency as shown in figure 22.12.

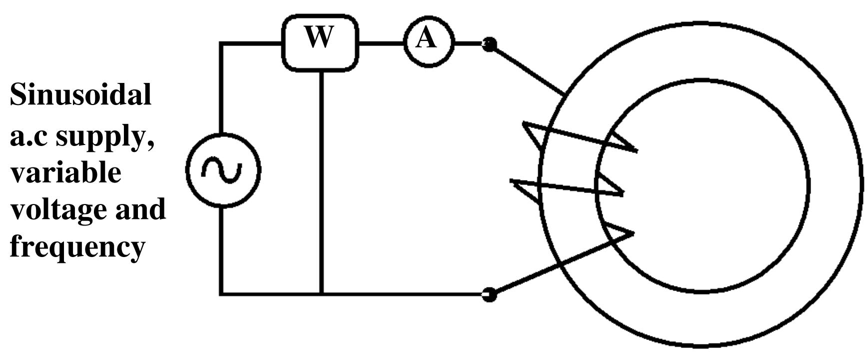


Figure 22.12: Core loss measurement.

Let A be the cross sectional area of the core and let winding resistance of the coil be negligibly small (which is usually the case), then equating the applied rms voltage to the induced rms voltage of the coil we get:

$$V \approx \sqrt{2}\pi f \varphi_{max} N$$

$$\text{Or, } V = \sqrt{2}\pi f B_{max} AN$$

$$\text{So, } B_{max} = \frac{V}{\sqrt{2}\pi f AN}$$

$$\therefore B_{max} \propto \frac{V}{f}$$

The above result i.e., $B_{max} \propto \frac{V}{f}$ is important because it tells us that to keep B_{max} constant at rated value at lower frequency of operation, applied voltage should be proportionately decreased. In fact, from the knowledge of N (number of turns of the coil) and A (cross sectional area of the core), V (supply voltage) and f (supply frequency) one can estimate the maximum value of the flux density from the relation $B_{max} = \frac{V}{\sqrt{2}\pi f AN}$. This point has been further discussed in the future lesson on transformers.

Now coming back to the problem of separation of core loss into its components: we note that there are three unknowns, namely K_h, K_e and n (Steinmetz's exponent) to be determined in the equation $P_{core} = K_h f B_{max}^n + K_e f^2 B_{max}^2$. LHS of this equation is nothing but the wattmeter reading of the experimental set up shown in Figure 22.12. Therefore, by noting down the wattmeter readings corresponding to three different applied voltages and frequencies, we can have three independent algebraic equations to solve for K_h, K_e and n. However, to simplify the steps in solving of the equations two readings may be taken at same flux density (keeping $\frac{V}{f}$ ratio constant) and the third one at different flux density. To understand this, solve the following problem and verify the answers given.

For a magnetic circuit, following results are obtained.

Frequency	B_{max}	Core loss
50 Hz	1.2 T	115 W
30 Hz	1.2 T	60.36 W
30 Hz	1.4 T	87.24 W

Estimate the constants, K_h, K_e and n and separate the core loss into hysteresis and eddy losses at the above frequencies and flux densities.

The answer of the problem is:

Frequency	B_{max}	Core loss	Hyst loss	Eddy loss
50 Hz	1.2 T	115 W	79 W	36 W
30 Hz	1.2 T	60.36 W	47.4 W	12.96 W
30 Hz	1.4 T	87.24 W	69.6 W	17.64 W

22.6 Inductor

One can make an inductor *L*, by having several turns *N*, wound over a core as shown in figure 22.13. In an *ideal* inductor, as we all know, no power loss takes place. Therefore, we must use a very good magnetic material having negligible B-H loop area. Also we must see that the operating point lies in the linear zone of the B-H characteristic in order to get a constant value of the inductance. This means μ_r may be assumed to be constant. To make eddy current loss vanishingly small, let us assume the lamination thickness is extremely small and the core material has a very high resistivity ρ. Under these assumptions let us derive an expression for the inductance *L*, in order to have a feeling on the factors it will depend upon. Let us recall that inductance of a coil is defined as the flux linkage with the coil when 1 A flows through it.

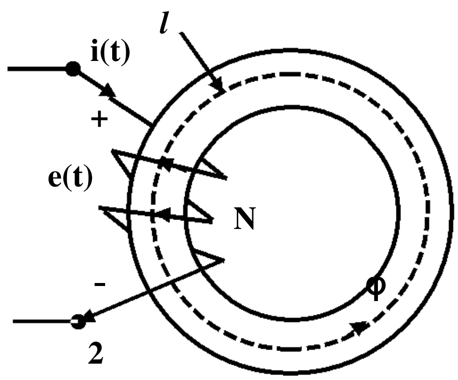


Figure 22.13: An inductor.

Let φ be the flux produced when *i* A flows through the coil. Then by definition:

$$\text{Total flux linkage} = N\varphi$$

$$\therefore \text{inductance is } L = \frac{N\varphi}{i} \text{ by definition.}$$

$$= \frac{N B A}{i} \quad \because \varphi = B \times A$$

$$= \frac{N \mu_0 \mu_r H A}{i} \quad \because B = \mu_0 \mu_r H$$

$$= \mu_0 \mu_r \frac{N H A}{i}$$

$$= \mu_0 \mu_r \frac{N \frac{Ni}{l} A}{i} \quad \text{putting } H = \frac{Ni}{l}$$

$$\textbf{Finally, } L = \mu_0 \mu_r \frac{N^2 A}{l}$$

The above equation relates *inductance* with the dimensions of the magnetic circuit, number of turns and permeability of the core in the similar way as we relate resistance of a wire, with the dimensions of the wire and the resistivity (recall, $R = \rho \frac{\ell}{A}$). It is important to note that L is directly proportional to the square of the number of turns, directly proportional to the sectional area of the core, directly proportional to the permeability of the core and inversely proportional to the mean length of the flux path. In absence of any core loss and linearity of B-H characteristic, Energy stored during increasing current from 0 to I is exactly equal to the energy returned during decreasing current from I to 0. From our earlier studies we know for increasing current:

$$\text{Voltage induced in the coil } e = N A \frac{dB}{dt}$$

$$\text{Energy absorbed in time } dt \text{ is } dW = e\, i\, dt = NA \frac{dB}{dt} i\, dt$$

$$= N A \frac{dB}{dt} i\, dt$$

$$= N\, i\, A\, d B$$

$$= A\, l\, H\, d B$$

$$\text{Energy absorbed to reach } I \text{ or } B = A\, l \int_0^B H\, dB$$

$$= A\, l \int_0^B \frac{B}{\mu_0 \mu_r} dB$$

$$= A\, l \frac{B^2}{2 \mu_0 \mu_r}$$

$$\textbf{Energy stored per unit volume} = \frac{B^2}{2 \mu_0 \mu_r}$$

By expressing B in terms of current, I in the above equation one can get a more familiar expression for energy stored in an inductor as follows:

$$\text{Energy absorbed to reach } I \text{ or } B = A\, l \frac{B^2}{2 \mu_0 \mu_r}$$

$$= A\, l \frac{(\mu_0 \mu_r H)^2}{2 \mu_0 \mu_r}$$

$$= A\, l \frac{\mu_0 \mu_r H^2}{2}$$

$$= A\, l \mu_0 \mu_r \frac{(N I)^2}{2 l^2}$$

$$= \frac{1}{2} \frac{\mu_0 \mu_r A N^2}{l} I^2$$

$$\therefore \textbf{Energy stored in the inductor} = \tfrac{1}{2} L I^2$$

22.7 Force between two opposite faces of the core across an air gap

In a magnetic circuit involving *air gap*, magnetic force will exist between the parallel faces across the air gap of length x. The situation is shown for a magnetic circuit in figure 22.14 (i) . Direction of the lines of forces will be in the clockwise direction and the left face will become a north pole and the right face will become a south pole. Naturally there will be force of *attraction* F_a between the faces. Except for the fact that this force will develop stress in the core, no physical movement is possible as the structure is *rigid*.

$$\text{Let the flux density in the air gap be} = B$$

$$\text{energy stored per unit volume in the gap} = \frac{B^2}{2\mu_0}$$

$$\text{gap volume} = A\,x$$

$$\text{Total energy stored} = \frac{B^2}{2\mu_0} \times \text{gap volume}$$

$$= \frac{B^2}{2\mu_0} A\,x \qquad (22.2)$$

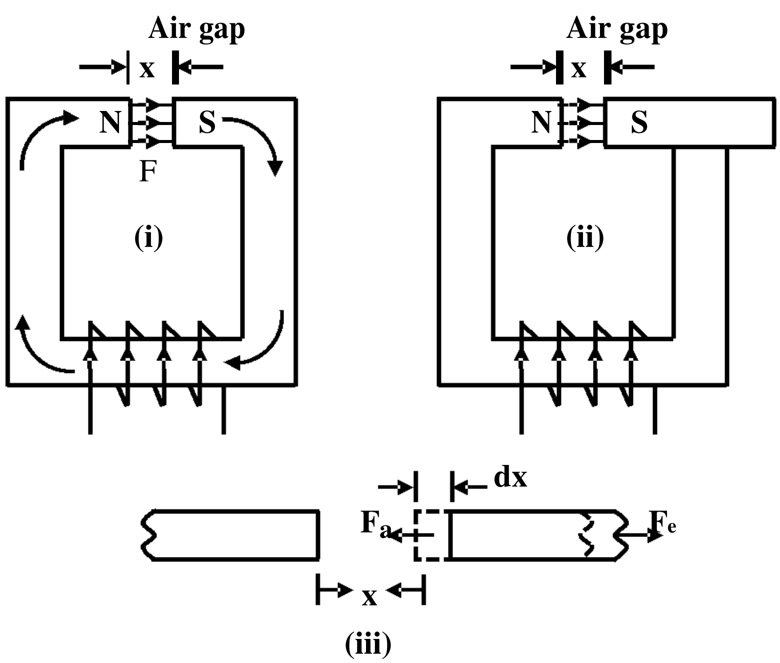


Figure 22.14: Force between parallel faces.

Easiest way to derive expression for F_a is to apply law of conservation of energy by using the concept of *virtual work*. To do this, let us *imagine* that right face belongs to a freely moving structure with initial gap x as in figure 22.14 (ii). At this gap x, we have find F_a. Obviously if we want to displace the moving structure by an elemental distance dx to the right, we have apply a force F_e toward right. As dx is very small tending to 0, we can assume B to remain unchanged. The magnitude of this external force F_e has to be same as the prevailing force of attraction F_a between the faces. Where does the energy expended by the external agency go? It will go to *increase* the energy stored in the gap as its volume increase by $A\,dx$. Figure 22.14 (iii) shows an expanded view of the gap portion for clarity. Let us put it in mathematical steps as follows:

energy stored per unit volume in the gap $= \dfrac{B^2}{2\mu_0}$

initial gap volume $= A\,x$

Total energy stored, $W_x = \dfrac{B^2}{2\mu_0} \times$ gap volume

$W_x = \dfrac{B^2}{2\mu_0} A\,x$

let the external force applied be $= F_e$

let the force of attraction be as $= F_a$

explained above, F_e work done $= F_a$

by external agency $= F_e\,dx = F_a\,dx$

increase in the volume of the gap $= A\,(x + dx) - Ax = A\,dx$

increase in stored energy $= \dfrac{B^2}{2\mu_0} A\,dx$

but work done by external agency = increase in stored energy

$F_a\,dx = \dfrac{B^2}{2\mu_0} A\,dx$

or, desired force of attraction $\boldsymbol{F_a} = \dfrac{\boldsymbol{B^2}}{\boldsymbol{2\mu_0}} \boldsymbol{A}$

22.8 Tick the correct answer

1. If the number of turns of a coil wound over a core is halved, the inductance of the coil will become:

   (A) doubled. (B) halved. (C) quadrapuled. (D) ¼ th

2. The expression for eddy current loss per unit volume in a thin ferromagnetic plate of thickness τ is:

   (A) $\frac{1}{6\rho}\pi^2 f^2 B_{max}^2 \tau^2$ (B) $\frac{\rho}{6}\pi^2 f^2 B_{max}^2 \tau^2$

   (C) $\frac{1}{6\rho}\pi^2 f^2 B_{max} \tau^2$ (D) $\frac{1}{6\rho}\pi^2 f B_{max}^2 \tau^2$

3. As suggested by Steinmetz, hyteresis loss in a ferromagnetic material is proportional to:

   (A) $f^n B_{max}$ (B) $f B^n_{max}$

   (C) $f^2 B_{max}^2$ (D) $f B_{max}{}^n$

   where, n may very between 1.5 to 2.5 depending upon material.

4. The eddy current loss in a magnetic circuit is found to be 100 W when the exciting coil is energized by 200 V, 50 Hz source. If the coil is supplied with 180 V, 54 Hz instead, the *eddy current loss* will become

   (A) 90 W (B) 81 W (C) 108 W (D) 50 W

5. A magnetic circuit draws a certain amount of alternating sinusoidal exciting current producing a certain amount of alternating flux in the core. If an air gap is introduced in the core path, the exciting current will:

   (A) increase. (B) remain same. (C) decrease. (D) vanish

22.9 Solve the following

1. The area of the hysteresis loop of a 1200 cm^3 ferromagnetic material is 0.9 cm^2 with B_{max} = 1.5 T. The scale factors are 1cm ≡ 10A/m along x-axis and 1 cm = 0.8T along y-axis. Find the power loss in watts due to hysteresis if this material is subjected to an 50 Hz alternating flux density with a peak value 1.5 T.

2. Calculate the core loss per kg in a specimen of alloy steel for a maximum density of 1.1 T and a frequency of 50 Hz, using 0.4 mm plates. Resistivity ρ is 24 $\mu\Omega$-cm; density is 7.75 g/cm^3; hysteresis loss 355 J/m^3 per cycle.

3. (a) A linear magnetic circuit has a mean flux length of 100 cm and uniform cross sectional area of 25 cm^2. A coil of 100 turns is wound over it and carries a current of 0.5 A. If relative permeability of the core is 1000, calculate the inductance of the coil and energy stored in the coil.

   (b) In the magnetic circuit of part (a), if an air gap of 2 mm length is introduced calculate (i) the energy stored in the air gap (ii) energy stored in core and (iii) force acting between the faces of the core across the gap for the same coil current.

4. An iron ring with a mean diameter of 35 cm and a cross section of 17.5 cm^2 has 110 turns of wire of negligible resistance. (a) What voltage must be applied to the coil at 50 Hz to obtain a maximum flux density of 1.2 T; the excitation required corresponding to this density 450 AT/m? Find also the inductance. (b) What is the effect of introducing a 2 mm air gap?

5. A coil wound over a core, is designed for 200 V (rms), 50 Hz such that the operating point is on the knee of the B-H characteristic of the core. At this rated voltage and frequency the value of the exciting current is found to be 1 A. Give your comments on the existing current if the coil is energized from:

   (a) 100 V, 25 Hz supply.
   (b) 200 V, 25 Hz supply.

UNIT-3

Transformer

Three Phase Transformer

Contents

26.1 Goals of the lesson

Three phase system has been adopted in modern power system to generate, transmit and distribute power all over the world. In this lesson, we shall first discuss how three number of *single phase transformers* can be connected for 3-phase system requiring change of voltage level. Then we shall take up the construction of a 3-phase transformer as a single unit. *Name plate rating* of a three phase transformer is explained. Some basic connections of a 3-phase transformer along with the idea of *vector grouping* is introduced.

Key Words: bank of three phase transformer, vector group.

After going through this section students will be able to answer the following questions.

- Point out one important advantage of connecting a *bank* of 3-phase transformer.
- Point out one disadvantage of connecting a *bank* of 3-phase transformer.
- Is it possible to transform a 3-phase voltage, to another level of 3-phase voltage by using two identical single phase transformers? If yes, comment on the total kVA rating obtainable.
- From the name plate rating of a 3-phase transformer, how can you get individual coil rating of both HV and LV side?
- How to connect successfully 3 coils in delta in a transformer?

26.2 Three phase transformer

It is the three phase system which has been adopted world over to generate, transmit and distribute electrical power. Therefore to change the level of voltages in the system three phase transformers should be used.

Three number of identical single phase transformers can be suitably connected for use in a three phase system and such a three phase transformer is called a *bank of three phase transformer*. Alternatively, a three phase transformer can be constructed as a single unit.

26.3 Introducing basic ideas

In a single phase transformer, we have only two coils namely primary and secondary. Primary is energized with single phase supply and load is connected across the secondary. However, in a 3-phase transformer there will be 3 numbers of primary coils and 3 numbers of secondary coils. So these 3 primary coils and the three secondary coils are to be properly connected so that the voltage level of a balanced 3-phase supply may be changed to another 3-phase balanced system of different voltage level.

Suppose you take three identical transformers each of rating 10 kVA, 200 V / 100 V, 50 Hz and to distinguish them call them as A, B and C. For transformer-A, primary terminals are marked as A_1 A_2 and the secondary terminals are marked as a_1 a_2. The markings are done in such a way that A_1 and a_1 represent the dot (•) terminals. Similarly terminals for B and C transformers are marked and shown in figure 26.1.

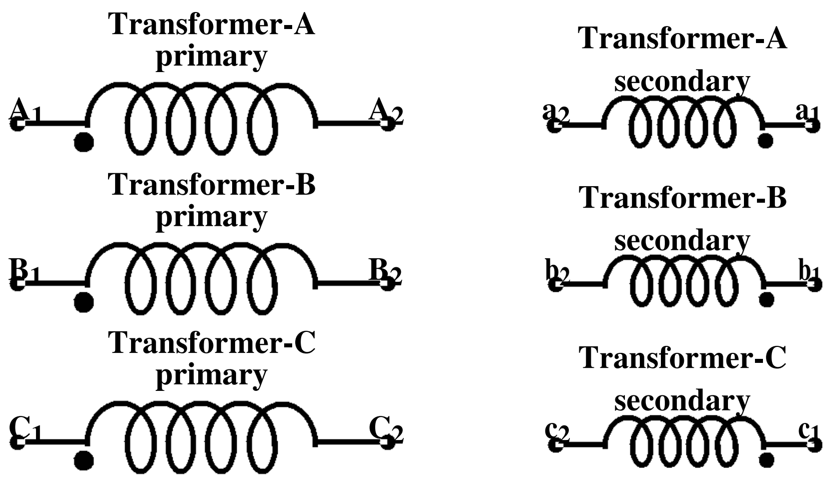


Figure 26.1: Terminal markings along with dots

It may be noted that individually each transformer will work following the rules of single phase transformer i.e, induced voltage in a_1a_2 will be in phase with applied voltage across A_1A_2 and the ratio of magnitude of voltages and currents will be as usual decided by a where $a = N_1/N_2$ = 2/1, the turns ratio. This will be true for transformer-B and transformer-C as well i.e., induced voltage in b_1b_2 will be in phase with applied voltage across B_1B_2 and induced voltage in c_1c_2 will be in phase with applied voltage across C_1C_2.

Now let us join the terminals A_2, B_2 and C_2 of the 3 primary coils of the transformers and no inter connections are made between the secondary coils of the transformers. Now to the free terminals A_1, B_1 and C_1 a balanced 3-phase supply with phase sequence A-B-C is connected as shown in figure 26.2. Primary is said to be connected in star.

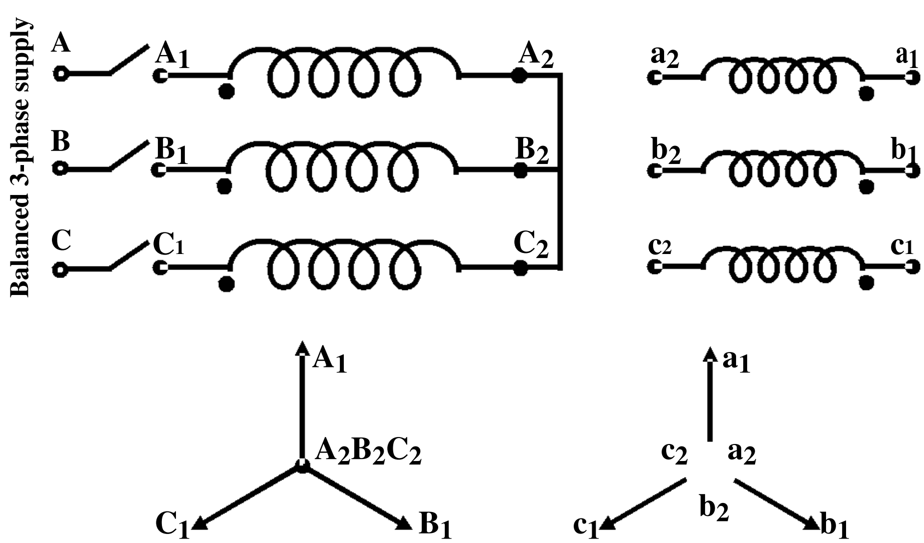


Figure 26.2: Star connected primar y with secondary coils left alone.

If the line voltage of the supply is $V_{LL} = 200\sqrt{3}$ V , the magnitude of the voltage impressed across each of the primary coils will be $\sqrt{3}$ times less i.e., 200 V. However, the phasors $\overline{V}_{A_1A_2}$,

$\overline{V}_{B_1B_2}$ and $\overline{V}_{C_1C_2}$ will be have a mutual phase difference of 120º as shown in figure 26.2. Then from the fundamental principle of single phase transformer we know, secondary coil voltage $\overline{V}_{a_1a_2}$ will be parallel to $\overline{V}_{A_1A_2}$; $\overline{V}_{b_1b_2}$ will be parallel to $\overline{V}_{B_1B_2}$ and $\overline{V}_{c_1c_2}$ will be parallel to $\overline{V}_{C_1C_2}$. Thus the secondary induced voltage phasors will have same magnitude i.e., 100 V but are displaced by 120º mutually. The secondary coil voltage phasors $\overline{V}_{a_1a_2}$, $\overline{V}_{b_1b_2}$ and $\overline{V}_{c_1c_2}$ are shown in figure 26.2.
Since the secondary coils are not interconnected, the secondary voltage phasors too have been shown independent without any interconnections between them. In contrast, the terminals A_2, B_2 and C_2 are physically joined forcing them to be equipotential which has been reflected in the primary coil voltage phasors as well where phasor points A_2, B_2 and C_2 are also shown joined. Coming back to secondary, if a voltmeter is connected across any coil i.e., between a_1 and a_2 or between b_1 and b_2 or between c_1 and c_2 it will read 100 V. However, voltmeter will not read anything if connected between a_1 and b_1 or between b_1 and c_1 or between c_1 and a_1 as open circuit exist in the paths due to no physical connections between the coils.

Imagine now the secondary coil terminals a_2, b_2 and c_2 are joined together physically as shown in figure 26.3. So the secondary coil phasors should not be shown isolated as a_2, b_2 and c_2 become equipotential due to shorting of these terminals. Thus, the secondary coil voltage phasors should not only be parallel to the respective primary coil voltages but also a_2, b_2 and c_2 should be equipotential. Therefore, shift and place the phasors $\overline{V}_{a_1a_2}$, $\overline{V}_{b_1b_2}$ and $\overline{V}_{c_1c_2}$ in such a way that they remain parallel to the respective primary coil voltages and the points a_2, b_2 and c_2 are superposed.

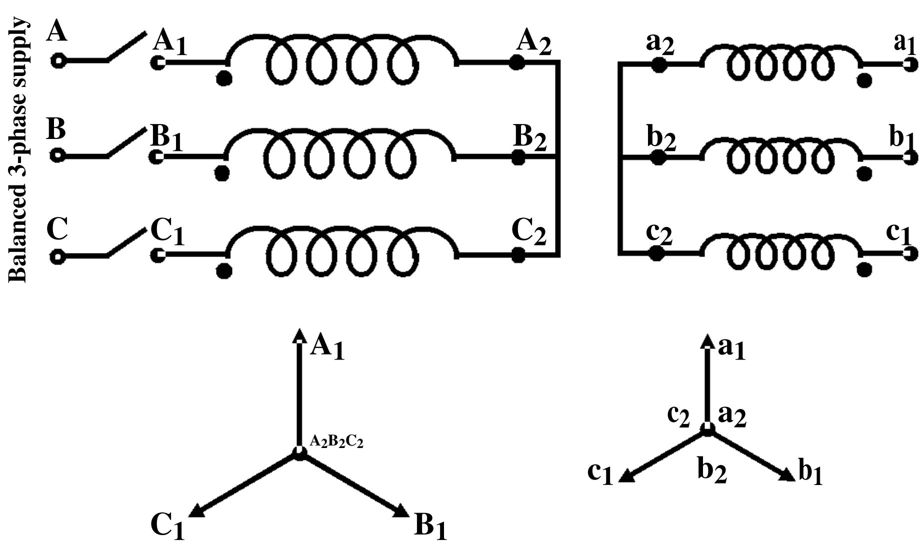


Figure 26.3: Both primary & secondary are star connected.

Here obviously, if a voltmeter is connected between a_1 and b_1 or between b_1 and c_1 or between c_1 and a_1 it will read corresponding phasor lengths a_1b_1 or b_1c_1 or c_1a_1 which are all equal to 200 $\sqrt{3}$ V. Thus, $\overline{V}_{a_1b_1}$, $\overline{V}_{b_1c_2}$ and $\overline{V}_{c_2a_1}$ are of same magnitude and displaced mutually by 120º to form a balanced 3-phase voltage system. Primary 3-phase line to line voltage of 200 $\sqrt{3}$ V is therefore stepped down to 3-phase, 100$\sqrt{3}$ V line to line voltage at the secondary. The junction of A_2, B_2 and C_2 can be used as *primary neutral* and may be denoted by *N*. Similarly the junction of a_2, b_2 and c_2 may be denoted by *n* for secondary neutral.

26.3.1 A wrong star-star connection

In continuation with the discussion of the last section, we show here a deliberate wrong connection to highlight the importance of proper terminal markings of the coils with dots (•). Let us start from the figure 26.2 where the secondary coils are yet to be connected. To implement star connection on the secondary side, let us assume that someone joins the terminals a_2, b_1 and c_2 together as shown in figure 26.4.

The question is: is it a valid star connection? If not why? To answer this we have to interconnect the secondary voltage phasors in accordance with the physical connections of the coils. In other words, shift and place the secondary voltage phasors so that a_2, b_1 and c_2 overlap each other to make them equipotential. The lengths of phasors $V_{a_1 a_2}$, $V_{b_1 b_2}$ and $V_{c_1 c_2}$ are no doubt, same and equal to 100 V but they do not maintain 120º mutual phase displacement between them as clear from figure 26.4. The magnitude of the line to line voltages too will not be equal. From simple geometry, it can easily be shown that

$$\left|\bar{V}_{a_1 b_2}\right| = \left|\bar{V}_{b_2 c_1}\right| \neq \left|\bar{V}_{c_1 a_1}\right|$$

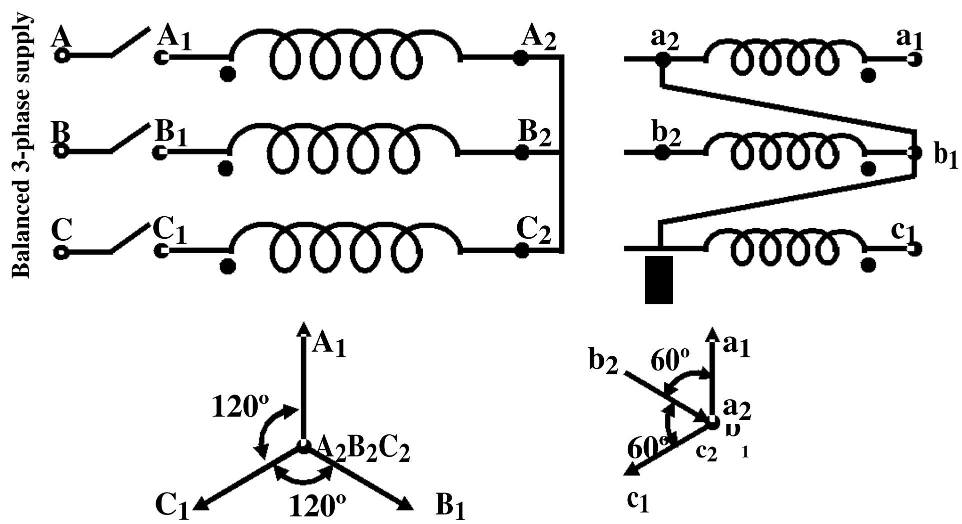


Figure 26.4: Both primary & secondary are star connected.

Thus both the phase as well as line voltages are not balanced 3-phase voltage. Hence the above connection is useless so far as transforming a balanced 3-phase voltage into another level of balanced 3-phase voltage is concerned.

Appropriate polarity markings with letters along with dots (•) are essential in order to make various successful 3-phase transformer connections in practice or laboratory.

26.3.2 Bank of three phase transformer

In the background of the points discussed in previous section, we are now in a position to study different connections of 3-phase transformer. Let the discussion be continued with the same three single phase identical transformers, each of rating 10kVA, 200V / 100V, 50Hz,. These transformers now should be connected in such a way, that it will change the level of a balanced three phase voltage to another balanced three phase voltage level. The three primary and the three secondary windings can be connected in various standard ways such as *star / star* or *star / delta* or *delta / delta* or in *delta / star* fashion. Apart from these, *open delta* connection is also used in practice.

Star-star connection

We have discussed in length in the last section, the implementation of star-star connection of a 3-phase transformer. The connection diagram along with the phasor diagram are shown in figure 26.5 and 26.6.

As discussed earlier, we need to apply to the primary terminals ($A_1B_1C_1$) a line to line voltage of 200 $\sqrt{3}$ V so that rated voltage (200 V) is impressed across each of the primary coils of the individual transformer. This ensures 100 V to be induced across each of the secondary coil and the line to line voltage in the secondary will be 100 $\sqrt{3}$ V. Thus a 3-phase line to line voltage of 200 $\sqrt{3}$ V is stepped down to a 3-phase line to line voltage of 100 $\sqrt{3}$ V. Now we have to calculate how much load current or kVA can be supplied by this bank of three phase transformers without over loading any of the single phase transformers. From the individual rating of each transformer, we know maximum allowable currents of HV and LV windings are respectively I_{HV} = 10000/200 = 50A and I_{LV} = 10000/100 = 100A. Since secondary side is connected in star, line current and the winding currents are same. Therefore total kVA that can be supplied to a balanced 3-phase load is $\sqrt{3}V_{LL}I_L = \sqrt{3}\left(\sqrt{3}100\right)100 = 30$ kVA. While solving problems, it is not necessary to show all the terminal markings in detail and a simple and popular way of showing the same star-star connection is given in figure 26.7.

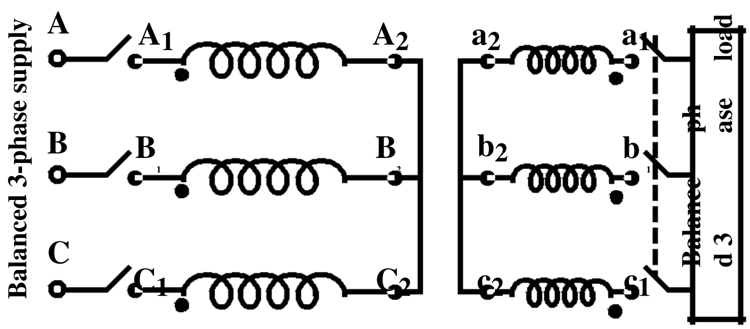


Figure 26.5: Star/star Connection.

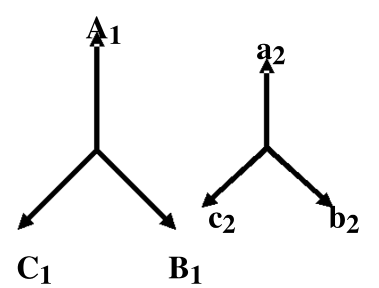


Figure 26.6: Phasor diagram.

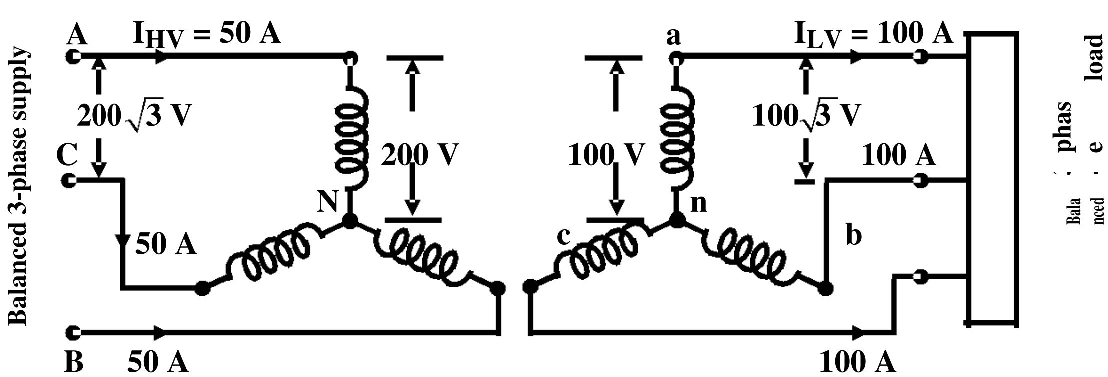


Figure 26.7: Simplified way of showing star-star connection

Star-delta connection

To connect windings in delta, one should be careful enough to avoid dead short circuit. Suppose we want to carry out star / delta connection with the help of the above single phase transformers. HV windings are connected by shorting A_2, B_2 and C_2 together as shown in the figure 26.8. As we know, in delta connection, coils are basically connected in series and from the junction points, connection is made to supply load. Suppose we connect quite arbitrarily (without paying much attention to terminal markings and polarity), a_1 with b_2 and b_1 with c_1. Should we now join a_2 with c_2 by closing the switch S, to complete the delta connection? As a rule, we should not join (i.e., put short circuit) between any two terminals if potential difference exists between the two. It is equivalent to put a short circuit across a voltage source resulting into very large circulating current. Therefore before closing S, we must calculate the voltage difference between a_2 with c_2. To do this, move the secondary voltage phasors such that a_1 and b_2 superpose as well as b_1 with c_1 superpose – this is because a_1 and b_2 are physically joined to make them equipotential; similarly b_1 and c_1 are physically joined so as to make them equipotential. The phasor diagram is shown in figure 26.9. If a voltmeter is connected across S (i.e., between a_2 and c_2), it is going to read the length of the phasor $\bar{V}_{a2\,c2}$. By referring to phasor diagram of figure 26.9, it can be easily shown that the voltage across the switch S, under this condition is $V_{a2\,c2} = 100 + 2\cos 60^{\circ} 100 = 200\text{V}$. So this connection is not proper and the switch S should not be closed.

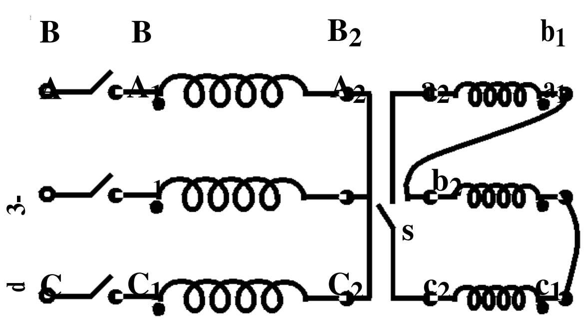


Figure 26.8: Incomplete Connection.

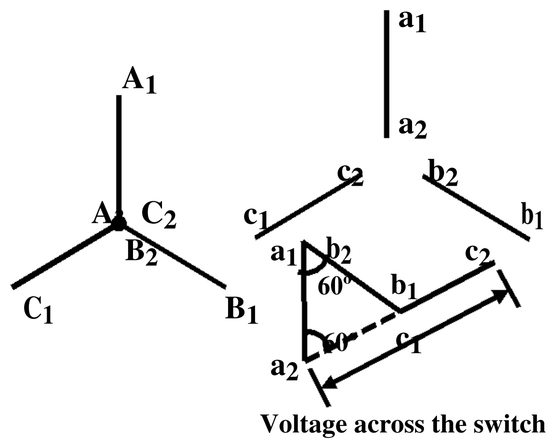


Figure 26.9: Phasor diagram.

Another alternative way to attempt delta connection in the secondary could be: join a_1 with b_2 and b_1 with c_2. Before joining a_2 with c_1 to complete delta connection, examine the open circuit voltage $\bar{V}_{a2\,c1}$. Following the methods described before it can easily be shown that $V_{a2\,c1}$ = 0, which allows to join a_2 with c_1 without any circulating current. So this, indeed is a correct delta connection and is shown in figure 26.10 where a_1 is joined with b_2, b_1 is joined with c_2 and c_1 is joined with a_2. The net voltage acting in the closed delta in this case is zero. Although voltage exists in each winding, the resultant sum becomes zero as they are 120° mutually apart. The output terminals are taken from the junctions as a, b and c for supplying 3-phase load. The corresponding phasor diagram is shown in figure 26.11.

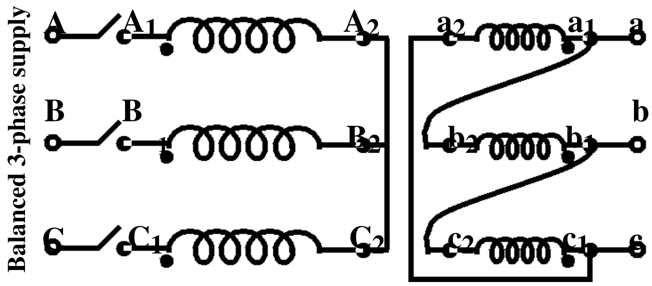


Figure 26.10: Star/delta Connection.

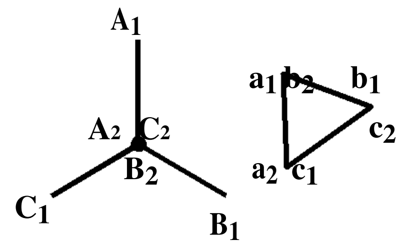


Figure 26.11: Phasor diagram.

Here also we can calculate the maximum kVA this star / delta transformer can handle without over loading any of the constituents transformers. In this case the secondary line to line voltage is same as the winding voltage i.e., 100V, but the line current which can be supplied to the load is $100\sqrt{3}$. Because it is at this line current, winding current becomes the rated 100A. Therefore total load that can be supplied is $\sqrt{3} V_{LL} I_L = \sqrt{3}\,100\left(\sqrt{3}\,100\right)$ VA = 30kVA . Here also total kVA is 3 times the kVA of each transformer. The star-delta connection is usually drawn in a simplified manner for problem solving and easy understanding as shown in figure 26.12.

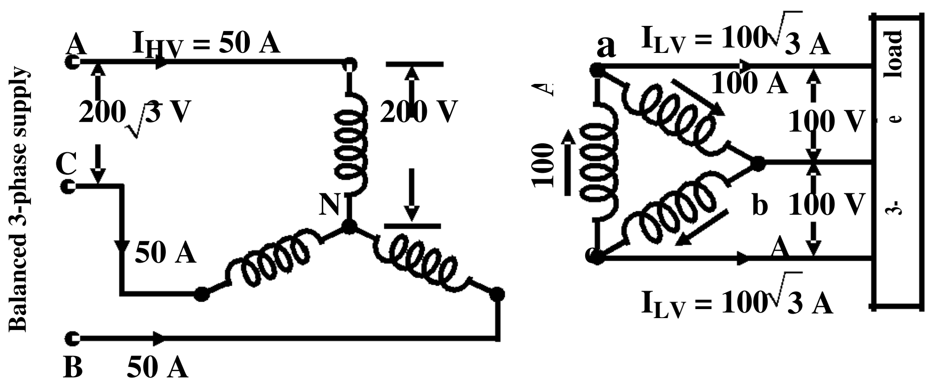


Figure 26.12: Simplified way of showing star-star connection

Another valid delta connection on the LV side is also possible by joining a_2 with b_1, b_2 with c_1 and c_2 with a_1. It is suggested that the reader tries other 3-phase connections and verify that the total KVA is 3-times the individual KVA of each transformer. However, we shall discuss about delta / delta and open delta connection.

Delta / delta and open delta connection

Here we mention about the delta/delta connection because, another important and useful connection namely open delta connection can be understood well. Valid delta connection can be implemented in the usual way as shown in the figure 26.13. The output line to line voltage will be 100V for an input line voltage of 200V. From the secondary one can draw a line current of $100\sqrt{3}$ A which means a total of 30 kVA can be supplied without overloading any of the Version individual transformers. A simplified representation of the delta-delta connection is shown in figure 26.15 along with the magnitude of the currents in the lines and in the coils of HV and LV side.

Let us now imagine that the third transformer C be removed from the circuit as shown in the second part of the figure 26.13. In effect now two transformers are present. If the HV sides is energized with three phase 200V supply, in the secondary we get 3-phase balanced 100V supply which is clear from the phasor diagram shown in figure 26.14. Although no transformer winding exist now between A_2 & B_1 on the primary side and between a_2 & b_1 on the secondary side, the voltage between A_2 & B_1 on the primary side and between a_2 & b_1 on the secondary side exist. Their phasor representation are shown by the dotted line confirming balanced 3-phase supply. But what happens to kVA handling capacity of the open delta connection? Is it 20 kVA, because two transformers are involved? Let us see. The line current that we can allow to flow in the secondary is 100A (and not $100\sqrt{3}$ as in delta / delta connection). Therefore total maximum kVA handled is given by $\sqrt{3}V_{LL} I_L = \left(\sqrt{3}\ 100\ 100\right)$ VA = 17.32 kVA , which is about 57.7% of the delta connected system. This is one of the usefulness of using bank of 3-phase transformers and connecting them in delta-delta. In case one of them develops a fault, it can be removed from the circuit and power can be partially restored.

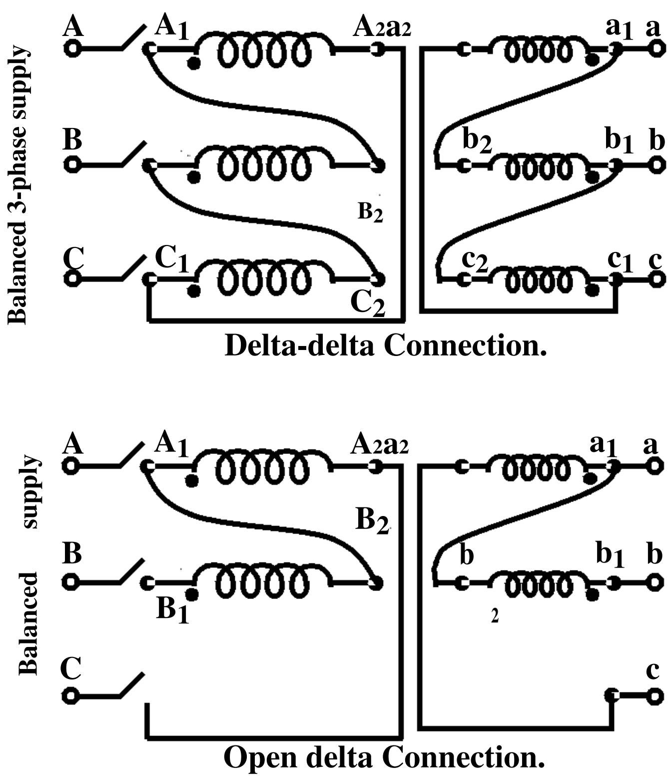


Figure 26.13: Delta/delta and open delta connection.

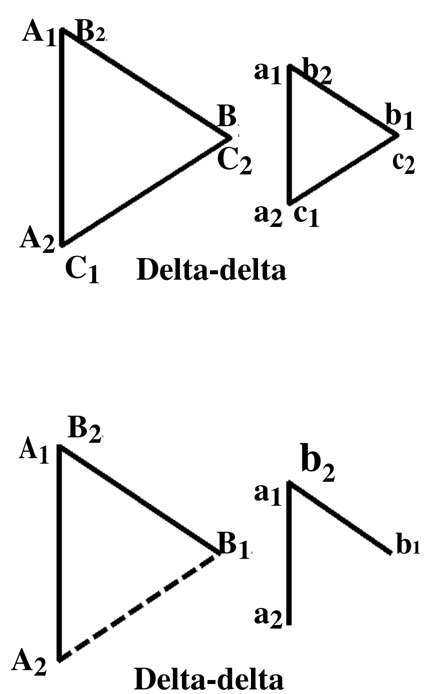


Figure 26.14: Phasor diagram

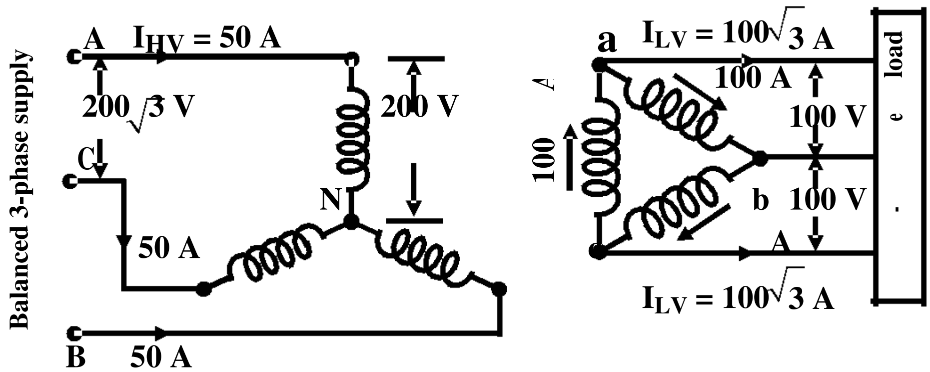


Figure 26.15: Simplified way of showing delta-delta connection

26.3.3 3-phase transformer- a single unit

Instead of using three number of single phase transformers, a three phase transformer can be constructed as a single unit. The advantage of a single unit of 3-phase transformer is that the cost is much less compared to a bank of single phase transformers. In fact all large capacity transformers are a single unit of three phase transformer.

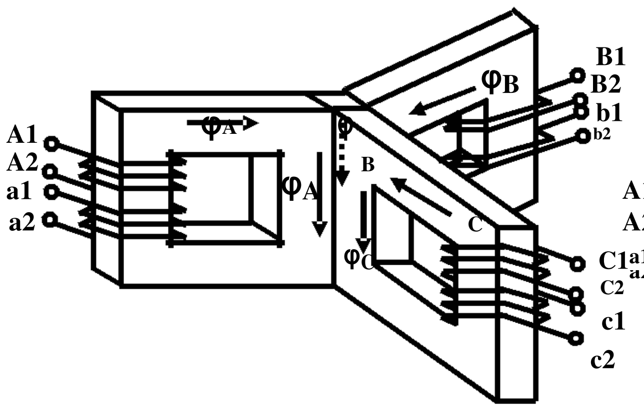


Figure 26.16: A conceptual three phase transformer.

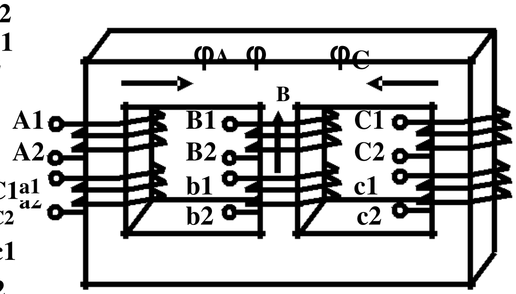


Figure 26.17: A practical core type three phase transformer.

To understand, how it is constructed let us refer to figure 26.16. Here three, single phase transformers are so placed that they share a common central limb. The primary and the secondary windings of each phase are placed on the three outer limbs and appropriately marked. If the primary windings are connected to a balanced 3-phase supply (after connecting the windings in say star), the fluxes $\varphi_A(t)$, $\varphi_B(t)$ and $\varphi_C(t)$ will be produced in the cores differing in time phase mutually by 120°. The return path of these fluxes are through the central limb of the core structure. In other words the central limb carries sum of these three fluxes. Since instantaneous sum of the fluxes, $\varphi_A(t)+ \varphi_B(t)+ \varphi_C(t) = 0$, no flux lines will exist in the central limb at any time. As such the central common core material can be totally removed without affecting the working of the transformer. Immediately we see that considerable saving of the core material takes place if a 3-phase transformer is constructed as a single unit. The structure however requires more floor area as the three outer limbs protrudes outwardly in three different directions.

A further simplification of the structure can be obtained by bringing the limbs in the same plane as shown in the figure 26.17. But what do we sacrifice when we go for this simplified structure? In core structure of figure 26.16, we note that the reluctance seen by the three fluxes are same, Hence magnetizing current will be equal in all the three phases. In the simplified core structure of figure 26.17, reluctance encountered by the flux φ_B is different from the reluctance encountered by fluxes φ_A and φ_C, Hence the magnetizing currents or the no load currents drawn will remain slightly unbalanced. This degree of unbalanced for no load current has practically no influence on the performance of the loaded transformer. Transformer having this type of core structure is called the *core type* transformer.

26.4 Vector Group of 3-phase transformer

The secondary voltages of a 3-phase transformer may undergo a *phase shift* of either +30° leading or -30° lagging or 0° i.e, no phase shift or 180° reversal with respective line or phase to neutral voltages. On the name plate of a three phase transformer, the vector group is mentioned. Typical representation of the vector group could be Yd1 or Dy11 etc. The first capital latter Y indicates that the primary is connected in star and the second lower case latter d indicates delta connection of the secondary side. The third numerical figure conveys the angle of phase shift based on *clock convention*. The minute hand is used to represent the primary phase to neutral voltage and always shown to occupy the position 12. The hour hand represents the secondary phase to neutral voltage and may, depending upon phase shift, occupy position other than 12 as shown in the figure 26.18.

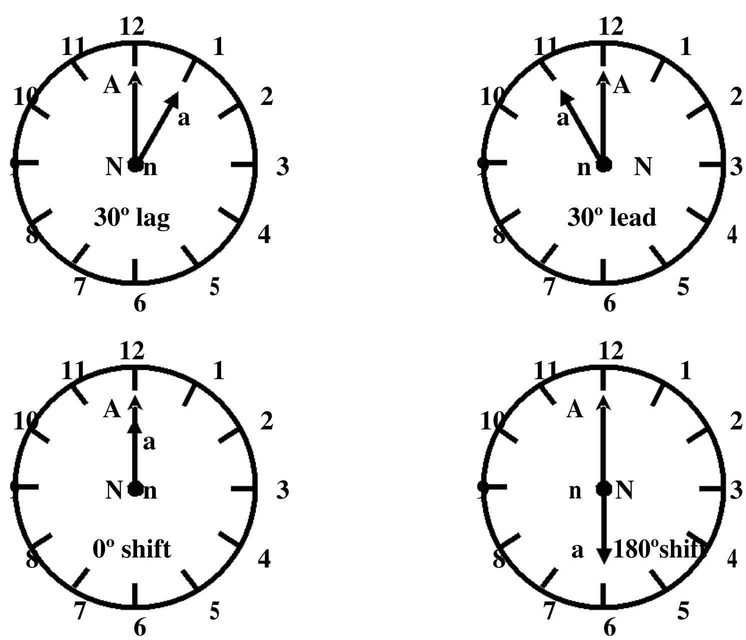


Figure 26.18: Clock convention representing vector groups.

The angle between two consecutive numbers on the clock is 30°. The star-delta connection and the phasor diagram shown in the figures 26.10 and 26.11 correspond to Yd_1. It can be easily seen that the secondary *a* phase voltage to neutral *n* (artificial in case of delta connection) leads the *A* phase voltage to neutral N by 30°. However the star delta connection shown in the figure 26.19 correspond to Yd_{11}.

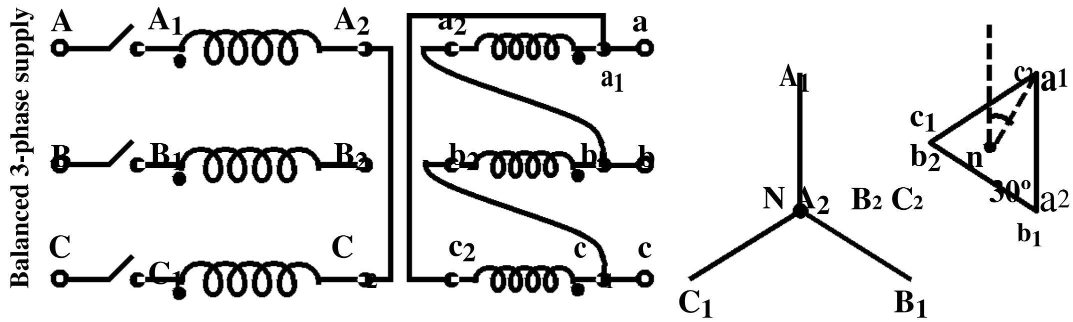


Figure 26.19: Connection and phasor diagram for *$Y\,d_{11}$*.

26.5 Tick the correct answer

1. The secondary line to line voltage of a star-delta connected transformer is measured to be 400 V. If the turns ratio between the primary and secondary coils is 2 : 1, the applied line to line voltage in the primary is:

   (A) 462 V (B) 346 V (C) 1386 V (D) 800 V

2. The secondary line to line voltage of a delta-delta connected transformer is measured to be 400 V. If the turns ratio between the primary and secondary coils is 2 : 1, the applied line to line voltage in the primary is:

   (A) 462 V (B) 346 V (C) 1386 V (D) 800 V

3. The secondary line to line voltage of a delta-star connected transformer is measured to be 400 V. If the turns ratio between the primary and secondary coils is 2 : 1, the applied line to line voltage in the primary is:

   (A) 800 V (B) 500 V (C) 1386 V (D) 462 V

4. The secondary line current of a star-delta connected transformer is measured to be 100 A. If the turns ratio between the primary and secondary coils is 2 : 1, the line current in the primary is:

   (A) 50 A (B) 28.9 A (C) 57.7 A (D) 60 A

5. The secondary line current of a delta-star connected transformer is measured to be 100 A. If the turns ratio between the primary and secondary coils is 2 : 1, the line current in the primary is:

   (A) 86.6 A (B) 50 A (C) 60 A (D) 57.7 A

6. The primary line current of an open delta connected transformer is measured to be 100 A. If the turns ratio between the primary and secondary coils 2 : 1, the line current in the primary is:

   (A) 173.2 A (B) 200 A (C) 150 A (D) 50 A

7. Two single-phase transformers, each of rating 15 kVA, 200 V / 400 V, 50 Hz are connected in open delta fashion. The arrangement can supply safely, a balanced 3-phase load of:

   (A) 45 kVA (B) 25.9 kVA (C) 30 kVA (D) 7.5 kVA

8. In figure 26.20 showing an incomplete 3-phase transformer connection, the reading of the voltmeter will be:

   (A) 0 V (B) 173.2 V (C) 346.4 V (D) 300 V

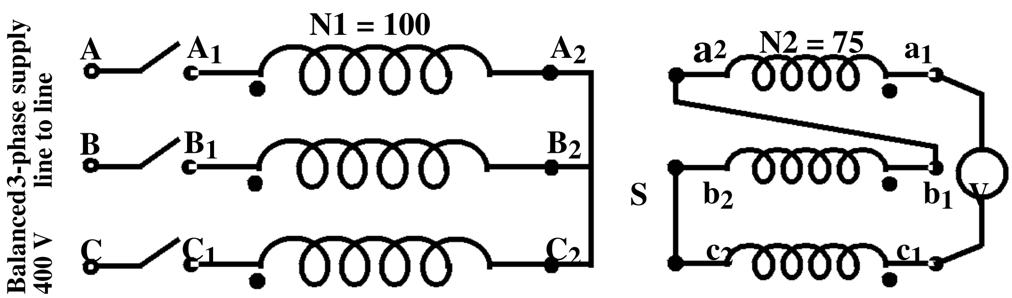


Figure 26.20:

26.6 Problems

1. Three number of single phase *ideal* transformers, each of rating. 10kVA, 200V / 100V, 50Hz is connected in star/delta fashion to supply a balanced three phase 20 kW, 0.8 power factor load at 100V(line to line). Draw a circuit diagram for this. Calculate (i) what line to line voltage should be applied to the primary side? (ii) Calculate the line and phase currents on the secondary and primary sides and indicate them on the diagram.

2. How two identical single phase transformers each of rating 5kVA, 200V/100V, 50Hz be used to step down a balanced 3-phase, 200V supply to a balanced 3-phase, 100V supply? Explain with circuit and phasor diagrams. Calculate also the maximum kVA that can be supplied from this connection.

3. A balanced 3-phase load of 20kW, 0.8 power factor lagging is to be supplied at a line to line voltage of 110V. However, a balanced 3-phase voltage of 381V (line to line) is available. Using three numbers of identical single phase ideal transformers each of rating 10kVA, 220V/110V, 50Hz make an arrangement such that the above load can be supplied. Draw the circuit diagram and show the magnitude of currents in the *lines* and in the *windings* of the transformers on both LV and HV side.

4. Refer to the following figure 26.21 which shows the windings of a 3-phase transformer. Primary turns per phase is 250. Each phase has got two *identical* secondary windings each having 100 turns. The primary windings are connected in star by shorting A_2, B_2 and C_2 and supplied from a balanced 3-phase 1000 V (line to line), 50 Hz source.

   a) If the secondary coils are connected by joining a_2 with b_3 and b_4 with c_1 then calculate $V_{a_1 c_1}$.

   b) All the 6 coils are connected in series in the following way:

a_2 joined with b_2	b_1 joined with c_2
c_1 joined with c_4	c_3 joined with b_4
b_3 joined with a_3	

   Draw the phasor diagram and calculate the voltage $V_{a_1 a_4}$

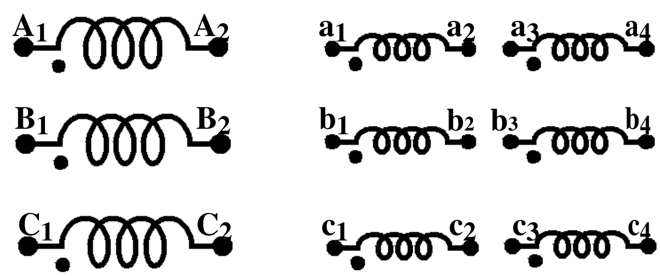


Figure 26.21: 3-phase transformer with two secondary coils per phase

UNIT-3

Transformer

Auto-Transformer

Contents

27.1 Goals of the lesson

In this lesson we shall learn about the working principle of another type of transformer called *autotransformer* and its uses. The differences between a *2-winding* and an *autotransformer* will be brought out with their relative advantages and disadvantages. At the end of the lesson some objective type questions and problems for solving are given. *Key Words*: tapping's, conducted VA, transformed VA.
After going through this section students will be able to understand the following.

1. Constructional differences between a 2-winding transformer and an autotransformer.
2. Economic advantages/disadvantages between the two types.
3. Relative advantages/disadvantages of the two, based on technical considerations.
4. Points to be considered in order to decide whether to select a 2-winding transformer or an autotransformer.
5. The difference between an *autotransformer* and *variac* (or dimmerstat).
6. The use of a 2-winding transformer as an *autotransformer*.
7. The connection of three identical single phase transformers to be used in 3-phase system.

27.2 Introduction

So far we have considered a *2-winding* transformer as a means for changing the level of a given voltage to a desired voltage level. It may be recalled that a *2- winding* transformer has two separate magnetically coupled coils with no electrical connection between them. In this lesson we shall show that change of level of voltage can also be done quite effectively by using a single coil only. The idea is rather simple to understand. Suppose you have a single coil of 200 turns (= N_{BC}) wound over a iron core as shown in figure 27.1. If we apply an a.c voltage of 400 V, 50 Hz to the coil (between points B and C), voltage per turn will be 400/200 = 2 V. If we take out a wire from one end of the coil say C and take out another wire tapped from any arbitrary point E, we would expect some voltage available between points E and C. The magnitude of the voltage will obviously be $2 \cdot N_{EC}$ where N_{EC} is the number of turns between points E and C. If tapping has been taken in such a way that N_{EC} = 100, voltage between E and C would be 200 V. Thus we have been able to change 400 V input voltage to a 200 V output voltage by using a single coil only. Such transformers having a single coil with suitable tapings are called *autotransformers*.

It is possible to connect a conventional *2-winding transformer* as an *autotransformer* or one can develop an *autotransformer* as a single unit.

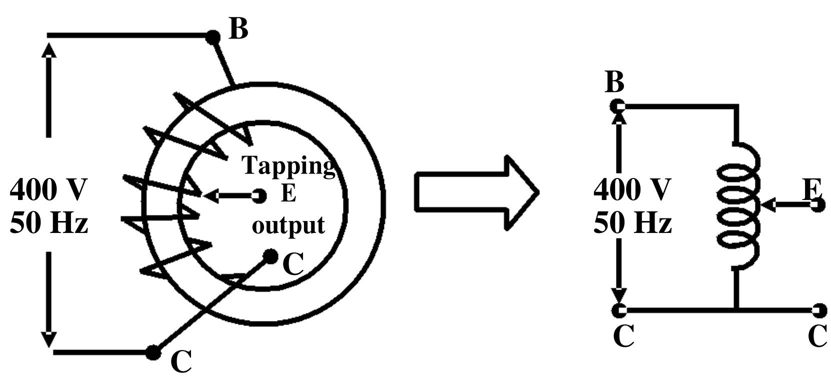


Figure 27.1: Transformer with a single coil.

27.3 2-winding transformer as Autotransformer

Suppose we have a single phase 200V/100V, 50Hz, 10kVA two winding transformer with polarity markings. Then the coils can be connected in various ways to have voltage ratios other than 2 also, as shown in figure 27.2.

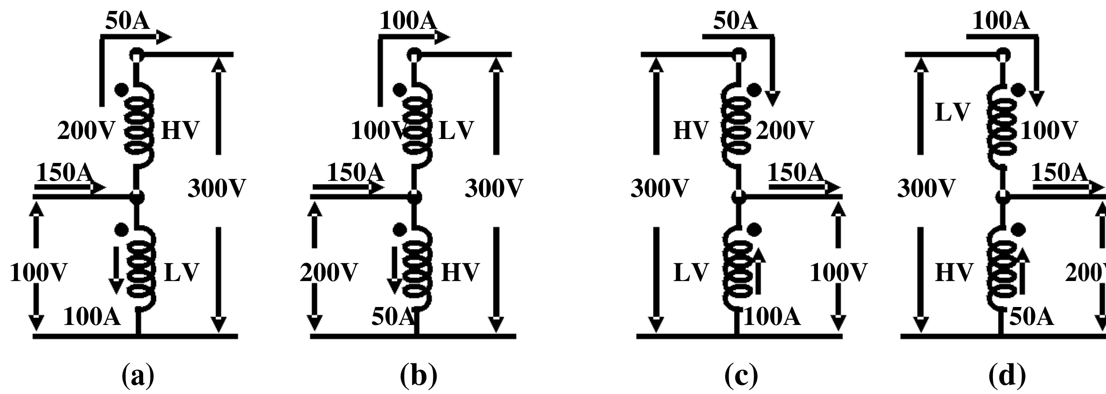


Figure 27.2: A two winding transformer connected as an autotransformer in various ways.

Let us explain the one of the connections in figure 27.2(a) in detail. Here the LV and the HV sides are connected in additive series. For rated applied voltage (100V) across the LV winding, 200V must be induced across the HV winding. So across the whole combination we shall get a voltage of 300V. Thus the input voltage is stepped up by a factor of 3 (300 V/100 V). Now how much current can be supplied to a load at 300 V? From the given rating of the transformer we know, $I_{HV\text{ rated}}$ = 50 A and $I_{LV\text{ rated}}$ = 100 A. Therefore for safe operation of the transformer, these rated currents should not be exceeded in HV and LV coils. Since the load is in series with the HV coil, 50A current can be safely supplied. But a current of 50A in the HV demands that the LV winding current must be 100A and in a direction as shown, in order to keep the flux in the core constant. Therefore by applying KCL at the junction, the current drawn from the supply will be 150A. Obviously the kVA handled by the transformer is 30 kVA and without overloading either of the windings. It may look a bit surprising because as a two winding transformer its rating is only 10 kVA. The explanation is not far to seek. Unlike a two winding transformer, the coils here are connected *electrically*. So the kVA transferred from supply to the load side takes place both *inductively* as well as *conductively* – 10kVA being transferred inductively and remaining 20kVA transferred conductively. The other connections shown in (b), (c) and (d) of figure 27.2 can similarly explained and left to the reader to verify.

27.4 Autotransformer as a single unit

Look at the figure 27.3 where the constructional features of an auto transformer is shown. The core is constructed by taking a rectangular long strip of magnetic material (say CRGO) and rolled to give the radial thickness. Over the core, a continuous single coil is wound the free terminals of which are marked as C and A. A *carbon brush* attached to a manually rotating handle makes contact with different number of turns and brought out as a terminal, marked E. The number of turns between E & C, denoted by N_{EC} can be varied from zero to a maximum of total number of turns between A & C i.e, N_{AC}. The output voltage can be varied smoothly from zero to the value of the input voltage simply by rotating the handle in the clockwise direction.

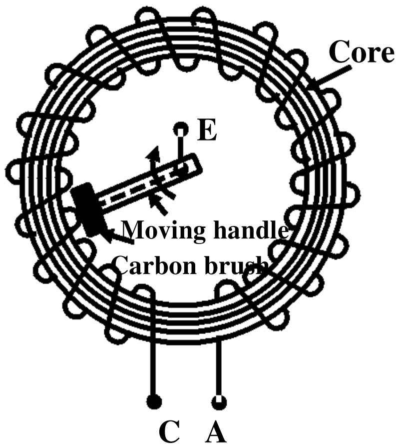


Figure 27.3: Autotransformer or Variac.

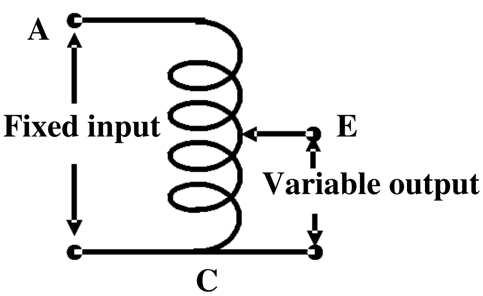


Figure 27.4: Schematic representation of autotransformer.

This type of autotransformers are commercially known as *varic* or *dimmerstat* and is an important piece of equipment in any laboratory.

Now we find that to change a given voltage V_1 to another level of voltage V_2 and to transfer a given KVA from one side to the other, we have *two choices* namely by using a *Two Winding Transformer* or by using an *Autotransformer*. There are some advantages and disadvantages associated with either of them. To understand this aspect let us compare the two types of transformers in equal terms. Let,

Input voltage $= V_1$

Output voltage required across the load $= V_2$

Rated current to be supplied to the load $= I_2$

Current drawn from the supply at rated condition $= I_1$

KVA to be handled by both types of transformers $= V_1 I_1 = V_2 I_2$

The above situation is pictorially shown in figures 27.5(a) and (b). Let for the two winding transformer,

For the two winding transformer:

Primary number of turns $= N_1$

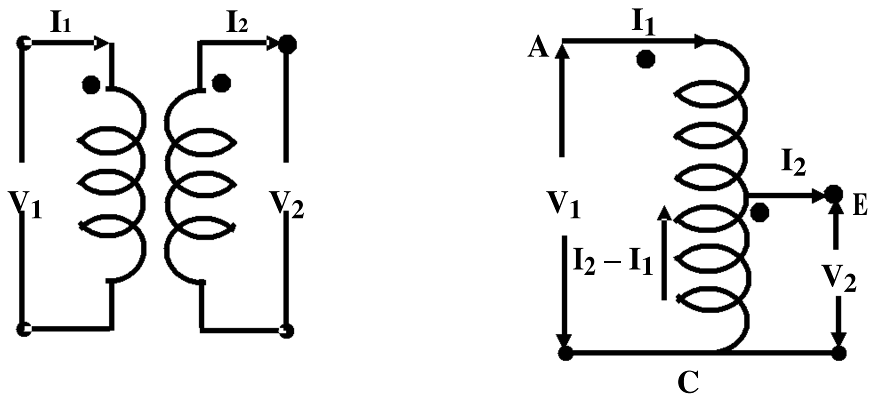
Secondary number of turns $= N_2$

For the autotransformer:

Number of turns between A & C $= N_1$

Number of turns between E & C $= N_2$

Therefore, number of turns between A & E $= N_1 - N_2$



(a) **(b)**

Figure 27.5: A two winding transformer and an autotransformer of same rating.

Let us now right down the mmf balance equation of the transformers.

For the two winding transformer:

$$\text{MMF balance equation is } N_1 I_1 = N_2 I_2$$

For the autotransformer:

$$\text{MMF balance equation is } (N_1 - N_2) I_1 = N_2 (I_2 - I_1)$$

$$\text{or, } N_1 I_1 = N_2 I_2$$

It may be noted that in case of an autotransformer, the portion EC is common between the primary and the secondary. At loaded condition current flowing through N_{EC} is $(I_2 - I_1)$. Therefore, compared to a two winding transformer lesser cross sectional area of the conductor in the portion EC can be chosen, thereby saving copper. We can in fact find out the ratio of amount of copper required in two types of transformers noting that the volume of copper required will be proportional to the product of current and the number of turns of a particular coil. This is because, length of copper wire is proportional to the number of turns and crossectional area of wire is proportional to the current value i.e.,

Volume of copper length of the wire × cross sectional area of copper wire

$N \times I$

Hence,

$$\frac{\text{Amount of copper required in an autotransformer}}{\text{Amount of copper required in a two winding transformer}} = \frac{(N_1 - N_2) I_1 + N_2 (I_2 - I_1)}{N_1 I_1 + N_2 I_2}$$

$$\text{Noting that } N_1 I_1 = N_2 I_2 \quad = \frac{2N_1 I_1 - 2N_2 I_1}{2N_1 I_1}$$

$$= \frac{N_1 - N_2}{\quad}$$

$= 1 - \frac{1}{a}$ where, a is the turns ratio.

Here we have assumed that N_1 is greater than N_2 i.e., a is greater than 1. The savings will of course be appreciable if the value of a is close to unity. For example if a = 1.2, copper required for autotransformer will be only 17% compared to a two winding transformer, i.e, saving will be about 83%. On the other hand, if a = 2, savings will be only 50%. Therefore, it is always economical to employ autotransformer where the voltage ratio change is close to unity. In fact autotransformers could be used with advantage, to connect two power systems of voltages say 11 kV and 15 kV.

Three similar single units of autotransformers could connected as shown in the figure 27.6 to get variable balanced three phase output voltage from a fixed three phase voltage. Such connections are often used in the laboratory to start 3-phase induction motor at reduced voltage.

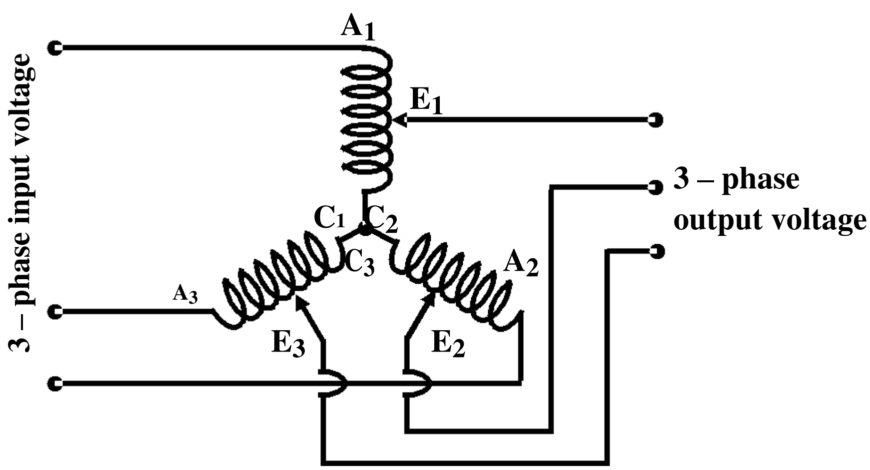


Figure 27.6: 3 – phase autotransformer connection

Apart from being economical, autotransformer has less leakage flux hence improved regulation. Copper loss in the common portion of the winding will be less, so efficiency will be slightly more. However its one major disadvantage is that it can not provide isolation between HV and LV side. In fact, due to an open circuit in the common portion between E & C, the voltage on the load side may soot up to dangerously high voltage causing damage to equipment. This unexpected rise in the voltage on the LV side is potentially dangerous to the personnel working on the LV side.

27.5 Tick the correct answers

1. Savings of copper, in an autotransformer will be significant over a two winding transformer of same rating when the ratio of the voltages is

   (A) ≈ 1 (B) >> 1 (C) = 1 (D) << 1

2. 110 V, 50 Hz single phase supply is needed from a 220 V, 50 Hz source. The ratio of weights of copper needed for a *two winding* and an *autotransformer* employed for the purpose is:

(A) 2 (B) 0.5 (C) 4 (D) 0.25

3. The two winding transformer and the autotransformer of the circuit shown in Figure 27.7 are *ideal*. The current in the section BC of the autotransformer is

   (A) 28 A from B to C (B) 12 A from C to B

   (C) 28 A from C to B (D) 12 A from B to C

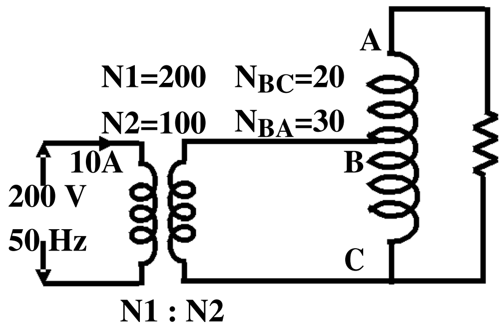


Figure 27.7:

4. A 22 kVA, 110 V/ 220 V, 50 Hz transformer is connected in such away that it steps up 110 V supply to 330 V. The maximum kVA that can be handled by the transformer is

   (A) 22 kVA (B) 33 kVA (C) 11 kVA (D) 5.5 kVA

27.6 Problems

1. The following figure 27.8 shows an ideal autotransformer with number of turns of various sections as N_{AB} = 100, N_{CB} = 60 and N_{DB} = 80. Calculate the current drawn from the supply and the input power factor when the supply voltage is 400V, 50Hz.

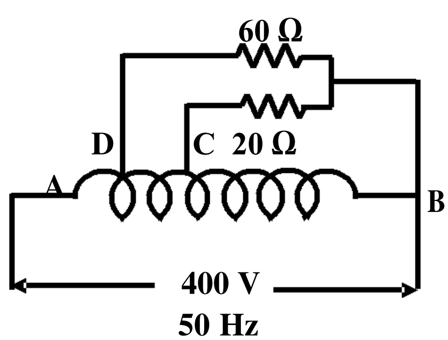


Figure 27.8:

2. An ideal autotransformer steps down a 400 V, single phase voltage to 200 V, single phase voltage. Across the secondary an impedance of (6 + j8)Ω is connected. Calculate the currents in all parts of the circuit.

3. Calculate the values of currents and show their directions in the various branches of a 3-phase, star connected autotransformer of ratio of 400 / 500 V and loaded with 600 kW at 0.85 lagging. Autotransformer may be considered to be ideal. It may be noted that, unless otherwise specified, voltage value of a 3-phase system corresponds to line to line voltage.

4. A delta-star connected 3-phase transformer is supplied with a balanced 3-phase, 400 V supply as shown in figure 27.9. A 3-phase auto transformer is fed from the output of the 3-phase transformer. Finally the at the secondary of the autotransformer a balanced 3-phase load is connected. The per phase primary and secondary turns of both the transformers are given in figure 27.9.

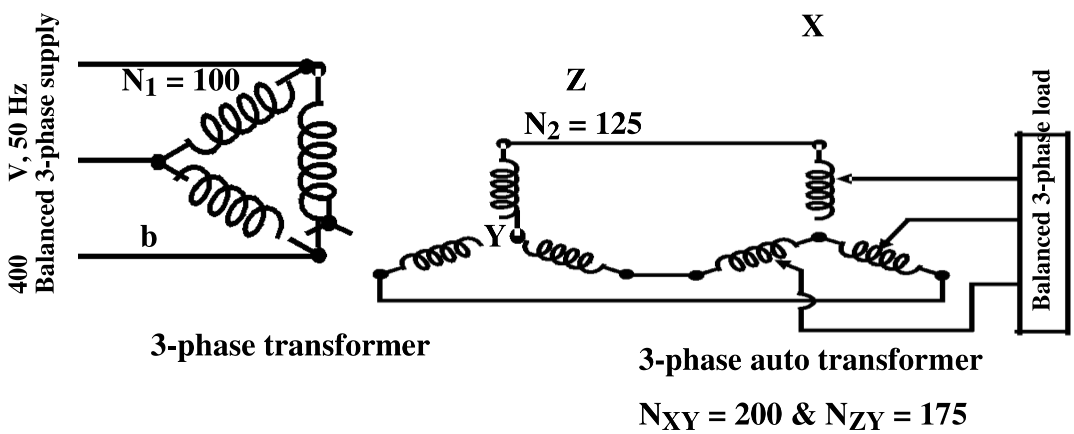


Figure 27.9:

Calculate line to line voltage at which the load receives power. If the load draws 10 A current, calculate currents (a) in the section XZ & ZY of the autotransformer and (b) line currents and coil currents of both the sides of the 3-phase, delta-star connected transformer.

UNIT-4

Three-phase Induction Motor

Rotating Magnetic Field in Three-phase Induction Motor

In the previous module, containing six lessons (23-28), mainly, the study of the single-phase two-winding Transformers – a static machine, fed from ac supply, has been presented. In this module, containing six lessons (29-34), mainly, the study of Induction motors, fed from balanced three-phase supply, will be described.

In this (first) lesson of this module, the formation of rotating magnetic field in the air gap of an induction motor, is described, when the three-phase balanced winding of the stator is supplied with three-phase balanced voltage. The balanced winding is of the same type, as given in lesson no. 18, for a three-phase ac generator.

Keywords: Induction motor, rotating magnetic field, three-phase balanced winding, and balanced voltage.

After going through this lesson, the students will be able to answer the following questions:

1. How a rotating magnetic field is formed in the air gap of a three-phase Induction motor, when the balanced winding of the stator is fed from a balanced supply?
2. Why does the magnitude of the magnetic field remain constant, and also what is the speed of rotation of the magnetic field, so formed? Also what is meant by the term 'synchronous speed'?

Three-phase Induction Motor

A three-phase balanced winding in the stator of the Induction motor (IM) is shown in Fig. 29.1 (schematic form). In a three-phase balanced winding, the number of turns in three windings, is equal, with the angle between the adjacent phases, say R & Y, is 120° (electrical). Same angle of 120° (elec.) is also between the phases, Y & B.

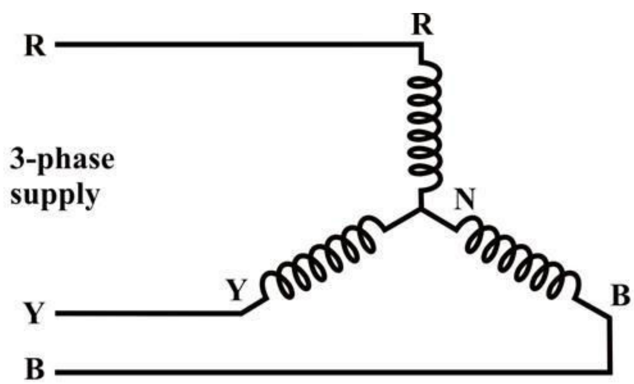


Fig. 29.1: Schematic diagram of the stator windings in a three-phase induction motor.

A three-phase balanced voltage, with the phase sequence as R-Y-B, is applied to the above winding. In a balanced voltage, the magnitude of the voltage in each phase, assumed to be in star in this case, is equal, with the phase angle of the voltage between the adjacent phases, say R & Y, being120° .

Rotating Magnetic Field

The three phases of the stator winding (balanced) carry balanced alternating (sinusoidal) currents as shown in Fig. 29.2.

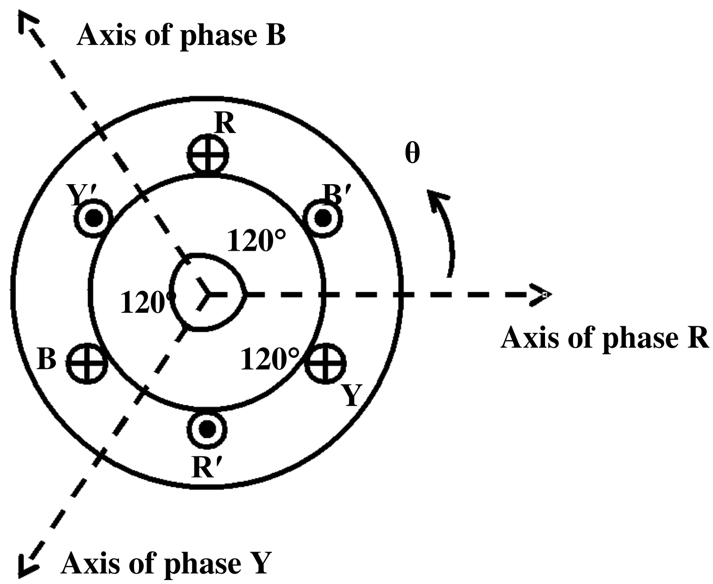


Fig. 29.2: The relative location of the magnetic axis of three phases.

$$i_R = I_m \cos \omega t$$
$$i_Y = I_m \cos (\omega t - 120°)$$
$$i_B = I_m \cos (\omega t + 120°) = I_m \cos (\omega t - 240°)$$

Please note that the phase sequence is R-Y-B. I_m is the maximum value of the phase currents, and, as the phase currents are balanced, the rms values are equal ($I_R = I_Y = I_B$).

Three pulsating mmf waves are now set up in the air- gap, which have a time phase difference of 120° from each other. These mmf's are oriented in space along the magnetic axes of the phases, R, Y & B, as illustrated by the concentrated coils in Fig. 29.2. Please note that 2-pole stator is shown, with the angle between the adjacent phases, R & Y as 120° , in both mechanical and electrical terms. Since the magnetic axes are located 120° apart in space from each other, the three mmf's are expresses mathematically as

$$F_R = F_m \cos \omega t \cos \theta$$
$$F_Y = F_m \cos (\omega t - 120°) \cos (\theta - 120°)$$
$$F_B = F_m \cos (\omega t + 120°) \cos (\theta + 120°)$$

wherein it has been considered that the three mmf waves differ progressively in time phase by 120° , i.e. $2\pi / 3$ rad (elect.), and are separated in space phase by 120° , i.e. $2\pi / 3$ rad (elect.). The resultant mmf wave, which is the sum of three pulsating mmf waves, is

$F = F_R + F_Y + F_B$

Substituting the values,

$F(\theta, t)$

$= F_m [\cos \omega t \cos \theta + \cos(\omega t - 120°) \cos(\theta - 120°) + \cos(\omega t + 120°) \cos(\theta + 120°)]$ The first term of this expression is

$\cos \omega t \cos \theta = 0.5 [\cos(\theta - \omega t) + \cos(\theta + \omega t)]$

The second term is

$\cos(\omega t - 120°) \cos(\theta - 120°) = 0.5 [\cos(\theta - \omega t) + \cos(\theta + \omega t - 240°)]$

Similarly, the third term can be rewritten in the form shown.

The expression is

$F(\theta, t) = 1.5 F_m \cos(\theta - \omega t)$

$+ 0.5 F_m \left[\cos(\theta + \omega t) + \cos(\theta + \omega t - 240°) + \cos(\theta + \omega t + 240°)\right]$ Note that

$\cos(\theta + \omega t - 240°) = \cos(\theta + \omega t + 120°)$, and

$\cos(\theta + \omega t + 240°) = \cos(\theta + \omega t - 120°)$.

If these two terms are added, then

$\cos(\theta + \omega t + 120°) + \cos(\theta + \omega t - 120°) = -\cos(\theta + \omega t)$

So, in the earlier expression, the second part of RHS within the capital bracket is zero. In other words, this part represents three unit phasors with a progressive phase difference of 120° , and therefore add up to zero. Hence, the resultant mmf is

$F(\theta, t) = 1.5 F_m \cos(\theta - \omega t)$

So, the resultant mmf is distributed in both space and time. It can be termed as a rotating magnetic field with sinusoidal space distribution, whose space phase angle changes linearly with time as ωt . It therefore rotates at a constant angular speed of ω rad (elect.)/s. This angular speed is called synchronous angular speed (ω_s).

The peak value of the resultant mmf is $F_{peak} = 1.5 F_m$. The value of F_m depends on No. of turns/phase, winding current, No. of poles, and winding factor. At $\omega t = 0$, i.e. when the current in R phase has maximum positive value, $F(\theta, 0) = 1.5 F_m \cos \theta$, i.e. the mmf wave has its peak value (at $\theta = 0$) lying on the axis of R phase, when it carries maximum positive current. At $\omega t = 2\pi / 3$ (120°) , the phase Y (assumed lagging) has its positive current maximum, so that the peak of the rotating magnetic field (mmf) lying on the axis of Y phase. By the same argument, the peak of the mmf coincides with the axis of phase B at $\omega t = 4\pi / 3$ (240°) . It is, therefore, seen that the resultant mmf moves from the axis of the leading phase to that of the lagging phase, i.e. from phase R towards phase Y, and then phase B, when the phase sequence of the currents is R-Y-B (R leads Y, and Y leads B). As described in brief later, the direction of rotation of the resultant mmf is reversed by simply changing the phase sequence of currents.

From the above discussion, the following may be concluded: Whenever a balanced three-phase winding with phases distributed in space so that the relative space angle between them is $2\pi / 3$ rad (elect.) $\left[120°\right]$, is fed with balanced three-phase currents with relative phase difference of $2\pi / 3$ rad (elect.) $\left[120°\right]$, the resultant mmf rotates in the air-gap at an angular speed of $\omega_s = 2\pi (f / p)$, where f is the frequency (Hz) of currents and p is No. of pairs of poles for which the winding is designed. The synchronous speed in rpm (r/min) is $N_s = \omega_s (60 / 2\pi) = 60\,(f / p)$. The direction of rotation of the mmf is from the leading phase axis to lagging phase axis. This is also valid for q-phase balanced winding, one value of which may be $q = 2$ (two). For a 2-phase balanced winding, the time and phase angles are $(\pi / 2)$ rad or 90° (elect.).

Alternatively, this production of rotating magnetic field can be shown by the procedure described. As stated earlier, the input voltage to three-phase balanced winding of the stator is a balanced one with the phase sequence (R-Y-B). This is shown in the sinusoidal voltage waveforms of the three phases, R, Y & B (Fig. 29.3).

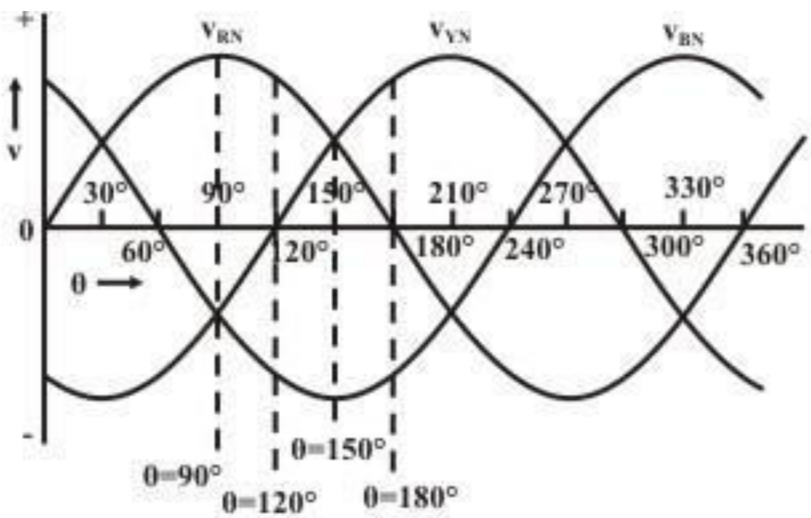


Fig. 29.3: Three-phase voltage waveforms with phase sequence R-Y-B.

A two-pole, three-phase balanced winding in the stator of IM is shown in Fig. 29.4(i)-(a-d), where the winding of each phase, say for example, ($R - R'$) is assumed to be concentrated in one slot each, both for forward and return conductors, with required no. as needed. Same is the case for other two phases. Please note that the angle of 120° is same in both mechanical (as shown) and also electrical terms, as no. of poles is two only. The two (forward and return) parts of the winding in each phase, say R are referred as R and R' respectively. So, also for two other phases, Y & B as shown.

Let us first consider, what happens at the time instant t_1 , of the voltage waveforms as given in Fig. 29.3. At this instant, the voltage in the R-phase is positive maximum (θ_1 = 90°), while the two other voltages in the phases, Y & B are half of the maximum value, and also negative. The three waveforms are represented by the following equations:

$e_{RN} = E_m \sin\theta$; $e_{YN} = E_m \sin(\theta - 120°)$; $e_{BN} = E_m \sin(\theta + 120°)$

where,

$\theta = \omega t$ (rad)

f = Supply frequency (Hz or c/s)

$\omega = 2\pi f$ = Angular frequency (rad/s)

E_m = Maximum value of the voltage, or induced emf in each phase

The currents in three windings are shown in Fig. 29.4(a)-(i). For ($\theta_1 = 90°$), the current in R-phase in positive maximum, while the currents in both Y and B phases are negative, with magnitude as half of maximum value (0.5). The fluxes due to the currents in the windings are shown in Fig. 29.4(a)-(ii). It may be noted that Φ_R is taken as reference, while Φ_Y leads Φ_R by $60°$ and Φ_B lags Φ_R by $60°$, as can be observed from the direction of currents in all three phases as given earlier. If the fluxes are added to find the resultant in phasor form, the magnitude is found as ($1.5 \cdot \Phi$). It may be noted that this magnitude is same as that found mathematically earlier. Its direction is also shown in same figure. The resultant flux is given by,

$$\Phi_R \angle 0° + \Phi_Y \angle 60° + \Phi_B \angle -60° = (\Phi + 2 \cdot (0.5 \cdot \Phi) \cdot \cos 60°) \cdot 1\angle 0° = (1.5 \cdot \Phi)\angle 0°$$

Now, let us shift to the instant, t_2 ($\theta_2 = 120°$) as shown in Fig. 29.3. The voltages in the two phases, R & B are ($\sqrt{3}/2 = 0.866$) times the maximum value, with R-phase as positive and B-phase as negative. The voltage in phase Y is zero. This is shown in Fig. 29.4(b)-(i). The fluxes due to the currents in the windings are shown in Fig. 29.4(b)-(ii). As given earlier, Φ_R is taken as reference here, while Φ_B lags Φ_R by $60°$. Please note the direction of the currents in both R and B phases. If the fluxes are added to find the resultant in phasor form, the magnitude is found as ($1.5 \cdot \Phi$). The direction is shown in the same figure. The resultant flux is given by,

$$\Phi_R \angle 0° + \Phi_B \angle -60° = (\Phi_R \angle 30° + \Phi_B \angle -30°) \cdot 1.0 \angle -30°$$
$$= (2 \cdot (0.866 \cdot \Phi) \cdot \cos 30°) \cdot 1\angle -30° = (1.5 \cdot \Phi)\angle -30°.$$

It may be noted that the magnitude of the resultant flux remains constant, with its direction shifting by $30° = 120° - 90°$ in the clockwise direction from the previous instant.

Similarly, if we now shift to the instant, t_3 ($\theta_3 = 150°$) as shown in Fig. 29.3. The voltage in the B- phase is negative maximum, while the two other voltages in the phases, R & Y, are half of the maximum value (0.5), and also positive. This is shown in Fig. 29.4(c)-(i). The fluxes due to the currents in the windings are shown in Fig. 29.4(c)-(ii). The reference of flux direction (Φ_R) is given earlier, and is not repeated here. If the fluxes are added to find the resultant in phasor form, the magnitude is found as ($1.5 \cdot \Phi$). The direction is shown in the same figure. The resultant flux is given by,

$$\Phi_R \angle 0° + \Phi_Y \angle -120° + \Phi_B \angle -60° = (\Phi_B \angle 0° + \Phi_R \angle 60° + \Phi_Y \angle -60°) \cdot 1.0 \angle -60°$$
$$= (\Phi + 2 \cdot (0.5 \cdot \Phi) \cdot \cos 60°) \cdot 1\angle -60° = (1.5 \cdot \Phi)\angle -60°.$$

It may be noted that the magnitude of the resultant flux remains constant, with the direction shifting by another 30° in the clockwise direction from the previous instant.

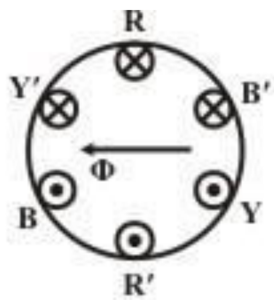
If we now consider the instant, t_4 ($\theta_4 = 180°$) as shown in Fig. 29.3. The voltages in the two phases, Y & B are ($\sqrt{3}/2 = 0.866$) times the maximum value, with Y-phase as positive and B-phase as negative. The voltage in phase R is zero. This is shown in Fig. 29.4(d)-(i). The fluxes due to the currents in the windings are shown in Fig. 29.4(d)-(ii).

If the fluxes are added to find the resultant in phasor form, the magnitude is found as $(1.5 \cdot \Phi)$. The direction is shown in the same figure. The resultant flux is given by,

$$\Phi_Y \angle -120° + \Phi_B \angle -60° = (\Phi_Y \angle -30° + \Phi_B \angle 30°) \cdot 1\angle -90°$$

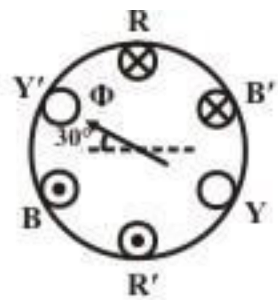
$$= (2 \cdot (0.866 \cdot \Phi) \cdot \cos 30°) \cdot 1\angle -90° = (1.5 \cdot \Phi) \angle -90° .$$

It may be noted that the magnitude of the resultant flux remains constant, with its direction shifting by another 30° in the clockwise direction from the previous instant.



$\theta = 90°$

Fig. 29.4(a)(i)



$\theta = 120°$

Fig. 29.4(b)(i)

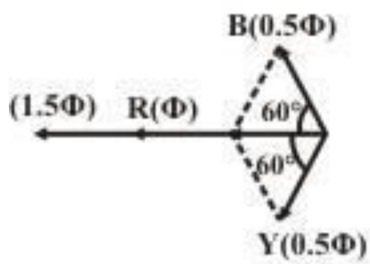


Fig. 29.4(a)(ii)

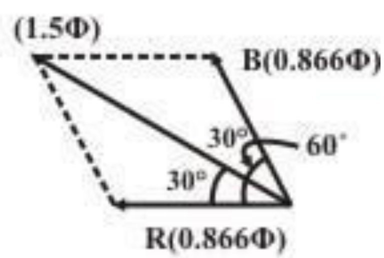
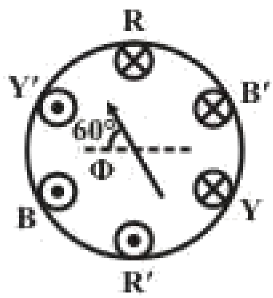
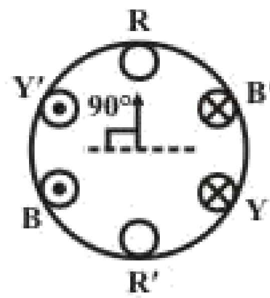


Fig. 29.4(b)(ii)



$\theta = 150°$

Fig. 29.4(c)(i)



$\theta = 180°$

Fig. 29.4(d)(i)

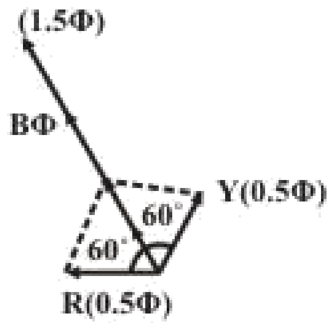


Fig. 29.4(c)(ii)

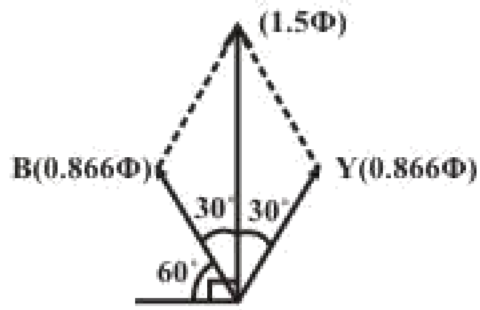


Fig. 29.4(d)(ii)

Fig. 29.4: (i) The currents in the three-phase balanced windings and (ii) The location (position) of the axis of resultant flux (or mmf), with the change in time ($\theta = \omega t$).

A study of the above shows that, as we move from t_1 ($\theta_1 = 90°$) to t_4 ($\theta_4 = 180°$) along the voltage waveforms (Fig. 29.3), the magnitude of the resultant flux remains constant at the value ($1.5 \cdot \Phi$) as shown in Fig. 29.4(ii)-(a-d). If another point, say t_5 ($\theta_5 = 210°$) is taken, it can be easily shown that the magnitude of the resultant flux at that instant remains same at ($1.5 \cdot \Phi$), which value is obtained mathematically earlier. If any other arbitrary point, t (θ) on the waveform is taken, the magnitude of the resultant flux at that instant remains same at ($1.5 \cdot \Phi$). Also, it is seen that the axis of the resultant flux moves through 90° , as the angle, θ changes from 90° to 180° , i.e. by the same angle of 90° . So, if we move through one cycle of the waveform, by 360° (electrical), the axis of the resultant flux also moves through 360° (2-pole stator), i.e. one complete revolution. The rotating magnetic field moves in the clockwise direction as shown, from phase R to phase Y. Please note that, for 2-pole configuration as in this case, the mechanical and electrical angles are same. So, the speed of the rotating magnetic field for this case is 50 rev/sec (rps), or 3,000 rev/min (rpm), as the supply frequency is 50 Hz or c/s, with its magnitude, i.e. resultant, remaining same.

Four-Pole Stator

A 4-pole stator with balanced three-phase winding (Fig. 29.5) is taken as an example. The winding of each phase (one part only), say for example, ($R_1 - R_1'$) is assumed to be concentrated in one slot each, both for forward and return conductors, with required no. as needed. Same is the case for other two phases. The connection of two parts of the winding in R-phase, is also shown in the same figure. The windings for each of three phases are in two parts, with the mechanical angle between the start of adjacent windings being 60° only, whereas the electrical angle remaining same at 120° . As two pairs of poles are there, electrically two cycles, i.e. 720° are there for one complete revolution, with each N-S pair for one cycle of 360° , but the mechanical angle is only 360° . If we move through one cycle of the waveform, by 360° (electrical), the axis of the resultant flux in this case moves through a mechanical angle of 180° , i.e. one pole pair (360° - elec.), or half revolution only. As stated earlier, for the resultant flux axis to make one complete revolution (360° - mech.), two cycles of the waveform (720° - elec.), are required, as No. of poles (p) is four (4). So, for the supply frequency of f = 50 Hz (c/s),
the speed of the rotating magnetic field is given by,

$n_s = (2 \cdot f) / p = f / (p / 2) = 50 / (4 / 2) = 25$ rev/sec (rps), or N_s = 1,500 rev/min (rpm).

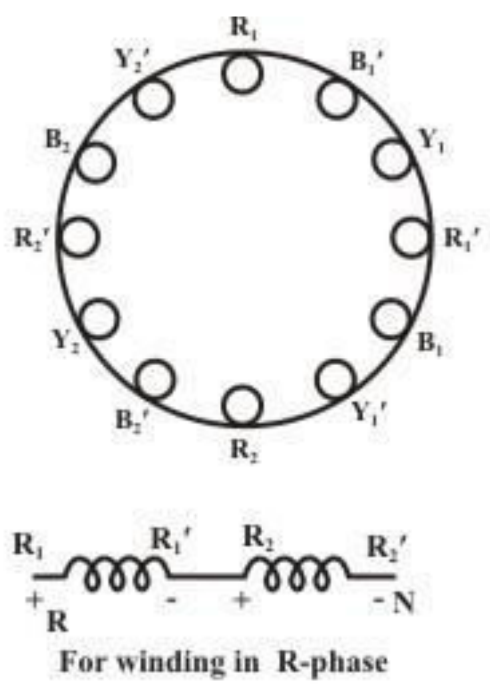


Fig. 29.5: Three-phase balanced windings in a 4-pole stator.

The relation between the synchronous speed, i.e. the speed of the rotating magnetic field, in rpm and the supply frequency in Hz, is given by

$N_s = (60 \cdot 2 \cdot f) / p = (120 \cdot f) / p$

To take another example of a 6 -pole stator, in which, for 50 Hz supply, the synchronous speed is 1,000 rpm, obtained by using the above formula.

The Reversal of Direction of Rotating Magnetic Field

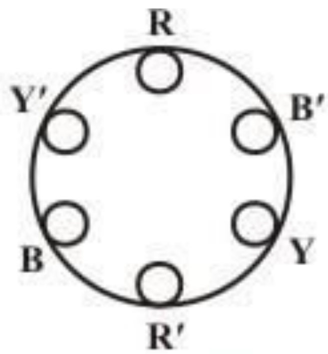


Fig. 29.6: Three-phase windings for Induction motor.

The direction of the rotating magnetic field is reversed by changing the phase sequence to R-B-Y, i.e. changing only the connection of any two of the three phases, and keeping the third one same. The schematic of the balanced three-phase winding for a 2-pole stator, with the winding of each phase assumed to be concentrated in one slot, is redrawn in Fig. 29.6, which is same as shown in Fig. 29.4(i) (a -d) . The space phase between the adjacent windings of any two phases (say R & Y, or R & B) is 120° , i.e. $2\pi/3$ rad (elect.), as a 2-pole stator is assumed. Also, it may be noted that, while the connection to phase R remains same, but the phases, Y and B of the winding are now connected to the phases, B and Y of the supply respectively. The waveforms for the above phase sequence (R-B-Y) are shown in Fig. 29.7. Please note that, the voltage in phase R leads the voltage in phase B, and the voltage in phase B leads the voltage in phase Y. As compared to the three waveforms shown in Fig. 29.3, the two waveforms of the phases Y & B change, while the reference phase R remains same, with the phase sequence reversed as given earlier. The currents in three phases of the stator winding are

$$i_R = I_m \cos \omega t$$

$$i_Y = I_m \cos(\omega t + 120°)$$

$$i_B = I_m \cos(\omega t - 120°) = I_m \cos(\omega t + 240°)$$

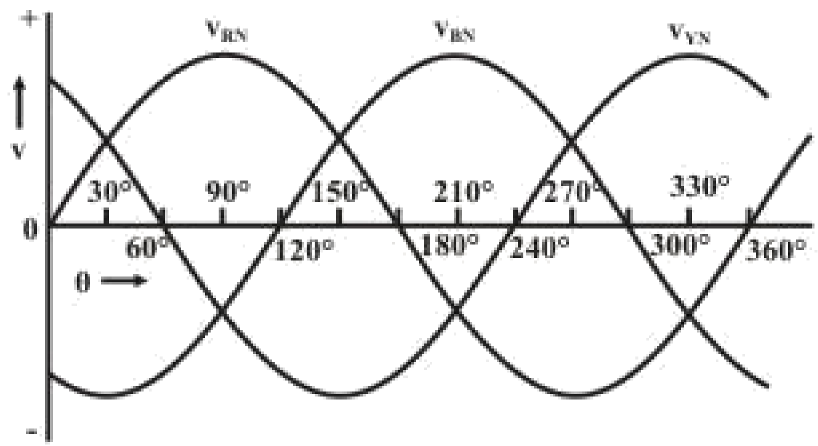


Fig. 29.7: Three phase voltage waveforms with phase sequence R-B-Y.

Without going into the details of the derivation, which has been presented in detail earlier, the resultant mmf wave is obtained as

$F(\theta, t) = F_R + F_Y + F_B$

$= F_m \left[\cos \omega t \cos \theta + \cos(\omega t + 120°) \cos(\theta - 120°) + \cos(\omega t - 120°) \cos(\theta + 120°)\right]$ As shown earlier, the first term of this expression is

$\cos \omega t \cos \theta = 0.5 [\cos(\theta + \omega t) + \cos(\theta - \omega t)]$

The second term is

$\cos(\omega t + 120°) \cos(\theta - 120°) = 0.5 [\cos(\theta + \omega t) + \cos(\theta - \omega t - 240°)]$

Similarly, the third term can be rewritten in the form shown.

The expression is

$F(\theta, t) = 1.5\, F_m \cos(\theta + \omega t)$

$+ 0.5\, F_m \left[\cos(\theta - \omega t) + \cos(\theta - \omega t - 240°) + \cos(\theta - \omega t + 240°)\right]$

As shown or derived earlier, the expression, after simplification, is

$F(\theta, t) = 1.5\, F_m \cos(\theta + \omega t)$

Note that the second part of the expression within square bracket is zero.

It can be shown that the rotating magnetic field now moves in the reverse (i.e., anticlockwise) direction (Fig. 29.6), from phase R to phase B (lagging phase R by 120°) , which is the reverse of earlier (clockwise) direction as shown in Fig. 29.4(i), as the phase sequence is reversed. This is also shown in the final expression of the resultant mmf wave, as compared to the one derived earlier. Alternatively, the reversal of direction of the rotating magnetic field can be derived by the procedure followed in the second method as given earlier.

In this lesson – the first one of this module, it has been shown that, if balanced three-phase voltage is supplied to balanced three-phase windings in the stator of an Induction motor, the resultant flux remains constant in magnitude, but rotates at the synchronous speed, which is related to the supply frequency and No. of poles, for which the winding (stator) has been designed. This is termed as rotating magnetic field formed in the air gap of the motor. The construction of three-phase induction motor (mainly two types of rotor used) will be described, in brief, in the next lesson, followed by the principle of operation.

UNIT-4

Three-phase Induction Motor

Construction and Principle of Operation of IM

In the previous, i.e. first, lesson of this module, the formation of rotating magnetic field in the air gap of an induction motor (IM), has been described, when the three-phase balanced winding of the stator is supplied with three-phase balanced voltage. The construction of the stator and two types of rotor − squirrel cage and wound (slip-ring) one, used for three-phase Induction motor will be presented. Also described is the principle of operation, i.e. how the torque is produced.

Keywords: Three-phase induction motor, cage and wound (slip-ring) rotor, synchronous and rotor speed, slip, induced voltages in stator winding and rotor bar/winding.

After going through this lesson, the students will be able to answer the following questions:

1. How would you identify the two types (cage and wound, or slip-ring) of rotors in three-phase induction motor?
2. What are the merits and demerits of the two types (cage and wound, or slip-ring) of rotors in IM?
3. How is the torque produced in the rotor of the three-phase induction motor?
4. How does the rotor speed differ from synchronous speed? Also what is meant by the term 'slip'?

Construction of Three-phase Induction Motor

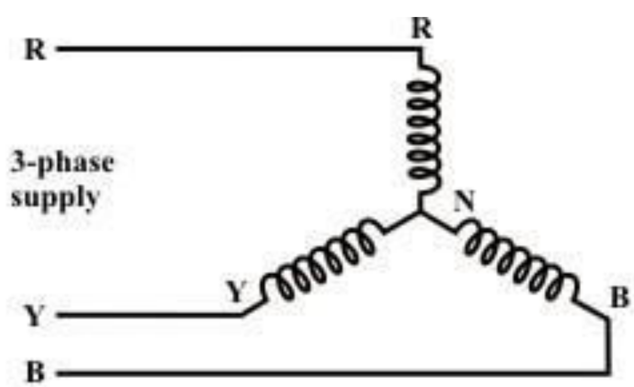


Fig. 30.1: Schematic diagram of the stator windings in a three-phase induction motor.

This is a rotating machine, unlike the transformer, described in the previous module, which is a static machine. Both the machines operate on ac supply. This machine mainly works as a motor, but it can also be run as a generator, which is not much used. Like all rotating machines, it consists of two parts − stator and rotor. In the stator (Fig. 30.1), the winding used is a balanced three-phase one, which means that the number of turns in each phase, connected in star/delta, is equal. The windings of the three phases are placed 120° (electrical) apart, the mechanical angle between the adjacent phases being $[\,(2 \cdot 120°) / p\,]$, where p is no. of poles. For a 4-pole (p = 4) stator, the mechanical angle between the winding of the adjacent phases, is $[(2 \cdot 120°) / 4] = 120° / 2 = 60°$, as shown in Fig. 29.4. The conductors, mostly multi-turn, are placed in the slots, which may be closed, or semi-closed, to keep the leakage inductance low. The start and return parts of the winding are placed nearly 180° , or $(180° - \beta)$ apart. The angle of short chording (β) is nearly equal to 30° , or close to that value. The short chording results in reducing the amount of copper used for the winding, as the length of the conductor needed for overhang part is reduced. There are also other advantages. The section of the stampings used for both stator and rotor, is shown in Fig. 30.2. The core is needed below the teeth to reduce the reluctance of the magnetic path, which carries the flux in the motor (machine). The stator is kept normally inside a support.

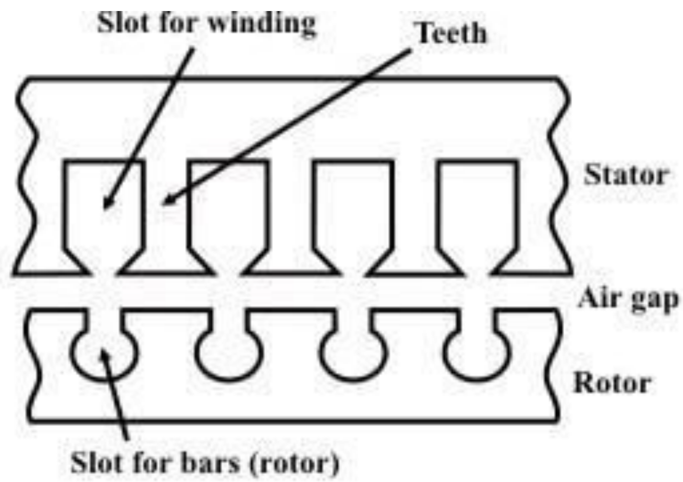


Fig. 30.2: Section for stamping of stator and rotor in IM (not to scale).

There are two types of rotor used in IM, viz. squirrel cage and wound (slip-ring) one. The cage rotor (Fig. 30.3a) is mainly used, as it is cheap, rugged and needs little or no maintainance. It consists of copper bars placed in the slots of the rotor, short circuited at the two ends by end rings, brazed with the bars. This type of rotor is equivalent to a wound (slip-ring) one, with the advantage that this may be used for the stator with different no. of poles. The currents in the bars of a cage rotor, inserted inside the stator, follow the pattern of currents in the stator winding, when the motor (IM) develops torque, such that no. of poles in the rotor is same as that in the stator. If the stator winding of IM is changed, with no. of poles for the new one being different from the earlier one, the cage rotor used need not be changed, thus, can be same, as the current pattern in the rotor bars changes. But the no. of poles in the rotor due to the above currents in the bars is same as no. of poles in the new stator winding. The only problem here is that the equivalent resistance of the rotor is constant. So, at the design stage, the value is so chosen, so as to obtain a certain value of the starting torque, and also the slip at full load torque is kept within limits as needed.

The other type of rotor, i.e., a wound rotor (slip ring) used has a balanced three-phase winding (Fig. 30.3b), being same as the stator winding, but no. of turns used depends on the voltage in the rotor. The three ends of the winding are brought at the three slip-rings, at which points external resistance can be inserted to increase the starting torque requirement. Other three ends are shorted inside. The motor with additional starting resistance is costlier, as this type of rotor is itself costlier than the cage rotor of same power rating, and additional cost of the starting resistance is incurred to increase the starting torque as required. But the slip at full load torque is lower than that of a cage rotor with identical rating, when no additional resistance is used, with direct short-circuiting at the three slip-ring terminals. In both types of rotor, below the teeth, in which bars of a cage rotor, or the conductors of the rotor winding, are placed, lies the iron core, which carries the flux as is the case of the core in the stator. The shaft of the rotor passes below the rotor core. For large diameter of the rotor, a spider is used between the rotor core and the shaft. For a wound (slip-ring) rotor, the rotor winding must be designed for same no. of poles as used for the stator winding. If the no. of poles in the rotor winding is different from no. of poles in the stator winding, no torque will be developed in the motor. It may be noted that this was not the case with cage rotor, as explained earlier.

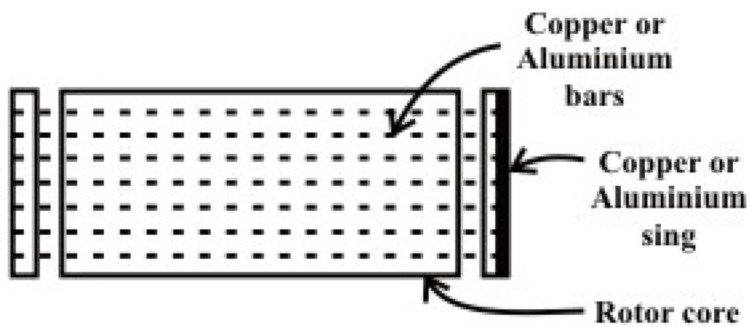


Fig. 30.3(a): Squirrel cage rotor of induction motor

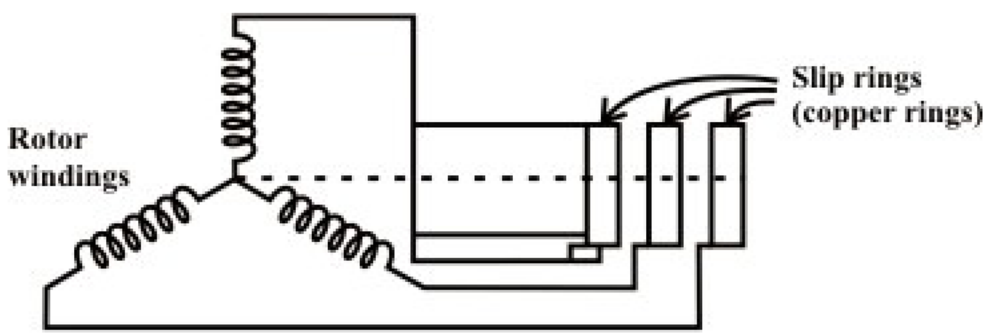


Fig. 30.3(b): Wound rotor (slip ring) of induction motor

The wound rotor (slip ring) shown in Fig. 30.3 (b) is shown as star-connected, whereas the rotor windings can also be connected in delta, which can be converted into its equivalent star configuration. This shows that the rotor need not always be connected in star as shown. The No. of rotor turns changes, as the delta-connected rotor is converted into star-connected equivalent. This point may be kept in mind, while deriving the equivalent circuit as shown in the next lesson (#31), if the additional resistance (being in star) is connected through the slip rings, in series with the rotor winding

Principle of Operation

The balanced three-phase winding of the stator is supplied with a balanced three-phase voltage. As shown in the previous lesson (#29), the current in the stator winding produces a rotating magnetic field, the magnitude of which remains constant. The axis of the magnetic field rotates at a synchronous speed ($n_s = (2 \cdot f) / p$), a function of the supply frequency (f), and number of poles (p) in the stator winding. The magnetic flux lines in the air gap cut both stator and rotor (being stationary, as the motor speed is zero) conductors at the same speed. The emfs in both stator and rotor conductors are induced at the same frequency, i.e. line or supply frequency, with No. of poles for both stator and rotor windings (assuming wound one) being same. The stator conductors are always stationary, with the frequency in the stator winding being same as line frequency. As the rotor winding is short-circuited at the slip-rings, current flows in the rotor windings. The electromagnetic torque in the motor is in the same direction as that of the rotating magnetic field, due to the interaction between the rotating flux produced in the air gap by the current in the stator winding, and the current in the rotor winding. This is as per Lenz's law, as the developed torque is in such direction that it will oppose the cause, which results in the current flowing in the rotor winding. This is irrespective of the rotor type used − cage or wound one, with the cage rotor, with the bars short-circuited by two end-rings, is considered equivalent to a wound one The current in the rotor bars interacts with the air-gap flux to develop the torque, irrespective of the no. of poles for which the winding in the stator is designed. Thus, the cage rotor may be termed as universal one. The induced emf and the current in the rotor are due to the relative velocity between the rotor conductors and the rotating flux in the air-gap, which is maximum, when the rotor is stationary ($n_r = 0.0$). As the rotor starts rotating in the same direction, as that of the rotating magnetic field due to production of the torque as stated earlier, the relative velocity decreases, along with lower values of induced emf and current in the rotor. If the rotor speed is equal that of the rotating magnetic field, which is termed as synchronous speed, and also in the same direction, the relative velocity is <u>zero</u>, which causes both the induced emf and current in the rotor to be reduced to zero. Under this condition, torque will <u>not</u> be produced. So, for production of positive (motoring) torque, the rotor speed must always be lower than the synchronous speed. The rotor speed is never equal to the synchronous speed in an IM. The rotor speed is determined by the mechanical load on the shaft and the total rotor losses, mainly comprising of copper loss.

The difference between the synchronous speed and rotor speed, expressed as a ratio of the synchronous speed, is termed as 'slip' in an IM. So, slip (s) in pu is

$$s = \frac{n_s - n_r}{n_s} = 1 - \frac{n_r}{n_s} \qquad \text{or, } n_r = (1 - s) \cdot n_s$$

where, n_s and n_r are synchronous and rotor speeds in rev/s.

In terms of $N_s = 60 \cdot n_s$ and $N_r = 60 \cdot n_r$, both in rev/min (rpm), slip is

$$s = (N_s - N_r) / N_s$$

If the slip is expressed in %, then $s = [(N_s - N_r) / N_s] \cdot 100$

Normally, for torques varying from no-load (≈ zero) to full load value, the slip is proportional to torque. The slip at full load is 4-5% (0.04-0.05).

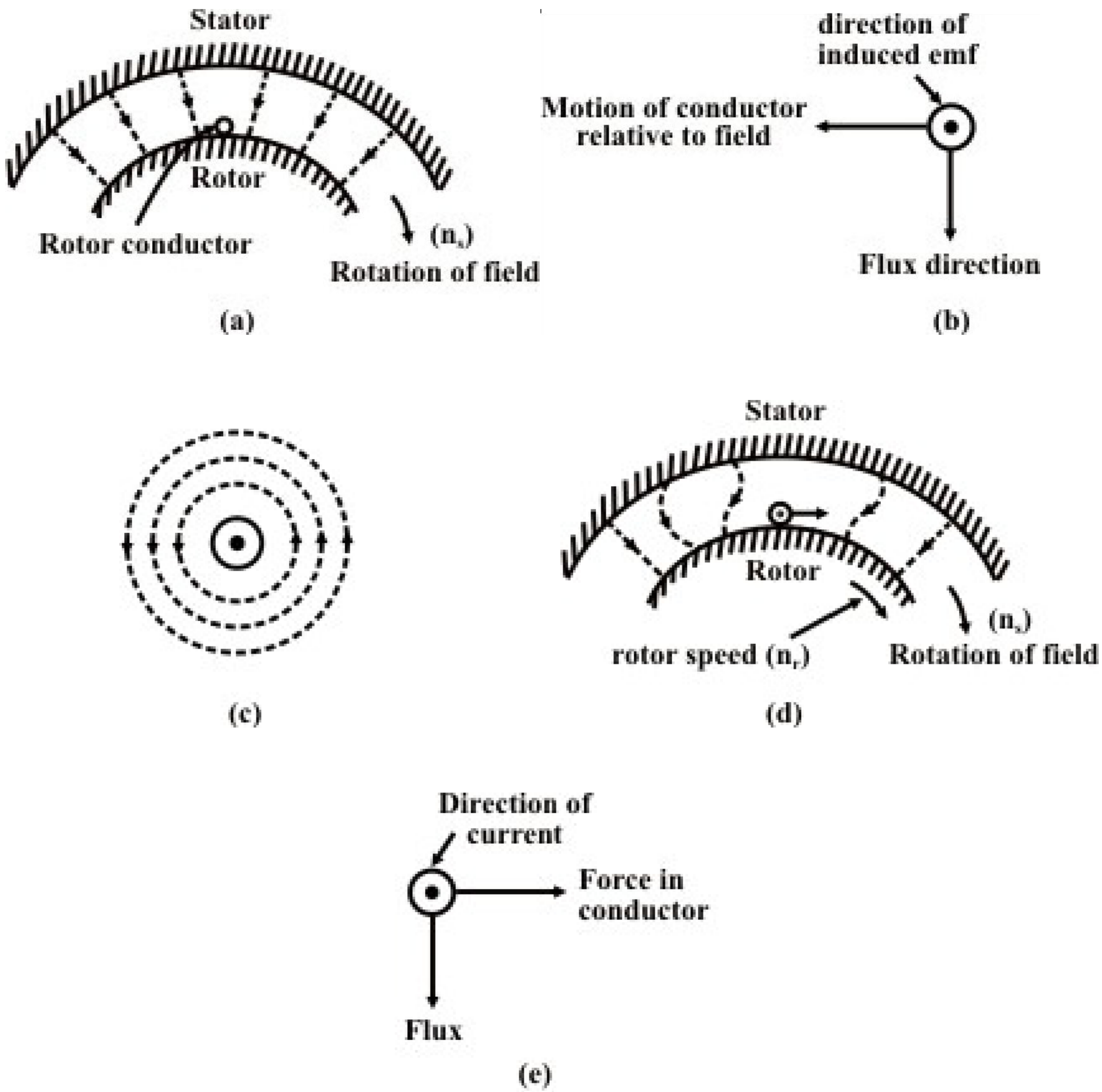


Fig. 30.4: Production of torque

An alternative explanation for the production of torque in a three-phase induction motor is given here, using two rules (right hand and left hand) of Fleming. The stator and rotor, along with air-gap, is shown in Fig. 30.4a. Both stator and rotor is shown there as surfaces, but without the slots as given in Fig, 30.2. Also shown is the path of the flux in the air gap. This is for a section, which is under North pole, as the flux lines move from stator to rotor. The rotor conductor shown in the figure is at rest, i.e., zero speed (stand-still). The rotating magnetic field moves past the conductor at synchronous speed in the clockwise direction. Thus, there is relative movement between the flux and the rotor conductor. Now, if the magnetic field, which is rotating, is assumed to be at standstill as shown in Fig. 30.4b, the conductor will move in the direction shown. So, an emf is induced in the rotor conductor as per Faraday's law, due to change in flux linkage. The direction of the induced emf as shown in the figure can be determined using Fleming's right hand rule.

As described earlier, the rotor bars in the cage rotor are short circuited via end rings. Similarly, in the wound rotor, the rotor windings are normally short-circuited externally via the slip rings. In both cases, as emf is induced in the rotor conductor (bar), current flows there, as it is short circuited. The flux in the air gap, due to the current in the rotor conductor is shown in Fig. 30.4c. The flux pattern in the air gap, due to the magnetic fields produced by the stator windings and the current carrying rotor conductor, is shown in Fig. 304d. The flux lines bend as shown there. The property of the flux lines is to travel via shortest path as shown in Fig. 30.4a. If the flux lines try to move to form straight line, then the rotor conductor has to move in the direction of the rotating magnetic field, but not at the same speed, as explained earlier. The current carrying rotor conductor and the direction of flux are shown in Fig. 30.4e. It is known that force is produced on the conductor carrying current, when it is placed in a magnetic field. The direction of the force on the rotor conductor is obtained by using Fleming's left hand rule, being same as that of the rotating magnetic field. Thus, the rotor experiences a motoring torque in the same direction as that of the rotating magnetic field. This briefly describes how torque is produced in a three-phase induction motor.

The frequency of the induced emf and current in the rotor

As given earlier, both the induced emf and the current in the rotor are due to the relative velocity between the rotor conductors and the rotating flux in the air-gap, the speed of which is the synchronous speed ($N_s = (120 \cdot f) / p$). The rotor speed is

$$N_r = (1 - s) \cdot N_s$$

The frequency of the induced emf and current in the rotor is

$$f_r = p \cdot (n_s - n_r) = s \cdot (p \cdot n_s) = s \cdot f$$

For normal values of slip, the above frequency is small. Taking an example, with full load slip as 4% (0.04), and supply (line) frequency as 50 Hz, the frequency (Hz) of the rotor induced emf and current, is $f_r = 0.04 \cdot 50.0 = 2.0$, which is very small, whereas the frequency (f) of the stator induced emf and current is 50 Hz, i.e. line frequency. At standstill, i.e. rotor stationary ($n_r = 0.0$), the rotor frequency is same as line frequency, as shown earlier, with slip [s = 1.0 (100%)]. The reader is requested to read the next lesson (#31), where some additional points are included in this matter. Also to note that the problems are given there (#31).

In this lesson – the second one of this module, the construction of a three-phase Induction Motor has been presented in brief. Two types of rotor – squirrel cage and wound (slip-ring) ones, along with the stator part, are described. Then, the production of torque in IM, when the balanced stator winding is fed from balanced three-phase voltage, with the balanced rotor winding in a wound one being short -circuited, is taken up. In the next lesson, the equivalent circuit per phase of IM will be derived first. Then, the complete power flow diagram is presented.

UNIT4

Three-phase Induction Motor

Torque-Slip (speed) Characteristics of Induction Motor (IM)

Instructional Objectives

- Derivation of the expression for the gross torque developed as a function of slip (speed) of Induction motor
- Sketch the above characteristics of torque-slip (speed), explaining the various features
- Derive the expression of maximum torque and the slip (speed) at which it occurs
- Draw the above characteristics with the variation in input (stator) voltage and rotor resistance

Introduction

In the previous, i.e. third, lesson of this module, starting with the formulas for the induced emfs per phase in both stator and rotor windings, the equivalent circuit per phase of the three-phase induction motor (IM), has been derived. The relation between the rotor input, rotor copper loss and rotor output (gross) are derived next. Finally, the various losses − copper losses (stator/rotor), iron loss (stator) and mechanical loss, including the determination of efficiency, and also power flow diagram, are presented. In this lesson, firstly, the torque-slip (speed) characteristics of IM, i.e., the expression of the gross torque developed as a function of slip, will be derived. This is followed by the sketch of the different characteristics, with the variations in input (stator) voltage and rotor resistance, along with the features. Lastly, the expression of maximum torque developed and the slip (speed) at which it occurs, are derived.

Keywords: The equivalent circuit per phase of IM, gross torque developed, torque-slip (speed) characteristics, maximum torque, slip at maximum torque, variation of the characteristics with changes in input (stator) voltage and rotor resistance.

Gross Torque Developed

The current per phase in the rotor winding (the equivalent circuit of the rotor, per phase is shown in Fig. 31.1) is (as given in earlier lesson (#31))

$$I_2 = \frac{s\,E_r}{\sqrt{(r_2)^2 + (s\,x_2)^2}} = \frac{E_r}{\sqrt{(r_2/s)^2 + (x_2)^2}}$$

Please note that the symbols used are same as given in the earlier lesson.

In a similar way, the output power (gross) developed (W) is the loss in the fictitious resistance in the equivalent circuit as shown earlier, which is

$$P_0 = 3\,(I_2)^2\,[r_2\,(1-s)/s] = \frac{3\,(s\,E_r)^2\,[r_2\,(1-s)/s]}{\left[(r_2)^2 + (s\,x_2)^2\right]} = \frac{3\,(E_r)^2\;r_2\,s\,(1-s)}{\left[(r_2)^2 + (s\,x_2)^2\right]}$$

The motor speed in rps is $n_r = (1-s)\,n_s$

The motor speed (angular) in rad/s is $\omega_r = (1-s)\,\omega_s$

The gross torque developed in $N\ m$ is

$$T_0 = \frac{P_0}{\omega_r} = \frac{3\,(E_r)^2\; r_2\, s\,(1-s)}{(1-s)\,\omega_s\left[(r_2)^2+(s\,x_2)^2\right]} = \frac{3\,(E_r)^2\; r_2\, s}{2\pi n_s\left[(r_2)^2+(s\,x_2)^2\right]}$$

The synchronous speed (angular) is $\omega_s = 2\pi\; n_s$

The input power to the rotor (or the power transferred from the stator via air gap) is the loss in the total resistance (r_2 / s), which is

$$P_i = 3\,(I_2)^2\,(r_2\;/\,s) = \frac{3(s\,E\,)^2\quad (r\,/\,s)}{\left[(r_2)^2+(s\;x_2)^2\right]} = \frac{3\,(E\,)^2\qquad r\;s}{\left[(r_2)^2+(s\;x_2)^2\right]}$$

The relationship between the input power and the gross torque developed is $P_i = \omega_s\, T_0$
So, the input power is also called as torque in synchronous watts, or the torque is

$$T_0 = P_i\,/\,\omega_s$$

Torque-slip (speed) Characteristics

The torque-slip or torque-speed characteristic, as per the equation derived earlier, is shown in Fig. 32.1. The slip is $s = (\omega_s - \omega_r)\,/\,\omega_s = (n_s - n_r)\,/\,n_s = 1 - (n_r\,/\,n_s)$. The range of speed, n_r is between 0.0 (standstill) and n_s (synchronous speed). The range of slip is between 0.0 ($n_r = n_s$) and 1.0 ($n_r = 0.0$).

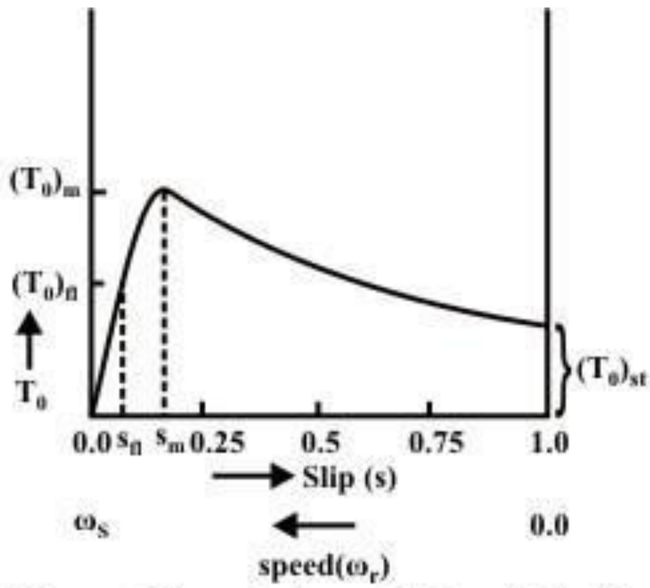


Fig. 32.1: Terque-slip(speed) characteristics of Induction Motor

For low values of slip, $r_2 \gg (s\;x_2)$. So, torque is

$$T_0 = \frac{3\,(E_r)^2\quad r_2\, s}{\omega_s\,(r_2)^2} = \frac{3(E_r)^2}{\omega_s}\;\frac{s}{r_2}$$

This shows that $T_0\;\; s$, the characteristic being linear. The following points may be noted. The output torque developed is zero (0.0), at $s = 0.0$, or if the motor is rotated at synchronous speed ($n_r = n_s$). This has been described in lesson No. 30, when the principle of operation was presented. Also, the slip at full load (output torque = $(T_0)_{fl}$) is normally 4-5% ($s_{fl} = 0.04 - 0.05$), the full load speed of IM being 95-96% of synchronous speed ($(n_r)_{fl} = (1 - s_{fl})\, n_s = (0.05 - 0.96)\, n_s$).

For large values of slip, $r_2 << (s\ x_2)$. So, torque is

$$T_0 = \frac{3\,(E_r)^2}{\omega_s}\,\frac{r_2\, s}{(s\ x_2)^2} = \frac{3\,(E_r)^2}{\omega_s}\,\frac{r_2}{s\,(x_2)^2}$$

This shows that, $T_0\ (1/s)$, the characteristic being hyperbolic. The starting torque ($s = 1.0$, or $n_r = 0.0$) developed, along with starting current, is discussed later.

So, starting from low value of slip ($s > 0.0$), at which torque is proportional to slip, whereas for large values of slip ($s < 1.0$), torque is inverse proportional to slip, both being derived earlier. In the characteristic shown, it may be observed that torque reaches a maximum value, which can be obtained in the following way. The relation between torque and slip is

$$T_0 = \frac{K\ r_2\ s}{\left[(r_2)^2 + (s\ x_2)^2\right]} \quad \text{where, } K = 3\,(E_r)^2/\omega_s$$

$$\text{or, } \frac{1}{T_0} = \frac{\left[(r_2)^2 + (s\ x_2)^2\right]}{K\ r_2\ s} = \frac{1}{K}\left[\frac{r_2}{s} + \frac{s\,(x_2)^2}{r_2}\right]$$

To determine the maximum value of torque (T_0) in terms of slip, the minimum value of its inverse ($1/T_0$) need be determined from the relation,

$$\frac{d}{ds}\left(\frac{1}{T_0}\right) = \frac{1}{K}\left[-\frac{r_2}{s^2} + \frac{(x_2)^2}{r_2}\right] = 0$$

from which $s^2 = \dfrac{(r_2)^2}{(x_2)^2}$ or, $s = r_2 / x_2$.

Please note that, for motoring condition as shown earlier, slip, s is positive (+ve), as $n_r < n_s$. At this slip, $s = s_m$, $r_2 = s_m\, x_2$. This may be termed as slip at maximum torque.

The motor speed is [$(n_r)_m = (1 - s_m)\, n_s$]. This value of slip is small, for normal wound rotor (or slip ring) IM, without any additional resistance inserted in the rotor circuit. This value is higher in the case of squirrel cage IM. Substituting the value of s, the maximum value of torque is

$$T_{0m} = \frac{K}{2x_2} = \frac{3\,(E_r)^2}{\omega_s}\,\frac{1}{2x_2}$$

which shows that it is independent of r_2 . The maximum torque is also termed as pull-out torque. If the load torque on the motor exceeds this value, the motor will stall, i,e. will come to standstill condition.

The values of maximum torque and the slip at that torque, can be obtained by using

$$\frac{d}{ds}(T_0) = 0.0$$

which is not shown here.

It may be observed from the torque-slip characteristic (Fig. 32.1), or described earlier, that the output torque developed increases, if the slip increases from 0.0 to s_m, or the motor speed decreases from n_s to $(n_r)_m$. This ensures stable operation of IM in this region ($0.0 < s < s_m$), for constant load torque. But the output torque developed decreases, if the slip increases from s_m to 1.0, or the motor speed decreases from $(n_r)_m$ .to zero (0.0). This results in unstable operation of IM in this region ($s_m < s < 1.0$), for constant load torque. However, for fan type loads with the torque as (T_L $(n_r)^2$), stable operation of IM is achieved in this region ($s_m < s < 1.0$).

Starting Current and Torque

The starting current (rotor) is

$$(I_2)_{st} = \frac{E_r}{\sqrt{(r_2)^2 + (x_2)^2}}$$

as slip at starting ($n_r = 0.0$) is 1.0, which is the same at standstill (or stalling condition). The magnitude of the induced voltage per phase in the stator winding is nearly same as input voltage per phase fed to the stator, if the voltage drop in the stator impedance, being small, is neglected, i.e. $V_s \approx E_s$. As shown in the earlier lesson (#31), the ratio of the induced emfs per phase in the stator and rotor winding can be taken as the ratio of the effective turns in two windings, i.e. $E_s / E_r = T_s' / T_r'$, where $T_s' = k_{ws} \, T_s$ and $T_r' = k_{wr} \, T_r$. The winding factor for the stator winding is $k_{ws} = k_{ds} \, k_{ps}$. Same formula is used for the above factor in the rotor winding, assuming it to be wound rotor one.

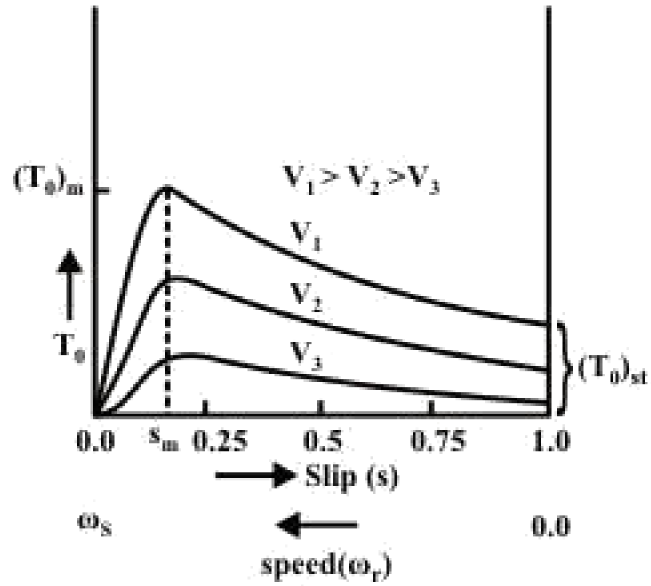
The starting current in the stator winding can be shown as $(I_s)_{st} = (I_r)_{st} \, (T_r' / T_s')$, neglecting the no load current. This current is normally large, much greater than full load current. This current is reduced by using starters in both types (cage and wound rotor) of IM, which will be taken up in the next lesson.

The starting torque in $N \cdot m$ is

$$(T_0)_{st} = 3 \, ((I_2)_{st})^2 \, r_2 / \omega_s = \frac{3 \, (E_r)^2 \, r_2}{\omega_s \, [(r_2)^2 + (x_2)^2]}$$

This expression is obtained substituting $s = 1.0$ in the expression of T_0 derived earlier. If the starter is used, the starting torque is also reduced, as is the case with starting current.

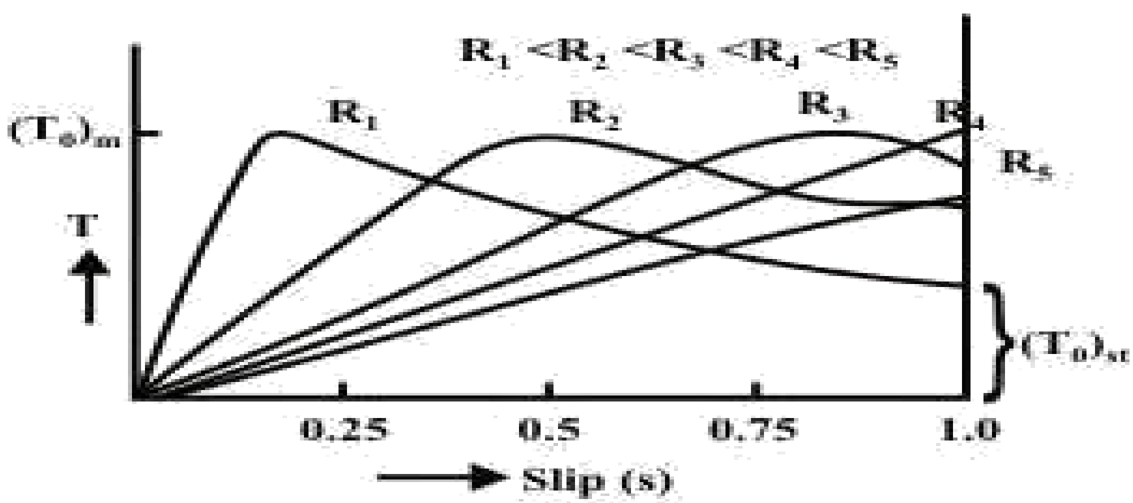
Torque-slip (speed) Characteristics,
with variation in input (stator) voltage and rotor circuit resistance



(a)

Fig. 32.2: Terque-speed characteristics for (a) variation in input(stator)voltage.

The set of torque-slip characteristics with variation in input (stator) voltage is shown in Fig. 32.2a. The point to note that the torque at a given slip decreases with the decrease in input (stator) voltage, as $T_0 \; V^2$. The characteristics shown are for decreasing stator voltages ($V_1 > V_2 > V_3$). The speed decreases or the slip increases with constant load torque, as the input (stator) voltage decreases. The region for stable operation with constant load torque remains same ($0.0 < s < s_m$), as given earlier. But again, stable operation can be obtained in the region ($s_m < s < 1.0$), with fan type loads with the torque as ($T_L \; (n_r)^2$). Another problem is that the maximum or pull-out torque decreases as (T_0 $)_m \; V^2$, where V is input (stator) voltage, which is a drawback with constant load torque operation..



(b)

Fig. 32.2: variation in total rotor resistance.

The set of torque-slip characteristics with variation in rotor circuit resistance is shown in Fig. 32.2b. The characteristics shown are for increasing rotor circuit resistances ($r_2 > R_2 > R_3 > R_4$). The point to note that, the maximum torque remains same for all the characteristics. This has been shown earlier that the maximum torque depends on rotor reactance only, but not on rotor circuit resistance. Only the slip at maximum torque increases with the increase in rotor circuit resistance. So, for constant load torque operation, the slip increases or the speed decreases with the increase in rotor circuit resistance. The motor efficiency decreases, as the rotor copper loss increases with the increase in slip. The load torque remains same, but the output power decreases, as the speed decreases. Also, it may be observed that the starting torque increases with the increase in rotor circuit resistance, with the total rotor circuit resistance lower than rotor reactance. The starting torque is equal to the maximum torque, when the total rotor circuit resistance is equal to rotor reactance. If the rotor circuit resistance is more than rotor reactance, the starting torque decreases.

In this lesson − the fourth one of this module, the expression of gross torque developed, as a function of slip (speed), in IM has been derived first. The sketches of the different torque-slip (speed) characteristics, with the variations in input (stator) voltage and rotor resistance, are presented, along with the explanation of their features. Lastly, the expression of maximum torque developed and also the slip, where it occurs, have been derived. In the next lesson, the various types of starters used in IM will be presented, along with the need of the starters, followed by the comparison of the starting current and torque developed using the starters.

UNIT-4

Three-phase Induction Motor

Different Types of Starters for Induction Motor (IM)

Instructional Objectives

- Need of using starters for Induction motor
- Two (Star-Delta and Auto-transformer) types of starters used for Squirrel cage Induction motor
- Starter using additional resistance in rotor circuit, for Wound rotor (Slip-ring) Induction motor

Introduction

In the previous, i.e. fourth, lesson of this module, the expression of gross torque developed, as a function of slip (speed), in IM has been derived first. The sketches of the different torque-slip (speed) characteristics, with the variations in input (stator) voltage and rotor resistance, are presented, along with the explanation of their features. Lastly, the expression of maximum torque developed and also the slip, where it occurs, have been derived. In this lesson, starting with the need for using starters in IM to reduce the starting current, first two (Star- Delta and Auto-transformer) types of starters used for Squirrel cage IM and then, the starter using additional resistance in rotor circuit, for Wound rotor (Slip-ring) IM, are presented along with the starting current drawn from the input (supply) voltage, and also the starting torque developed using the above starters.

Keywords: Direct-on-Line (DOL) starter, Star- delta starter, auto-transformer starter, rotor resistance starter, starting current, starting torque, starters for squirrel cage and wound rotor induction motor, need for starters.

Direct-on-Line (DOL) Starters

Induction motors can be started Direct-on-Line (DOL), which means that the rated voltage is supplied to the stator, with the rotor terminals short-circuited in a wound rotor (slip-ring) motor. For the cage rotor, the rotor bars are short circuited via two end rings. Neglecting stator impedance, the starting current in the stator windings is (see lesson 32) is

$$(I_1)_{st} = \frac{E_r'}{\sqrt{(r_2')^2 + (x_2')^2}}$$

where,

$(I_1)_{st} = (I_2')_{st} = (I_2)_{st} / a =$ Starting current in the motor (stator)

$a = T_s' / T_r'$ = Effective turns ratio between stator and rotor windings

$E_s = E_r' = a\, E_r$ = Input voltage per phase to the motor (stator)

E_r = Induced emf per phase in the rotor winding

$r_2' = a^2\, r_2$ = Rotor resistance in terms of stator winding

$x_2' = a^2\, x_2$ = Rotor reactance at standstill in terms of stator winding

The input voltage per phase to the stator is equal to the induced emf per phase in the stator winding, as the stator impedance is neglected (also shown in the last lesson (#32)).

In the formula for starting current, no load current is neglected. It may be noted that the starting current is quite high, about 4-6 times the current at full load, may be higher, depending on the rating of IM, as compared to no load current.

The starting torque is ($(T_0)_{st}$ $[(I_1)_{st}]^2$), which shows that, as the starting current increases, the starting torque also increases. This results in higher accelerating torque (minus the load torque and the torque component of the losses), with the motor reaching rated or near rated speed quickly.

Need for Starters in IM

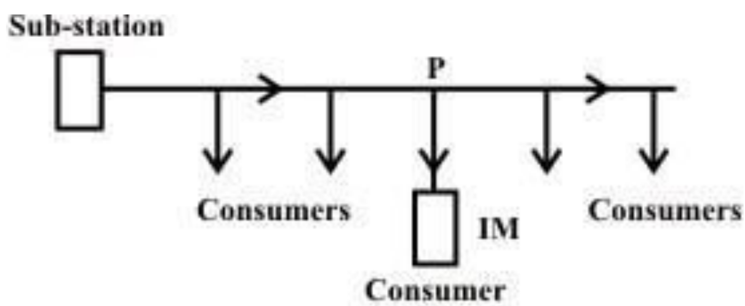


Fig. 33.1: The distribution line fed from substation for supply to various consumers.

The main problem in starting induction motors having large or medium size lies mainly in the requirement of high starting current, when started direct-on-line (DOL). Assume that the distribution line is starting from a substation (Fig. 33.1), where the supply voltage is constant. The line feeds a no. of consumers, of which one consumer has an induction motor with a DOL starter, drawing a high current from the line, which is higher than the current for which this line is designed. This will cause a drop (dip) in the voltage, all along the line, both for the consumers between the substation and this consumer, and those, who are in the line after this consumer. This drop in the voltage is more than the drop permitted, i.e. higher than the limit as per ISS, because the current drawn is more than the current for which the line is designed. Only for the current lower the current for which the line is designed, the drop in voltage is lower the limit. So, the supply authorities set a limit on the rating or size of IM, which can be started DOL. Any motor exceeding the specified rating, is not permitted to be started DOL, for which a starter is to be used to reduce the current drawn at starting.

Starters for Cage IM

The starting current in IM is proportional to the input voltage per phase (V_s) to the motor (stator), i.e. $(I_1)_{st}$ E_s, where, $|V_s| \approx |E_s|$, as the voltage drop in the stator impedance is small compared to the input voltage, or $|V_s| = |E_s|$, if the stator impedance is neglected. This has been shown earlier. So, in a (squirrel) cage induction motor, the starter is used only to decrease the input voltage to the motor so as to decrease the starting current. As described later, this also results in decrease of starting torque.

Star-Delta Starter

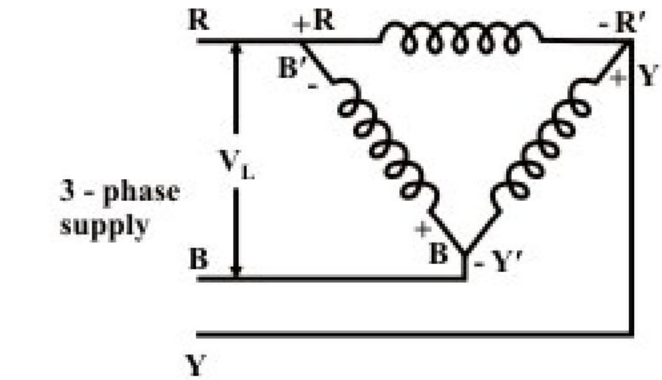


Fig. 33.2(a): Delta-connected stator winding of IM at run

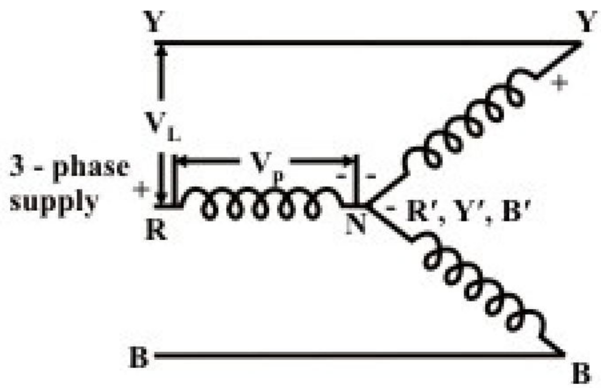


Fig. 33.2(b): same winding of IM, reconnected as star at start.

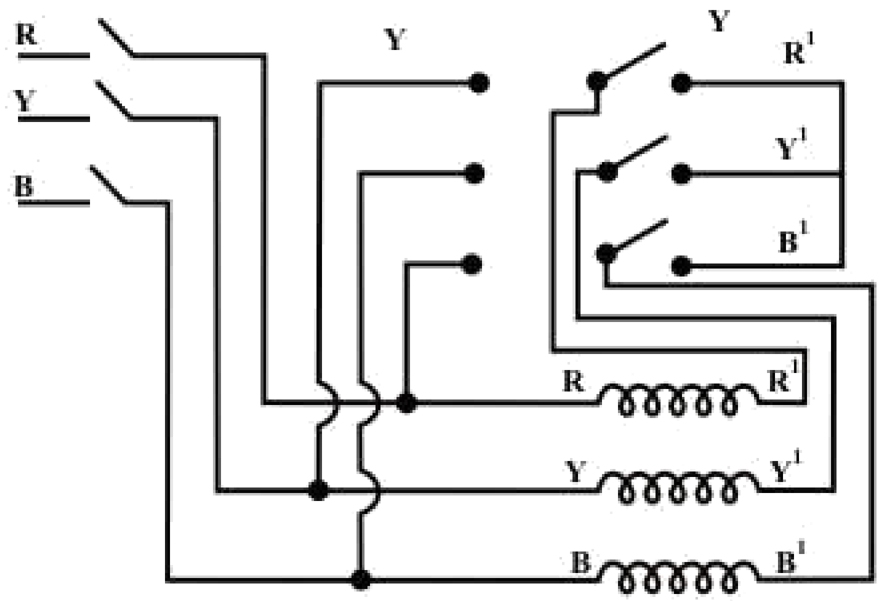
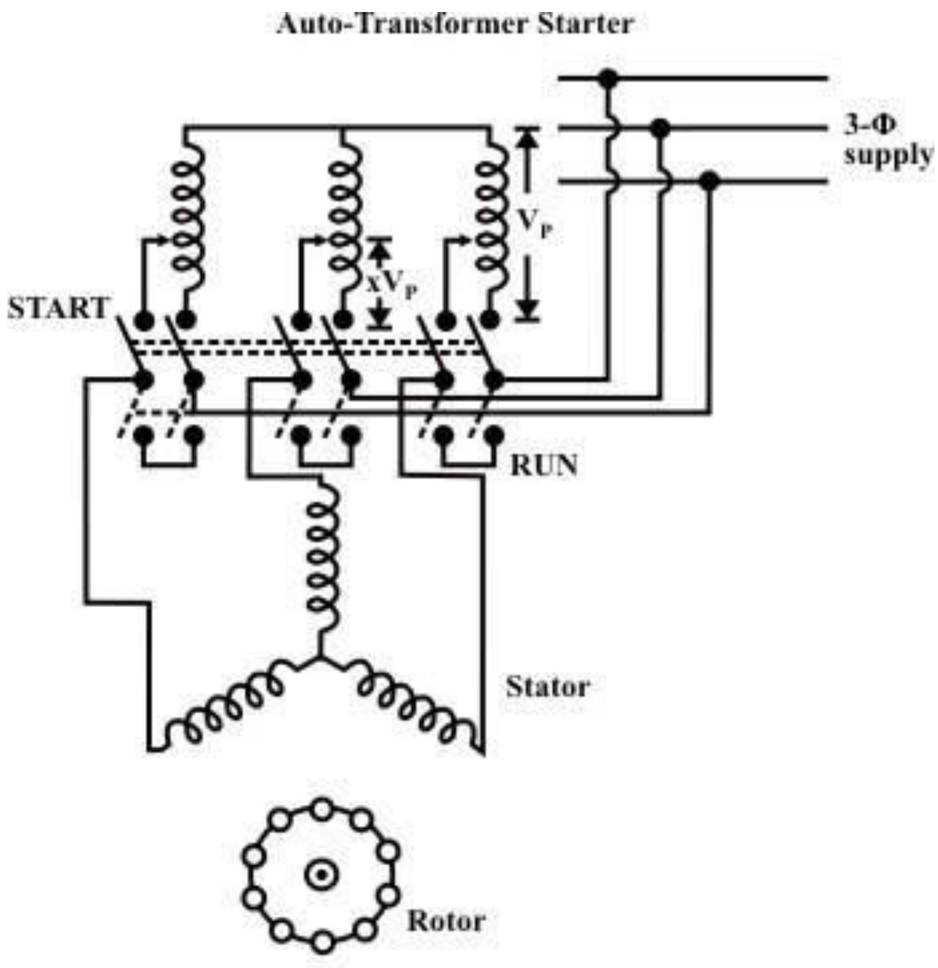


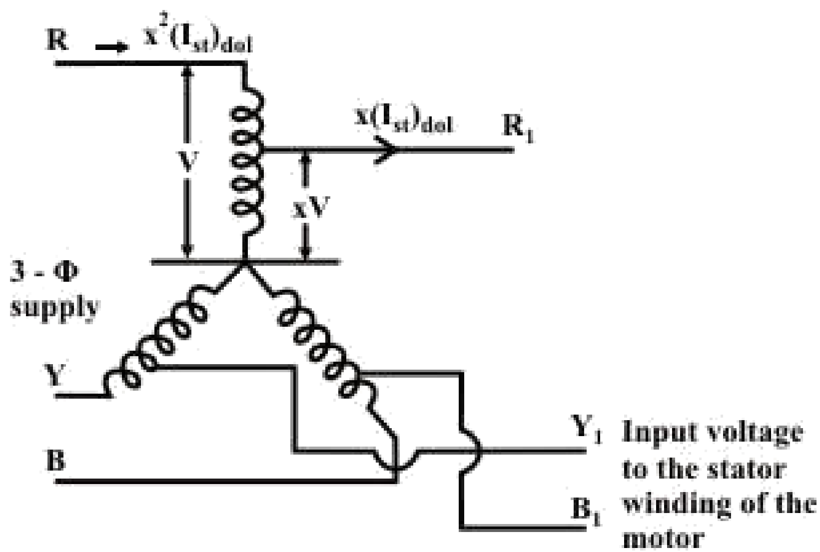
Fig. 33.2(c): T.P.D.T. switch used to first start the motor with the windings connected in star and then switch for delta connection in run position(star-delta starter).

This type is used for the induction motor, the stator winding of which is nominally delta-connected (Fig. 33.2a). If the above winding is reconnected as star (Fig. 33.2b), the voltage per phase supplied to each winding is reduced by $1/\sqrt{3}$ (0.577) . This is a simple starter, which can be easily reconfigured as shown in Fig. 33.2c. As the voltage per phase in delta connection is V_s , the phase current in each stator winding is (V_s / Z_s), where Z_s is the impedance of the motor per phase at standstill or start (stator impedance and rotor impedance referred to the stator, at standstill). The line current or the input current to the motor is [$(I_1)_{st} = \sqrt{3}\, V_s / Z_s$], which is the current, if the motor is started direct-on-line (DOL). Now, if the stator winding is connected as star, the phase or line current drawn from supply at start (standstill) is [$(V_s / Z_s) / \sqrt{3}$], which is ($1/3 = (1/\sqrt{3})^2$) of the starting current, if DOL starter is used. The voltage per phase in each stator winding is now ($V_s / \sqrt{3}$). So, the starting current using star-delta starter is reduced by 33.3%. As for starting torque, being proportional to the square of the current in each of the stator windings in two different connections as shown earlier, is also reduced by ($1/3 = (1/\sqrt{3})^2$), as the ratio of the two currents is ($1/\sqrt{3}$), same as that (ratio) of the voltages applied to each winding as shown earlier. So, the starting torque is reduced by 33.3%, which is a disadvantage of the use of this starter. The load torque and the loss torque, must be lower than the starting torque, if the motor is to be started using this starter. The advantage is that, no extra component, except that shown in Fig. 33.2c, need be used, thus making it simple. As shown later, this is an auto-transformer starter with the voltage ratio as 57.7%. Alternatively, the starting current in the second case with the stator winding reconnected as star, can be found by using star-delta conversion as given in lesson #18, with the impedance per phase after converting to delta, found as ($3 Z_s$), and the starting current now being reduced to (1/ 3) of the starting current obtained using DOL starter, with the stator winding connected in delta.

Auto-transformer Starter



(a)



(b)

Fig. 33.3: Auto-transformer starter for IM

An auto-transformer, whose output is fed to the stator and input is from the supply (Fig. 33.3), is used to start the induction motor. The input voltage of IM is $x\,V_s$, which is the output voltage of the auto-transformer, the input voltage being V_s. The output voltage/input voltage ratio is x, the value of which lies between 0.0 and 1.0 ($0.0 < x < 1.0$). Let $(I_1)_{st}$ be the starting current, when the motor is started using DOL starter, i.e applying rated input voltage. The input current of IM, which is the output current of auto-transformer, is $x\,(I_1)_{st}$, when this starter is used with input voltage as $x\,V_s$. The input current of auto-transformer, which is the starting current drawn from the supply, is $x^2\,(I_1)_{st}$, obtained by equating input and output volt-amperes, neglecting losses and assuming nearly same power factor on both sides. As discussed earlier, the starting torque, being proportional to the square of the input current to IM in two cases, with and without auto-transformer (i.e. direct), is also reduced by x^2, as the ratio of the two currents is x, same as that (ratio) of the voltages applied to the motor as shown earlier. So, the starting torque is reduced by the same ratio as that of the starting current. If the ratio is $x = 0.8\ (80\%)$, both starting current and torque are $x^2 = (0.8)^2 = 0.64\ (64\%)$ times the values of starting current and torque with DOL starting, which is nearly 2 times the values obtained using star-delta starter. So, the disadvantage is that starting current is increased, with the result that lower rated motor can now be started, as the current drawn from the supply is to be kept within limits, while the advantage is that the starting torque is now doubled, such that the motor can start against higher load torque. The star-delta starter can be considered equivalent to an auto-transformer starter with the ratio, $x = 0.577\ (57.7\%)$. If $x = 0.7\ (70\%)$, both starting current and torque are $x^2 = (0.7)^2 = 0.49 \approx 0.5\ (50\%)$ times the values of starting current and torque with DOL starting, which is nearly 1.5 times the values obtained using star-delta starter. By varying the value of the voltage ratio x of the auto-transformer, the

values of the starting current and torque can be changed. But additional cost of auto-transformer with intermittent rating is to be incurred for this purpose.

Rotor Resistance Starters for Slip-ring (wound rotor) IM

In a slip-ring (wound rotor) induction motor, resistance can be inserted in the rotor circuit via slip rings (Fig. 33.4), so as to increase the starting torque. The starting current in the rotor winding is

$$(I_2)_{st} = \frac{E_r}{\sqrt{(r_2 + R_{ext})^2 + (x_2)^2}}$$

where R_{ext} = Additional resistance per phase in the rotor circuit.

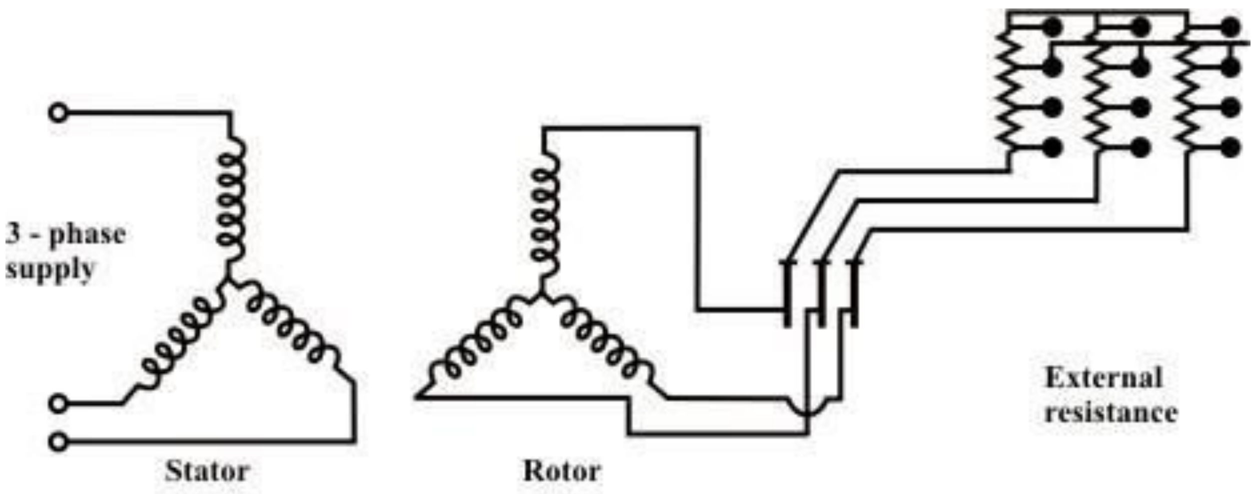


Fig. 33.4: Rotor resistance starter for IM

The input (stator) current is proportional to the rotor current as shown earlier. The starting current (input) reduces, as resistance is inserted in the rotor circuit. But the starting torque, [$(T_0)_{st} = 3\,[(I_2)_{st}]^2\,(r_2 + R_{ext})$] increases, as the total resistance in the rotor circuit is increased. Though the starting current decreases, the total resistance increases, thus resulting in increase of starting torque as shown in Fig. 32.2b, and also obtained by using the expression given earlier, for increasing values of the resistance in the rotor circuit. If the additional resistance is used only for starting, being rated for intermittent duty, the resistance is to be decreased in steps, as the motor speed increases. Finally, the external resistance is to be completely cut out, i.e. to be made equal to zero (0.0), thus leaving the slip-rings short-circuited. Here, also the additional cost of the external resistance with intermittent rating is to be incurred, which results in decrease of starting current, along with increase of starting torque, both being advantageous. Also it may be noted that the cost of a slip-ring induction is higher than that of IM with cage rotor, having same power rating. So, in both cases, additional cost is to be incurred to obtain the above advantages. This is only used in case higher starting torque is needed to start IM with high load torque. It may be observed from Fig. 32.2b that the starting torque increases till it reaches maximum value, i.e. ($(T_0)_{st} < (T_0)_m$), as the external resistance in

the rotor circuit is increased, the range of total resistance being [$r_2 < (r_2 + R_{ext}) < x_2$].

The range of external resistance is between zero (0.0) and ($x_2 - r_2$). The starting torque is equal to the maximum value, i.e. ($(T_0)_{st} = (T_0)_m$), if the external resistance inserted is equal to ($x_2 - r_2$). But, if the external resistance in the rotor circuit is increased further, i.e. [$R_{ext} > (x_2 - r_2)$], the starting torque decreases ($(T_0)_{st} < (T_0)_m$). This is, because the starting current decreases at a faster rate, even if the total resistance in the rotor circuit is increased.

In this lesson − the fifth one of this module, the direct-on-line (DOL) starter used for IM, along with the need for other types of starters, has been described first. Then, two types of starters − star-delta and auto-transformer, for cage type IM, are presented. Lastly, the rotor resistance starter for slip-ring (wound rotor) IM is briefly described. In the next (sixth and last) lesson of this module, the various types of single phase induction motors, along with the starting methods, will be presented.

UNIT-4

Three-phase Induction Motor

Starting Methods for Single-phase Induction Motor

Instructional Objectives

- Why there is no starting torque in a single-phase induction motor with one (main) winding in the stator?
- Various starting methods used in the single-phase induction motors, with the introduction of additional features, like the addition of another winding in the stator, and/or capacitor in series with it.

Introduction

In the previous, i.e. fifth, lesson of this module, the direct-on-line (DOL) starter used in three -phase IM, along with the need for starters, has been described first. Two types of starters − star-delta, for motors with nominally delta-connected stator winding, and auto-transformer, used for cage rotor IM, are then presented, where both decrease in starting current and torque occur. Lastly, the rotor resistance starter for slip-ring (wound rotor) IM has been discussed, where starting current decreases along with increase in starting torque. In all such cases, additional cost is to be incurred. In the last (sixth) lesson of this module, firstly it is shown that there is no starting torque in a single-phase induction motor with only one (main) winding in the stator. Then, the various starting methods used for such motors, like, say, the addition of another (auxiliary) winding in the stator, and/or capacitor in series with it.

Keywords: Single-phase induction motor, starting torque, main and auxiliary windings, starting methods, split-phase, capacitor type, motor with capacitor start/run.

Single-phase Induction Motor

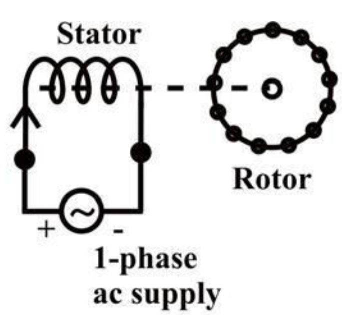


Fig. 34.1: Single Phase Induction Motor

The winding used normally in the stator (Fig. 34.1) of the single-phase induction motor (IM) is a distributed one. The rotor is of squirrel cage type, which is a cheap one, as the rating of this type of motor is low, unlike that for a three-phase IM. As the stator winding is fed from a single-phase supply, the flux in the air gap is alternating only, not a synchronously rotating one produced by a poly-phase (may be two- or three-) winding in the stator of IM. This type of alternating field cannot produce a torque ($(T_0)_{st} = 0.0$), if

the rotor is stationery ($\omega_r = 0.0$). So, a single-phase IM is not self-starting, unlike a three-phase one. However, as shown later, if the rotor is initially given some torque in either direction ($\omega_r \neq 0.0$), then immediately a torque is produced in the motor. The motor then accelerates to its final speed, which is lower than its synchronous speed. This is now explained using double field revolving theory.

Double field revolving theory

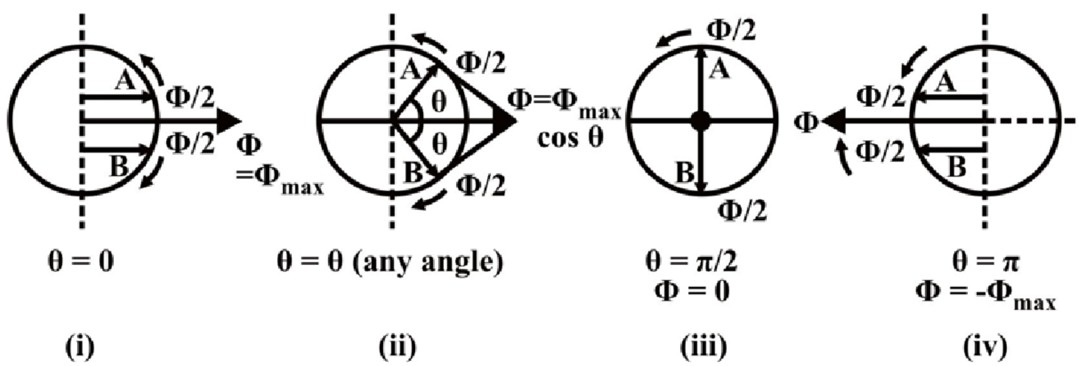


Fig. 34.2(a): Position of the pulsating and rotating in fluxes with change in angle (θ)

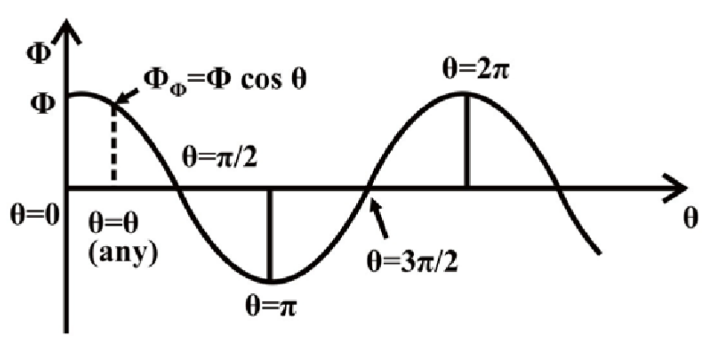


Fig. 34.2(b): Pulsating (sinusoidal) flux as a function of space angle (θ)

When the stator winding (distributed one as stated earlier) carries a sinusoidal current (being fed from a single-phase supply), a sinusoidal space distributed mmf, whose peak or maximum value pulsates (alternates) with time, is produced in the air gap. This sinusoidally varying flux (φ) is the sum of two rotating fluxes or fields, the magnitude of which is equal to half the value of the alternating flux ($\varphi / 2$), and both the fluxes rotating synchronously at the speed, ($n_s = (2 \cdot f) / P$) in opposite directions. This is shown in Fig. 34.2a. The first set of figures (Fig. 34.1a (i-iv)) show the resultant sum of the two rotating fluxes or fields, as the time axis (angle) is changing from $\theta = 0°$ to $\pi (180°)$. Fig. 34.2b

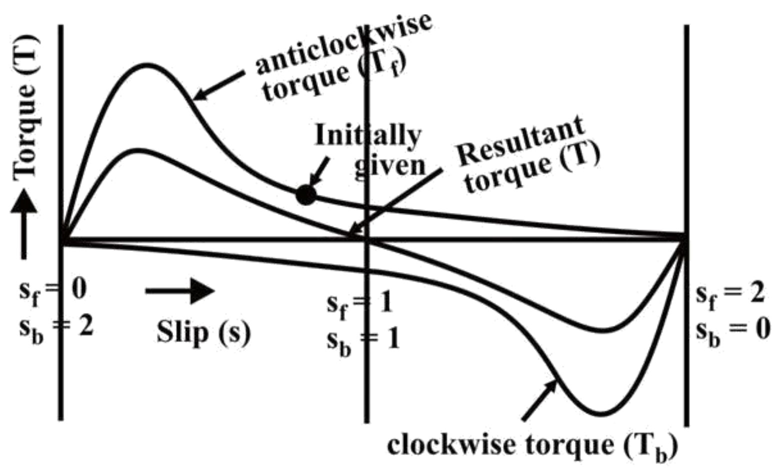


Fig. 34.3:Speed-torque characteristics of single phase induction motor

The flux or field rotating at synchronous speed, say, in the anticlockwise direction, i.e. the same direction, as that of the motor (rotor) taken as positive induces emf (voltage) in the rotor conductors. The rotor is a squirrel cage one, with bars short circuited via end rings. The current flows in the rotor conductors, and the electromagnetic torque is produced in the same direction as given above, which is termed as positive (+ve). The other part of flux or field rotates at the same speed in the opposite (clockwise) direction, taken as negative. So, the torque produced by this field is negative (-ve), as it is in the clockwise direction, same as that of the direction of rotation of this field. Two torques are in the opposite direction, and the resultant (total) torque is the difference of the two torques produced (Fig. 34.3). If the rotor is stationary ($\omega_r = 0.0$), the slip due to forward (anticlockwise) rotating field is $s_f = 1.0$. Similarly, the slip due to backward rotating field is also $s_b = 1.0$. The two torques are equal and opposite, and the resultant torque is 0.0 (zero). So, there is no starting torque in a single-phase IM.

But, if the motor (rotor) is started or rotated somehow, say in the anticlockwise (forward) direction, the forward torque is more than the backward torque, with the resultant torque now being positive. The motor accelerates in the forward direction, with the forward torque being more than the backward torque. The resultant torque is thus positive as the motor rotates in the forward direction. The motor speed is decided by the load torque supplied, including the losses (specially mechanical loss).

Mathematically, the mmf, which is distributed sinusoidally in space, with its peak value pulsating with time, is described as $F = F_{peak} \cos\theta$, θ (space angle) measured from the winding axis. Now, $F_{peak} = F_{\max} \cos\omega t$. So, the mmf is distributed both in space and time, i.e. $F = F_{\max} \cos\theta \cdot \cos\omega t$. This can be expressed as,

$$F = (F_{\max} / 2) \cdot \cos(\theta - \omega t) + (F_{\max} / 2) \cdot \cos(\theta + \omega t),$$

which shows that a pulsating field can be considered as the sum of two synchronously rotating fields ($\omega_s = 2\pi n_s$). The forward rotating field is, $F_f = (F_{\max} / 2) \cdot \cos(\theta - \omega t)$, and the backward rotating field is, $F_b = (F_{\max} / 2) \cdot \cos(\theta + \omega t)$. Both the fields have the same amplitude equal to $(F_{\max}/2)$, where $F_{\max}$ is the maximum value of the pulsating mmf along the axis of the winding.

When the motor rotates in the forward (anticlockwise) direction with angular speed ($\omega_r = 2\pi n_r$), the slip due to the forward rotating field is,

$$s_f = (\omega_s - \omega_r)/\omega_s = 1 - (\omega_r/\omega_s), \text{ or } \omega_r = (1 - s_f)\omega_s .$$

Similarly, the slip due to the backward rotating field, the speed of which is ($-\omega_s$), is,

$$s_b = (\omega_s + \omega_r)/\omega_s = 1 + (\omega_r/\omega_s) = 2 - s_b ,.$$

The torques produced by the two fields are in opposite direction. The resultant torque is,

$$T = T_f - T_b$$

It was earlier shown that, when the rotor is stationary, $T_f = T_b$, with both $s_f = s_b = 1.0$, as $\omega_r = 0.0$ or $n_r = 0.0$. Therefore, the resultant torque at start is 0.0 (zero).

Starting Methods

The single-phase IM has no starting torque, but has resultant torque, when it rotates at any other speed, except synchronous speed. It is also known that, in a balanced two-phase IM having two windings, each having equal number of turns and placed at a space angle of 90° (electrical), and are fed from a balanced two-phase supply, with two voltages equal in magnitude, at an angle of 90°, the rotating magnetic fields are produced, as in a three-phase IM. The torque- speed characteristic is same as that of a three-phase one, having both starting and also running torque as shown earlier. So, in a single-phase IM, if an auxiliary winding is introduced in the stator, in addition to the main winding, but placed at a space angle of 90° (electrical), starting torque is produced. The currents in the two (main and auxiliary) stator windings also must be at an angle of 90°, to produce maximum starting torque, as shown in a balanced two-phase stator. Thus, rotating magnetic field is produced in such motor, giving rise to starting torque. The various starting methods used in a single-phase IM are described here.

Resistance Split-phase Motor

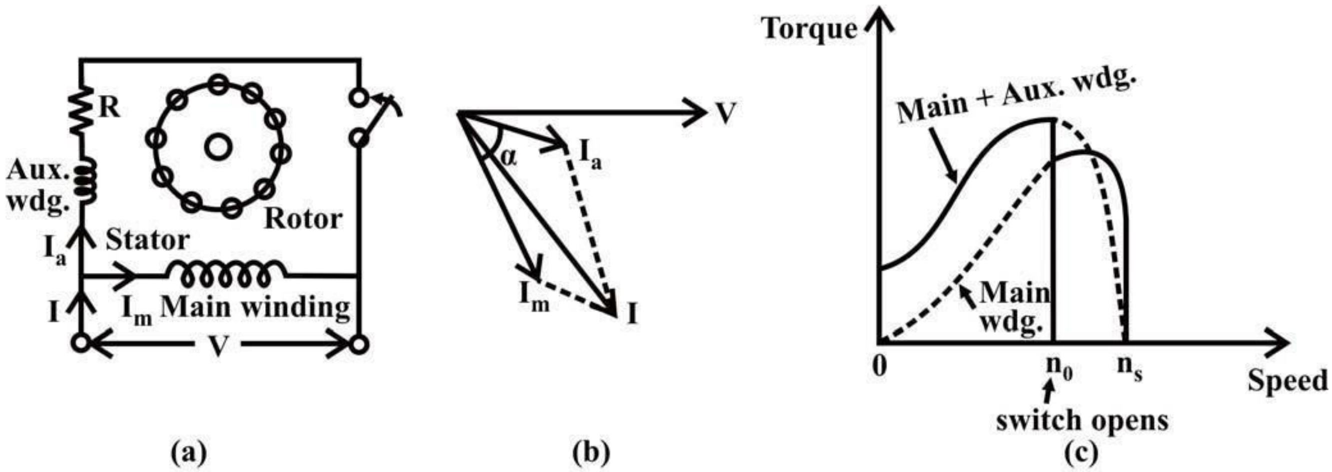


Fig. 34.4: Resistance Split-phase Induction Motor
(a) Schematic Diagram
(b) Phasor Diagram
(c) Torque-Speed characteristic

The schematic (circuit) diagram of this motor is given in Fig. 34.4a. As detailed earlier, another (auxiliary) winding with a high resistance in series is to be added along with the main winding in the stator. This winding has higher resistance to reactance (R_a / X_a) ratio as compared to that in the main winding, and is placed at a space angle of 90° from the main winding as given earlier. The phasor diagram of the currents in two windings and the input voltage is shown in Fig. 34.4b. The current (I_a) in the auxiliary winding lags the voltage (V) by an angle, φ_a , which is small, whereas the current (I_m) in the main winding lags the voltage (V) by an angle, φ_m , which is nearly 90° . The phase angle between the two currents is ($90° - \varphi_a$), which should be at least 30° . This results in a small amount of starting torque. The switch, S (centrifugal switch) is in series with the auxiliary winding. It automatically cuts out the auxiliary or starting winding, when the motor attains a speed close to full load speed. The motor has a starting torque of 100−200% of full load torque, with the starting current as 5-7 times the full load current. The torque-speed characteristics of the motor with/without auxiliary winding are shown in Fig. 34.4c. The change over occurs, when the auxiliary winding is switched off as given earlier. The direction of rotation is reversed by reversing the terminals of any one of two windings, but not both, before connecting the motor to the supply terminals. This motor is used in applications, such as fan, saw, small lathe, centrifugal pump, blower, office equipment, washing machine, etc.

Capacitor-start Motor

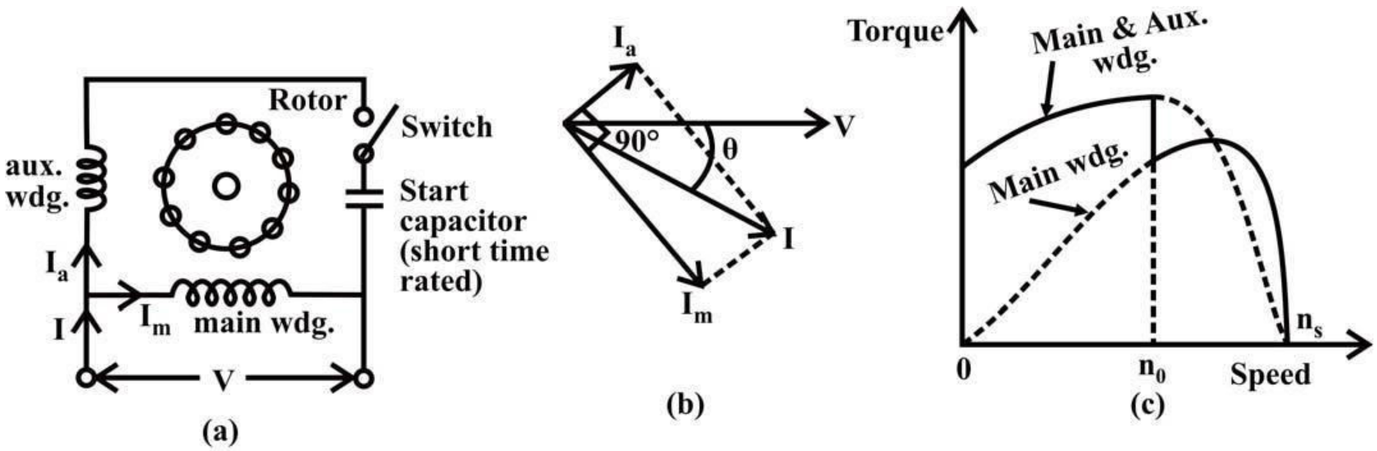
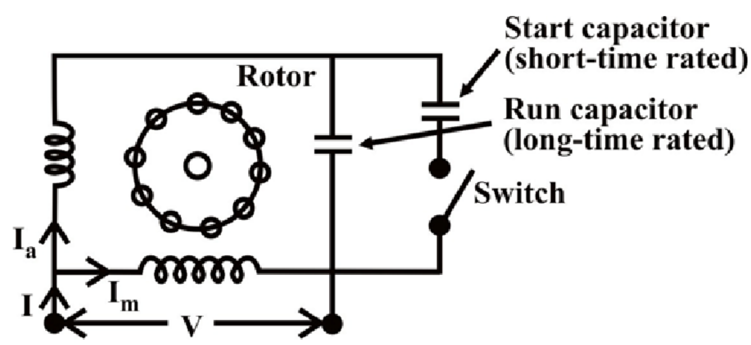


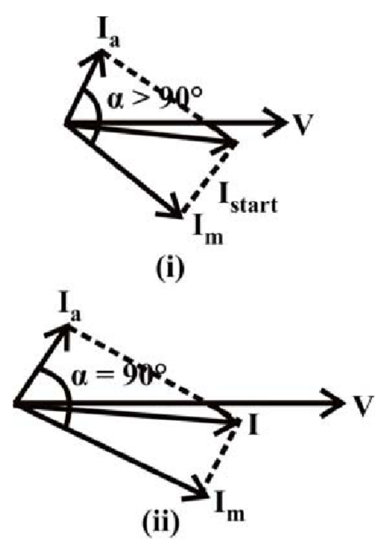
Fig. 34.5: Capacitor-Start Induction Motor
(a) Schematic Diagram
(b) Phasor Diagram
(c) Torque-Speed characteristic

The schematic (circuit) diagram of this motor is given in Fig. 34.5a. It may be observed that a capacitor along with a centrifugal switch is connected in series with the auxiliary winding, which is being used here as a starting winding. The capacitor may be rated only for intermittent duty, the cost of which decreases, as it is used only at the time of starting. The function of the centrifugal switch has been described earlier. The phasor diagram of two currents as described earlier, and the torque-speed characteristics of the motor with/without auxiliary winding, are shown in Fig. 34.5b and Fig. 34.5c respectively. This motor is used in applications, such as compressor, conveyor, machine tool drive, refrigeration and air-conditioning equipment, etc.

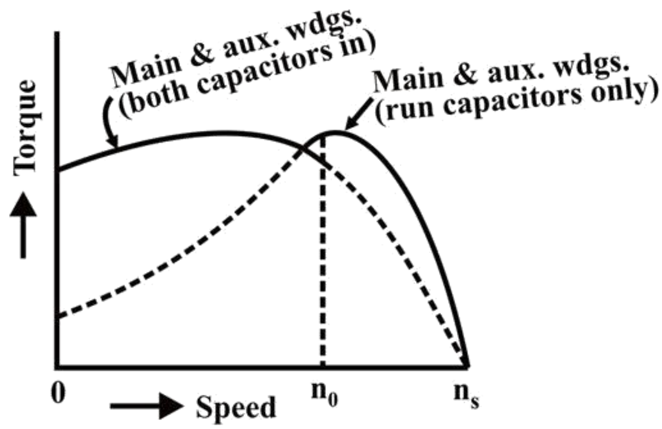
Capacitor-start and Capacitor-run Motor



(a)



(b)



(c)

Fig. 34.6: Capacitor-Start and Capacitor-run Induction Motor
(a) Schematic Diagram
(b) Phasor Diagrams
(c) Torque-Speed characteristics

In this motor (Fig. 34.6a), two capacitors − C_s for starting, and C_r for running, are used. The first capacitor is rated for intermittent duty, as described earlier, being used only for starting. A centrifugal switch is also needed here. The second one is to be rated for continuous duty, as it is used for running. The phasor diagram of two currents in both cases, and the torque-speed characteristics with two windings having different values of capacitors, are shown in Fig. 34.6b and Fig. 34.6c respectively. The phase difference between the two currents is ($\varphi_m + \varphi_a > 90°$) in the first case (starting), while it is 90° for second case (running). In the second case, the motor is a balanced two phase one, the two windings having same number of turns and other conditions as given earlier, are also satisfied. So, only the forward rotating field is present, and the no backward rotating field exists. The efficiency of the motor under this condition is higher. Hence, using two capacitors, the performance of the motor improves both at the time of starting and then running. This motor is used in applications, such as compressor, refrigerator, etc.

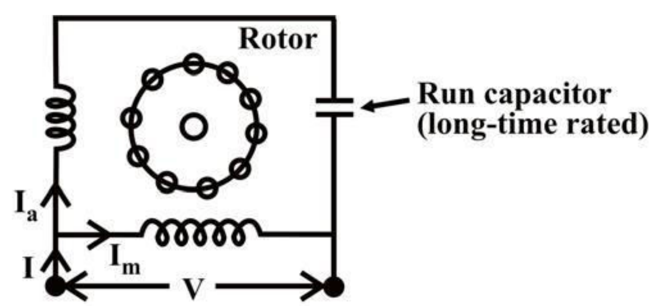


Fig. 34.7: Schematic Diagram of Capacitor-run Induction Motor

Beside the above two types of motors, a Permanent Capacitor Motor (Fig. 34.7) with the same capacitor being utilised for both starting and running, is also used. The power factor of this motor, when it is operating (running), is high. The operation is also quiet and smooth. This motor is used in applications, such as ceiling fans, air circulator, blower, etc.

UNIT-4

DC Machines

EMF & Torque Equation

Contents

37.1 Goals of the lesson

After going through the lesson, the student will be able to understand:

1. the factors on which induced voltage in the armature depend.
2. the factors on which the electromagnetic torque developed depend.
3. the derivation of the emf and torque equations.
4. that emf and torque equations are applicable to both generator and motor operations.
5. why the generated emf in motor called the back emf.
6. armature reaction, its ill effects and remedial measures.
7. the purpose of compensating winding – its location and connection.
8. the purpose of interpole – its location and connection.
9. the difference between the GNP (geometric neutral plane) and MNP (magnetic neutral plane).

37.2 Introduction

Be it motor or generator operations, the analysis of D.C machine performance center around two fundamental equations namely the emf equation and the torque equations. In fact both motoring and generating actions go together in d.c machines. For example in a d.c motor there will be induced voltage across the brushes in the same way as in a generator. The induced voltage in d.c motor is however called by a different name *back emf*. Thus the factors on which induced emf in generator depend will be no different from motor action. In fact the same emf equation can be employed to calculate induced emf for both generator and motor operation. In the same way same *torque equation* can be used to calculate electromagnetic torque developed in both motoring and generating actions.

In this lesson these two fundamental equations have been derived.

Field patterns along the air gap of the machine for both motor and generator modes are explained. The ill effects of armature mmf (for a loaded machine) is discussed and possible remedial measures are presented. Calculation of cross magnetizing and demagnetizing AT (ampere turns) of d.c machines with shifted brush are presented. Depending on your course requirement the derivation for these ATs with sifted brush may be avoided. Finally, the phenomenon of *armature reaction* and a brief account of commutation are presented.

37.3 EMF & Torque Equations

In this section we shall derive two most fundamental and important formulas (namely emf and torque equations) for d.c machine in general. These will be extensively used to analyse the performance and to solve problems on d.c machines.

37.3.1 EMF Equation

Consider a D.C generator whose field coil is excited to produce a flux density distribution along the air gap and the armature is driven by a prime mover at constant speed as shown in figure.

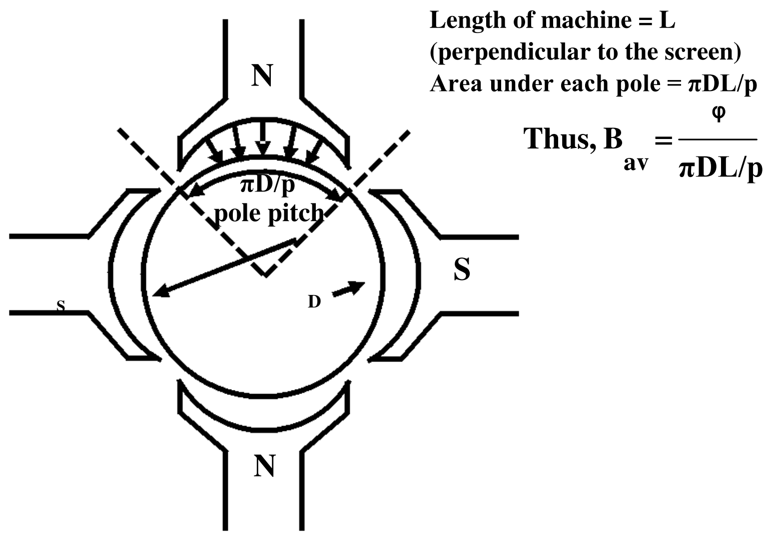


Figure 37.1: Pole pitch & area on armature surface per pole.

Let us assume a p polar d.c generator is driven (by a prime mover) at n rps. The excitation of the stator field is such that it produces a φ Wb flux per pole. Also let z be the total number of armature conductors and a be the number of parallel paths in the armature circuit. In general, as discussed in the earlier section the magnitude of the voltage from one conductor to another is likely to very since flux density distribution is *trapezoidal* in nature. Therefore, total average voltage across the brushes is calculated on the basis of average flux density B_{av}. If D and L are the rotor diameter and the length of the machine in meters then area under each pole is —

Hence average flux density in the gap is given by .

Average flux density B_{av}

$$= \frac{\varphi}{\left(\frac{\pi D}{p}\right)L}$$

$$= \frac{\varphi p}{\pi DL}$$

Induced voltage in a single conductor

$$= B_{av}Lv$$

Number of conductors present in each parallel path

$$= \frac{z}{a}$$

If v is the tangential velocity then, v

$$= \pi Dn$$

Therefore, total voltage appearing across the brushes

$$= \frac{z}{a} B_{av} Lv$$

$$= \frac{z}{a}\frac{\varphi p}{\pi DL} L\pi Dn$$

Thus voltage induced across the armature, E_A

$$= \frac{pz}{a}\varphi n \qquad (37.1)$$

We thus see that across the armature a voltage will be generated so long there exists some flux per pole and the machine runs with some speed. Therefore irrespective of the fact that the machine is operating as generator or as motor, armature has an induced voltage in it governed essentially by the above derived equation. This emf is called *back emf* for motor operation.

37.3.2 Torque equation

Whenever armature carries current in presence of flux, conductor experiences force which gives rise to the electromagnetic torque. In this section we shall derive an expression for the electromagnetic torque T_e developed in a d.c machine. Obviously T_e will be developed both in motor and generator mode of operation. It may be noted that the direction of conductor currents reverses as we move from one pole to the other. This ensures unidirectional torque to be produced. The derivation of the torque expression is shown below.

Let, I_a = Armature current

Average flux density $B_{av} = \dfrac{\varphi p}{\pi DL}$

Then, $\dfrac{I_a}{a}$ = Current flowing through each conductor.

Force on a single conductor $= B_{av}\dfrac{I_a}{a}L$

Torque on a single conductor $= B_{av}\dfrac{I_a}{a}L\dfrac{D}{2}$

Total electromagnetic torque developed, $T_e = zB_{av}\dfrac{I_a}{a}L\dfrac{D}{2}$

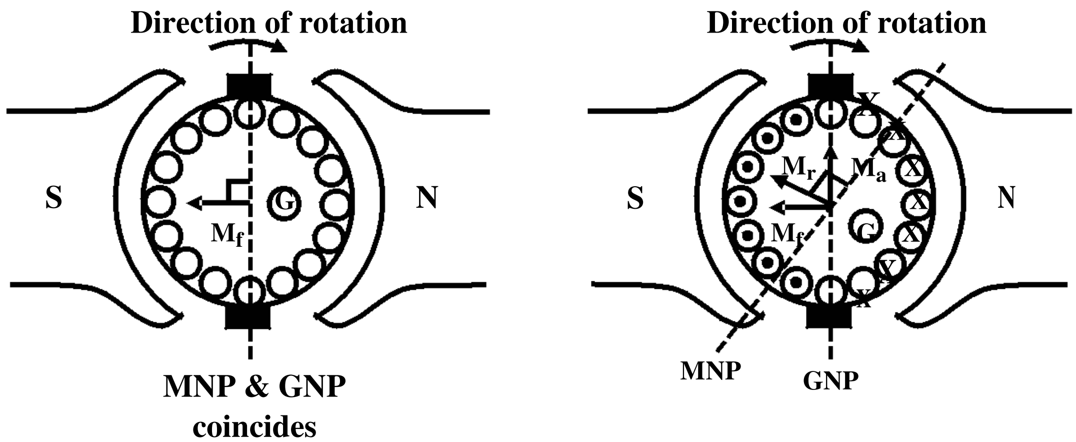
Putting the value of B_{av}, we get $T_e = \dfrac{pz}{2\pi a}\varphi I_a$ (37.2)

Thus we see that the above equation is once again applicable both for motor and generator mode of operation. The direction of the electromagnetic torque, T_e will be along the direction of rotation in case of motor operation and opposite to the direction of rotation in case of generator operation. When the machine runs steadily at a constant rpm then $T_e = T_{load}$ and $T_e = T_{pm}$, respectively for motor and generator mode.

The emf and torque equations are extremely useful and should be remembered by heart.

37.4 GNP and MNP

In a unloaded d.c machine field is produced only by the field coil as armature does not carry any current. For a unloaded generator, net field is equal to $\vec{M}_f$ produced by field coil alone and as shown in figure 37.2 (a). Then for a plane which is at right angles to $\vec{M}_f$, no field can exist along the plane, since $M_f \cos 90^0 = 0$. The plane along which there will be no field is called *Magnetic Neutral Plane* or MNP in short. The *Geometrical Neutral Plane* (GNP) is defined as a plane which is perpendicular to stator field axis. Thus for an unloaded generator GNP and MNP coincide. In a loaded generator, apart from $\vec{M}_f$, there will exist field produced by armature $\vec{M}_a$ as well making the resultant field $\vec{M}_r$ shifted as shown in figure 37.2 (b). Thus MNP in this case will be perpendicular to $\vec{M}_r$. Therefore it may be concluded that MNP for generator mode gets shifted along the direction of rotation of the armature.



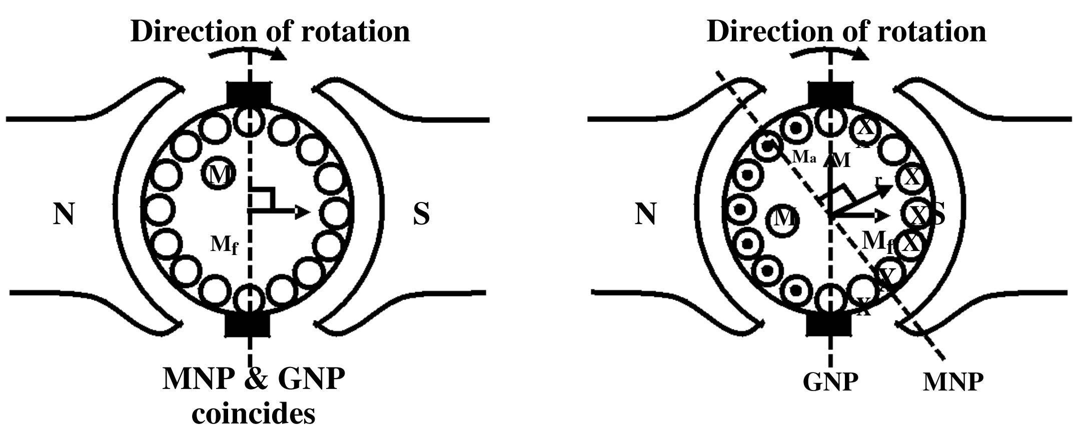
(a) Unloaded Generator **(b) Loaded Generator**

Figure 37.2: MNP & GNP : Generator mode.

The shift of MNP for a loaded motor will be in a direction opposite to the rotation as depicted in figure 37.3 (b). The explanation of this is left to the reader as an exercise.

37.5 Armature reaction

In a unloaded d.c machine armature current is vanishingly small and the flux per pole is decided by the field current alone. The uniform distribution of the lines of force get upset when armature too carries current due to loading. In one half of the pole, flux lines are concentrated and in the other half they are rarefied. Qualitatively one can argue that during loading condition flux per pole will remain same as in no load operation because the increase of flux in one half will be balanced by the decrease in the flux in the other half. Since it is the flux per pole which decides the emf generated and the torque produced by the machine, seemingly there will be no effect felt so far as the performance of the machine is concerned due to armature reaction. This in fact is almost true when the machine is lightly or moderately loaded.



(a) Unloaded motor **(b) Loaded motor**

Figure 37.3: MNP & GNP : Motor mode.

However at rated armature current the increase of flux in one half of the pole is rather less than the decrease in the other half due to presence of *saturation*. In other words there will be a net decrease in flux per pole during sufficient loading of the machine. This will have a direct bearing on the emf as well as torque developed affecting the performance of the machine.

Apart from this, due to distortion in the flux distribution, there will be some amount of flux present along the q-axis (brush axis) of the machine. This causes *commutation* difficult. In the following sections we try to explain armature reaction in somewhat detail considering motor and generator mode separately.

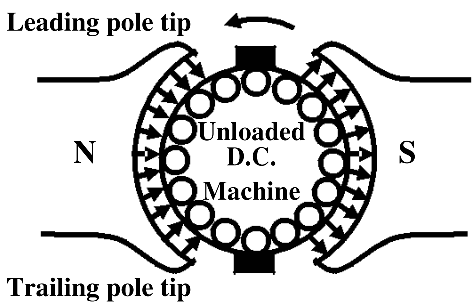


Figure 37.4: Flux lines during no load condition.

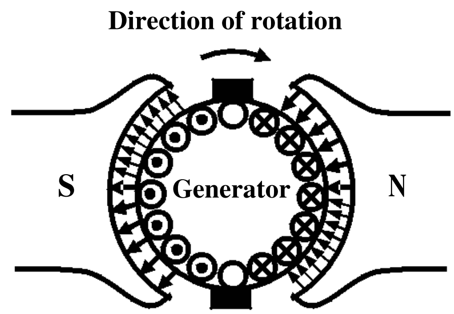


Figure 37.5: Flux lines for a loaded generator.

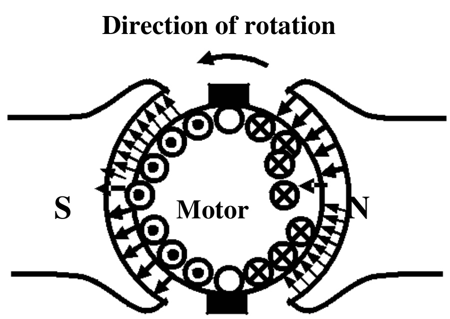


Figure 37.6: Flux lines for a loaded motor.

37.5.1 No Load operation

When a d.c machine operates absolutely under no load condition, armature current is zero. Under such a condition T_e developed is zero and runs at constant no load speed. In absence of any I_a, the flux per pole φ, inside the machine is solely decided by the field current and lines of force are uniformly distributed under a pole as shown in figure 37.4.

37.5.2 Loaded operation

A generator gets loaded when a resistance across the armature is connected and power is delivered to the resistance. The direction of the current in the conductor (either cross or dot) is decided by the fact that direction of T_e will be opposite to the direction of rotation. It is therefore obvious to see that flux per pole φ, developed in the generator should be decided not only by the mmf of the field winding alone but the armature mmf as well as the armature is carrying current now. By superposing the no load field lines and the armature field lines one can get the resultant field lines pattern as shown in Figures 37.5 and 37.7. The tip of the pole which is seen by a moving conductor first during the course of rotation is called the *leading pole tip* and the tip of the pole which is seen later is called the *trailing pole tip*. In case of generator mode we see that the lines of forces are concentrated near the trailing edge thereby producing torque in the opposite direction of rotation. How the trapezoidal no load field gets distorted along the air gap of the generator is shown in the Figure 37.7.

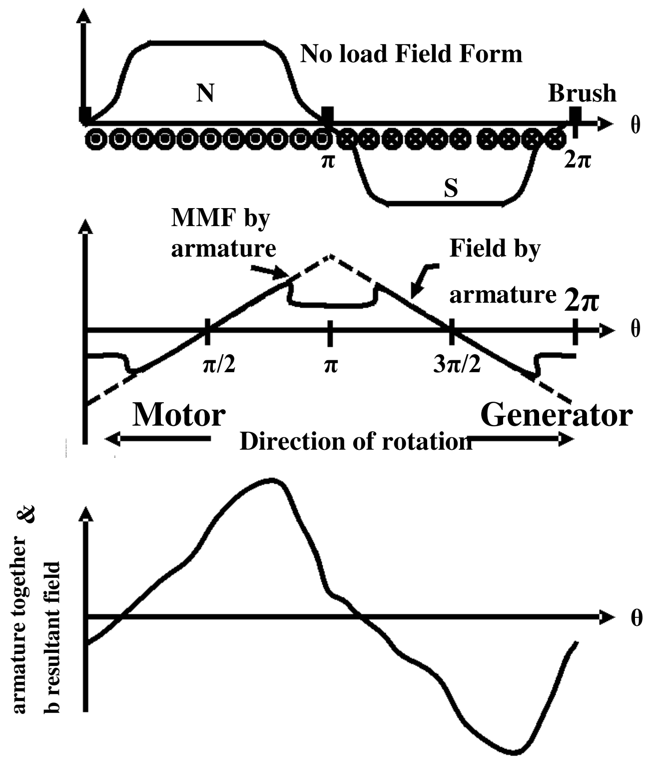


Figure 37.7: Effect of Armature Reaction

In this figure note that the armature mmf distribution is triangular in nature and the flux density distribution due to armature current is obtained by dividing armature mmf with the reluctance of the air gap. The reluctance is constant and small at any point under the pole. This means that the armature flux density will simply follow the armature mmf pattern. However, the reluctance in the q-axis region is quite large giving rise to small resultant flux of polarity same as the main pole behind in the q-axis.

In the same way one can explain the effect of loading a d.c motor by referring to Figures 37.6 and 37.7. Point to be noted here is that the lines of forces gets concentrated near the *leading* pole tip and rarefied near the *trailing* pole thereby producing torque along the direction of rotation. Also note the presence of some flux in the q-axis with a polarity same as main pole ahead.

37.6 Cross magnetising & Demagnetising AT

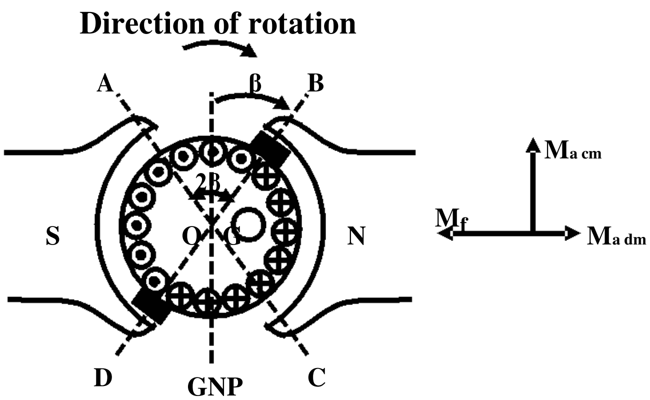
Usually the brushes in a d.c machine are along the GNP. The armature mmf $\vec{M}_a$ which acts always along the direction of the brush axis also acts along GNP. It may also be noted that $\vec{M}_a$ is at right angles to the field mmf $\vec{M}_f$ when brushes are not shifted. Thus $\vec{M}_a$ has cross magnetising effect on $\vec{M}_f$. Apparently $\vec{M}_a$ does not have any component opposing $\vec{M}_f$ directly.

The presence of cross magnetising armature mmf $\vec{M}_a$ distorts the no load field pattern caused by M_f.

The cross magnetising armature AT can be calculated as shown below.

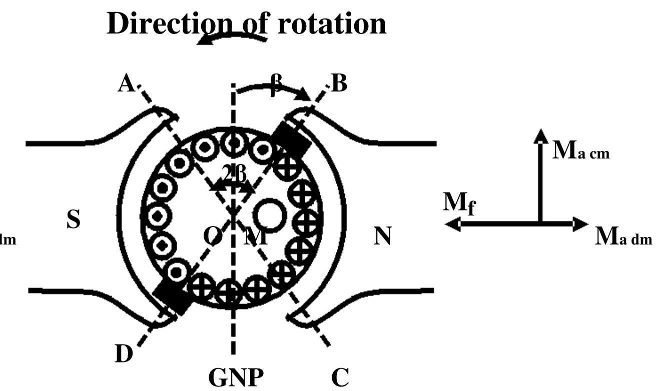
$$\text{Let, } P = \text{Number of poles}$$
$$z = \text{Total number of armature conductors}$$
$$a = \text{Number of parallel paths}$$
$$\text{Armature current} = I_a$$
$$\text{Current through armature conductor} = I_a / a$$
$$\text{Total Ampere conductors} = \frac{I_a}{a} z$$
$$\text{Total AT} = \frac{I_a}{a}\frac{z}{2}$$
$$\therefore \text{Armature AT/pole} = \frac{I_a z}{2aP}$$

Demagnetising by armature mmf can occur when a *deliberate* brush shift is introduced. Small brush shift is sometimes given to improve commutation. For generator brush shift is given in the forward direction (in the direction of rotation) while for motor mode the brush shift is given in the backward direction (opposite to the direction of rotation) as shown in figures 37.8 and 37.9.



Brush shift in forward direction for Generator

Figure 37.8:



Brush shift in backward direction for Motor

Figure 37.9:

Let the brush shift be $\beta°$ (mechanical) for all the brushes. Then as depicted in the figure 37.8 the conductors present within the angle 2 $\beta°$ (i.e., ∠AOB and ∠COD) will be responsible for demagnetization and conductors present within the angle (180° - 2 $\beta°$) (i.e., ∠AOD and ∠BOC) will be responsible for crossmagnetisation for a 2 polar machine.

Ampere turns for demagnetization can be calculated as follows:

$$\text{Number of conductors spread over } 360° = z$$
$$\text{Number of conductors spread over } 2\beta° = \frac{z}{360^\circ} 2\beta^\circ$$

Demagnetizing Ampere conductors contributed by 2 β° = $\frac{I_a}{a}\frac{z}{360^\circ}2\beta^\circ$

Since brushes are placed in the inter polar regions and there are P number of brush positions,

$$\therefore \text{Total number of conductors responsible for demagnetization} = \frac{z}{360^\circ}(P\,2\beta^\circ)$$

$$\text{Total Demagnetising Ampere conductors} = P\frac{I_a}{a}\frac{z}{360^\circ}2\beta^\circ$$

$$\text{Total Demagnetising Ampere turns} = \frac{PI_a}{2a}\frac{z}{360^\circ}2\beta^\circ$$

$$\text{Total Demagnetising Ampere turns per pole} = \frac{I_a z}{2a}\frac{2\beta^\circ}{360^\circ}$$

To find expression for the cross magnetising, replace 2 β° by (180° - 2 β°) in the above expression to get:

$$\text{Number of conductors responsible for cross magnetization} = \frac{z(360^\circ - P2\beta^\circ)}{360^\circ}$$

$$\text{Total cross Ampere turns} = \frac{I_a}{2a}\frac{z(360^\circ - P2\beta^\circ)}{360^\circ}$$

$$\text{Total cross Ampere turns per pole} = \frac{I_a}{2aP}\frac{z(360^\circ - P2\beta^\circ)}{360^\circ}$$

It may easily be verified that the sum of demagnetizing AT/pole and cross magnetising AT/pole is equal to total AT/pole as shown below:

$$\frac{I_a z}{2a}\frac{2\beta^\circ}{360^\circ} + \frac{I_a}{2aP}\frac{z(360^\circ - P2\beta^\circ)}{360^\circ} = \frac{I_a z}{2aP}$$

37.6.1 Commutation & Armature reaction

If we concentrate our attention to a single conductor, we immediately recognize that the direction of current reverses as it moves from the influence of one pole to the influence of the next opposite pole. This reversal of current in the conductor is called *commutation*. During no load operation when the conductor reaches the *magnetic neutral axis* or the q-axis, the induced voltage in it is zero as there is no flux is present in the q-axis. Also any coil present in this position and under going commutation, will get short circuited by the commutator segments and brushes. In other words we see that every coil will be short circuited whenever it undergoes commutation and fortunately at that time induced emf in the coil being zero, no circulating current will be present at least during no load condition. But as discussed earlier, flux in the quadrature axis will never be zero when the machine is loaded. Hence coil undergoing commutation will have circulating current causing problem so far as smooth commutation is concerned.

For small machines (up to few kilo watts) no special care is taken to avoid the armature reaction effects. However for large machines, to get rid of the ill effects of armature reaction one can use *compensating winding, inter poles* or both.

The basic idea of nullifying armature mmf is based on a very simple fact. We know that a magnetic field is produced in the vicinity when a conductor carries current. Naturally another conductor carrying same current but in the opposite direction if placed in close proximity of the first conductor, the resultant field in the vicinity will be close to zero. Additional winding called *compensating winding* is placed on the *pole faces* of the machine and connected in series with the armature circuit in such a way that the direction of current in compensating winding is opposite to that in the armature conductor as shown in Figure 37.10. It may be noted that compensating winding can not nullify the quadrature axis armature flux completely. Additional small poles called *inter poles* are provided in between the main poles in large machines to get rid of the commutation problem arising out of armature reaction.

Sectional view of a machine provided with both compensating and inter poles is shown in Figure 37.11 and the schematic representation of such a machine is shown in Figure 37.12.

Careful inspection of the figures mentioned reveal that the polarity of the inter pole should be same as that of the main pole ahead in case of generator and should be same as that of main pole behind in case of motor.

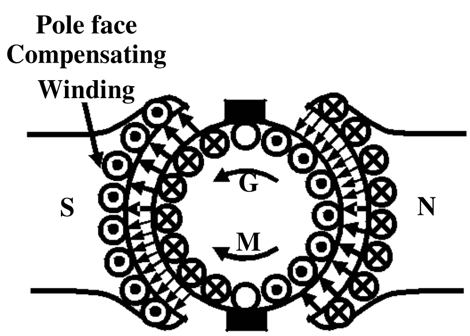


Figure 37.10: Position of compensating winding.

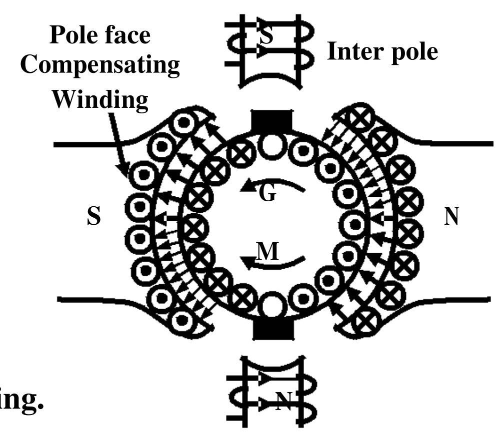


Figure 37.11: Inter pole coil.

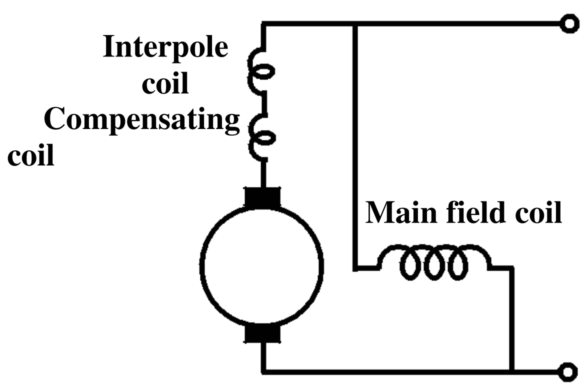


Figure 37.12: Interpole & compensating coil connection.

37.7 Tick the correct answer

1. A d.c generator is found to develop an armature voltage of 200 V. If the flux is reduced by 25% and speed is increased by 40%, the armature generated voltage will become:

   (A) 20 V (B) 107 V (C) 210 V (D) 373 V

2. A d.c motor runs steadily drawing an armature current of 15 A. To develop the same amount of torque at 20 A armature current, flux should be:

   (A) reduced by 25% (B) increased by 25%
   (C) reduced by 33% (D) increased by 33%

3. A d.c generator develops 200 V across its armature terminals with a certain polarity. To reverse the polarity of the armature voltage:

   (A) direction of field current should be reversed
   (B) direction of rotation should be reversed.
   (C) either of (A) and (B)
   (D) direction of both field current and speed should be reversed.

4. In a d.c shunt machine, the inter pole winding should be connected in

   (A) series with the armature.
   (B) series with the field winding.
   (C) parallel with the armature.
   (D) parallel with the field winding.

5. In a d.c shunt machine, compensating winding should be connected in

   (A) series with the armature.
   (B) series with the field winding.
   (C) parallel with the armature.
   (D) parallel with the field winding.

6. In a d.c generator, interpole coil should be connected in such a fashion that the polarity of the interpole is

   (A) same as that of main pole ahead.
   (B) same as that main pole behind.
   (C) either of (A) and (B).
   (D) dependent on armature current.

7. In a d.c motor, interpole coil should be connected in such a fashion that the polarity of the interpole is

(A) same as that of main pole ahead.
(B) same as that of main pole behind.
(C) either of (A) and (B).
(D) dependent on feild current.

37.8 Answer the following

1. Write down the expression for electromagnetic torque in a d.c motor. Now comment how the direction of rotation can be reversed.

2. Write down the expression for the generated voltage in a d.c generator. Now comment how can you reverse the polarity of the generated voltage.

3. Comment on the direction of electromagnetic torque in a d.c motor if both armature current and field current are reversed.

4. A 4-pole, lap wound, d.c machine has total number of 800 armature conductors and produces 0.03 Wb flux per pole when field is excited. If the machine is driven by a prime mover at 1000 rpm, calculate the generated emf across the armature. If the generator is loaded to deliver an armature current of 50 A, Calculate the prime mover and electromagnetic torques developed at this load current. Neglect frictional torque.

5. A 4-pole, lap wound, d.c machine has a total number of 800 armature conductors and an armature resistance of 0.4 Ω. If the machine is found to run steadily as motor at 1000 rpm and drawing an armature current of 10 A from a 220 V D.C supply, calculate the back emf, electromagnetic torque and the load torque.

6. Clearly mention the purpose of providing interpoles in large d.c machines.

7. Comment on the polarity of the interpole for motor and generator modes.

8. Why and what for, is the compensating winding provided in large d.c machines?

9. How are interpole coil and compensating windings connected in d.c machine?

UNIT-4

DC Machines

EMF & Torque Equation

Contents

37.1 Goals of the lesson

After going through the lesson, the student will be able to understand:

1. the factors on which induced voltage in the armature depend.
2. the factors on which the electromagnetic torque developed depend.
3. the derivation of the emf and torque equations.
4. that emf and torque equations are applicable to both generator and motor operations.
5. why the generated emf in motor called the back emf.
6. armature reaction, its ill effects and remedial measures.
7. the purpose of compensating winding – its location and connection.
8. the purpose of interpole – its location and connection.
9. the difference between the GNP (geometric neutral plane) and MNP (magnetic neutral plane).

37.2 Introduction

Be it motor or generator operations, the analysis of D.C machine performance center around two fundamental equations namely the emf equation and the torque equations. In fact both motoring and generating actions go together in d.c machines. For example in a d.c motor there will be induced voltage across the brushes in the same way as in a generator. The induced voltage in d.c motor is however called by a different name *back emf*. Thus the factors on which induced emf in generator depend will be no different from motor action. In fact the same emf equation can be employed to calculate induced emf for both generator and motor operation. In the same way same *torque equation* can be used to calculate electromagnetic torque developed in both motoring and generating actions.

In this lesson these two fundamental equations have been derived.

Field patterns along the air gap of the machine for both motor and generator modes are explained. The ill effects of armature mmf (for a loaded machine) is discussed and possible remedial measures are presented. Calculation of cross magnetizing and demagnetizing AT (ampere turns) of d.c machines with shifted brush are presented. Depending on your course requirement the derivation for these ATs with sifted brush may be avoided. Finally, the phenomenon of *armature reaction* and a brief account of commutation are presented.

37.3 EMF & Torque Equations

In this section we shall derive two most fundamental and important formulas (namely emf and torque equations) for d.c machine in general. These will be extensively used to analyse the performance and to solve problems on d.c machines.

37.3.1 EMF Equation

Consider a D.C generator whose field coil is excited to produce a flux density distribution along the air gap and the armature is driven by a prime mover at constant speed as shown in figure 37.1.

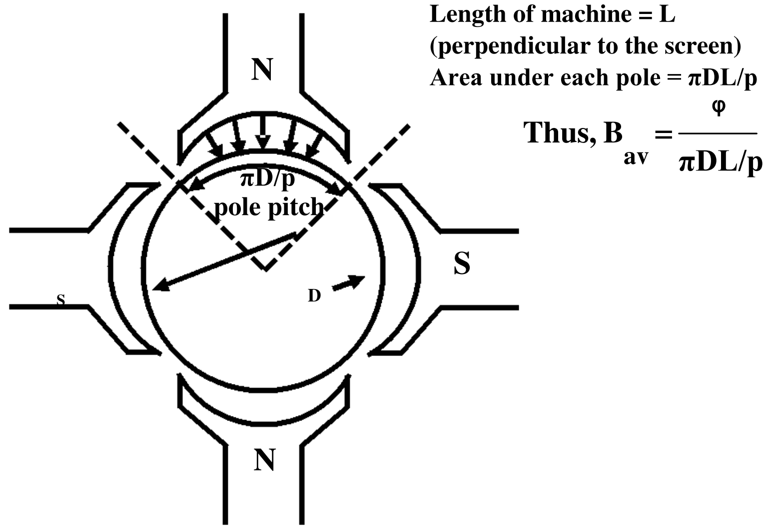


Figure 37.1: Pole pitch & area on armature surface per pole.

Let us assume a p polar d.c generator is driven (by a prime mover) at n rps. The excitation of the stator field is such that it produces a φ Wb flux per pole. Also let z be the total number of armature conductors and a be the number of parallel paths in the armature circuit. In general, as discussed in the earlier section the magnitude of the voltage from one conductor to another is likely to very since flux density distribution is *trapezoidal* in nature. Therefore, total average voltage across the brushes is calculated on the basis of average flux density B_{av}. If D and L are the rotor diameter and the length of the machine in meters then area under each pole is —

Hence average flux density in the gap is given by

$$\text{Average flux density } B_{av} = \frac{\varphi}{\left(\frac{\pi D}{p}\right)L} = \frac{\varphi p}{\pi DL}$$

$$\text{Induced voltage in a single conductor} = B_{av}Lv$$

$$\text{Number of conductors present in each parallel path} = \frac{z}{a}$$

$$\text{If } v \text{ is the tangential velocity then, } v = \pi Dn$$

$$\text{Therefore, total voltage appearing across the brushes} = \frac{z}{a} B_{av} Lv = \frac{z}{a}\frac{\varphi p}{\pi DL} L\pi Dn$$

$$\text{Thus voltage induced across the armature, } E_A = \frac{pz}{a}\varphi n \qquad (37.1)$$

We thus see that across the armature a voltage will be generated so long there exists some flux per pole and the machine runs with some speed. Therefore irrespective of the fact that the machine is operating as generator or as motor, armature has an induced voltage in it governed essentially by the above derived equation. This emf is called *back emf* for motor operation.

37.3.2 Torque equation

Whenever armature carries current in presence of flux, conductor experiences force which gives rise to the electromagnetic torque. In this section we shall derive an expression for the electromagnetic torque T_e developed in a d.c machine. Obviously T_e will be developed both in motor and generator mode of operation. It may be noted that the direction of conductor currents reverses as we move from one pole to the other. This ensures unidirectional torque to be produced. The derivation of the torque expression is shown below.

$$\text{Let, } I_a = \text{Armature current}$$

$$\text{Average flux density } B_{av} = \frac{\varphi p}{\pi DL}$$

$$\text{Then, } \frac{I_a}{a} = \text{Current flowing through each conductor.}$$

$$\text{Force on a single conductor} = B_{av}\frac{I_a}{a}L$$

$$\text{Torque on a single conductor} = B_{av}\frac{I_a}{a}L\frac{D}{2}$$

$$\text{Total electromagnetic torque developed, } T_e = zB_{av}\frac{I_a}{a}L\frac{D}{2}$$

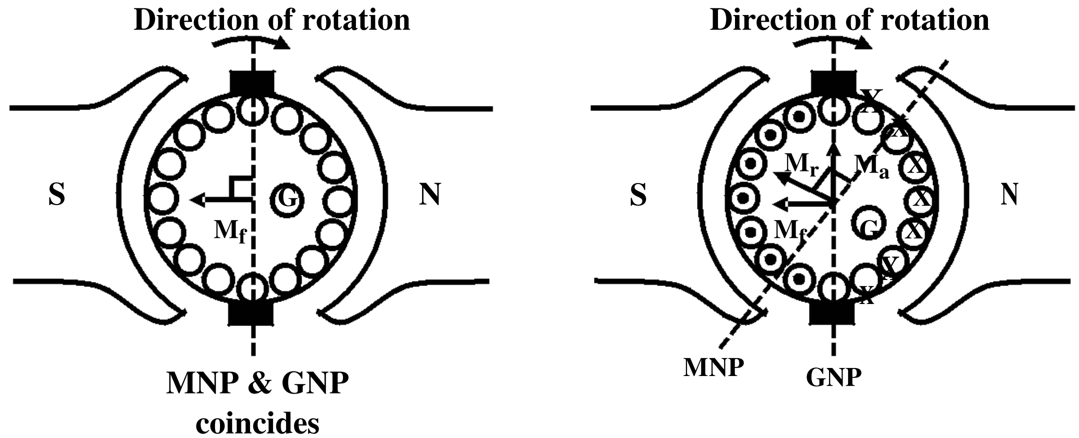
$$\text{Putting the value of } B_{av}\text{, we get } T_e = \frac{pz}{2\pi a}\varphi I_a \qquad (37.2)$$

Thus we see that the above equation is once again applicable both for motor and generator mode of operation. The direction of the electromagnetic torque, T_e will be along the direction of rotation in case of motor operation and opposite to the direction of rotation in case of generator operation. When the machine runs steadily at a constant rpm then $T_e = T_{load}$ and $T_e = T_{pm}$, respectively for motor and generator mode.

The emf and torque equations are extremely useful and should be remembered by heart.

37.4 GNP and MNP

In a unloaded d.c machine field is produced only by the field coil as armature does not carry any current. For a unloaded generator, net field is equal to $\vec{M}_f$ produced by field coil alone and as shown in figure 37.2 (a). Then for a plane which is at right angles to $\vec{M}_f$, no field can exist along the plane, since $M_f \cos 90^0 = 0$. The plane along which there will be no field is called *Magnetic Neutral Plane* or MNP in short. The *Geometrical Neutral Plane* (GNP) is defined as a plane which is perpendicular to stator field axis. Thus for an unloaded generator GNP and MNP coincide. In a loaded generator, apart from $\vec{M}_f$, there will exist field produced by armature $\vec{M}_a$ as well making the resultant field $\vec{M}_r$ shifted as shown in figure 37.2 (b). Thus MNP in this case will be perpendicular to $\vec{M}_r$. Therefore it may be concluded that MNP for generator mode gets shifted along the direction of rotation of the armature



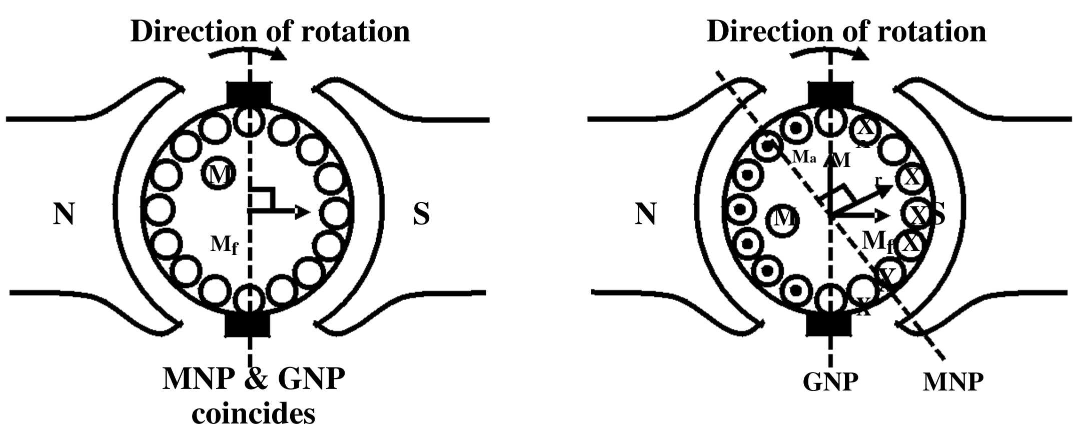
(a) Unloaded Generator (b) Loaded Generator

Figure 37.2: MNP & GNP : Generator mode.

The shift of MNP for a loaded motor will be in a direction opposite to the rotation as depicted in figure 37.3 (b). The explanation of this is left to the reader as an exercise.

37.5 Armature reaction

In a unloaded d.c machine armature current is vanishingly small and the flux per pole is decided by the field current alone. The uniform distribution of the lines of force get upset when armature too carries current due to loading. In one half of the pole, flux lines are concentrated and in the other half they are rarefied. Qualitatively one can argue that during loading condition flux per pole will remain same as in no load operation because the increase of flux in one half will be balanced by the decrease in the flux in the other half. Since it is the flux per pole which decides the emf generated and the torque produced by the machine, seemingly there will be no effect felt so far as the performance of the machine is concerned due to armature reaction. This in fact is almost true when the machine is lightly or moderately loaded.



(a) Unloaded motor (b) Loaded motor

Figure 37.3: MNP & GNP : Motor mode.

However at rated armature current the increase of flux in one half of the pole is rather less than the decrease in the other half due to presence of *saturation*. In other words there will be a net decrease in flux per pole during sufficient loading of the machine. This will have a direct bearing on the emf as well as torque developed affecting the performance of the machine.

Apart from this, due to distortion in the flux distribution, there will be some amount of flux present along the q-axis (brush axis) of the machine. This causes *commutation* difficult. In the following sections we try to explain armature reaction in somewhat detail considering motor and generator mode separately.

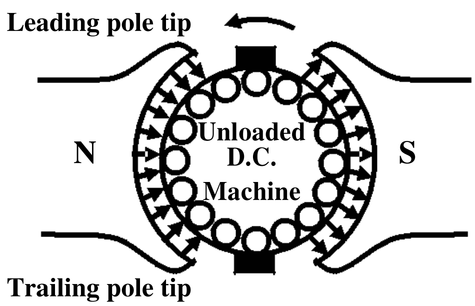


Figure 37.4: Flux lines during no load condition.

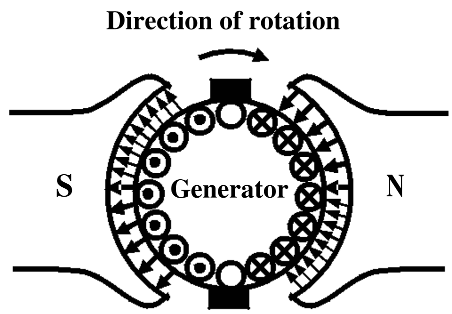


Figure 37.5: Flux lines for a loaded generator.

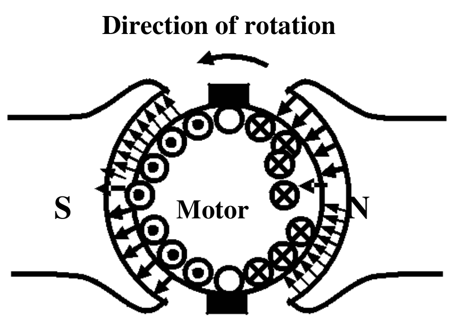


Figure 37.6: Flux lines for a loaded motor.

37.5.1 No Load operation

When a d.c machine operates absolutely under no load condition, armature current is zero. Under such a condition T_e developed is zero and runs at constant no load speed. In absence of any I_a, the flux per pole φ, inside the machine is solely decided by the field current and lines of force are uniformly distributed under a pole as shown in figure 37.4.

37.5.2 Loaded operation

A generator gets loaded when a resistance across the armature is connected and power is delivered to the resistance. The direction of the current in the conductor (either cross or dot) is decided by the fact that direction of T_e will be opposite to the direction of rotation. It is therefore obvious to see that flux per pole φ, developed in the generator should be decided not only by the mmf of the field winding alone but the armature mmf as well as the armature is carrying current now. By superposing the no load field lines and the armature field lines one can get the resultant field lines pattern as shown in Figures 37.5 and 37.7. The tip of the pole which is seen by a moving conductor first during the course of rotation is called the *leading pole tip* and the tip of the pole which is seen later is called the *trailing pole tip*. In case of generator mode we see that the lines of forces are concentrated near the trailing edge thereby producing torque in the opposite direction of rotation. How the trapezoidal no load field gets distorted along the air gap of the generator is shown in the Figure 37.7.

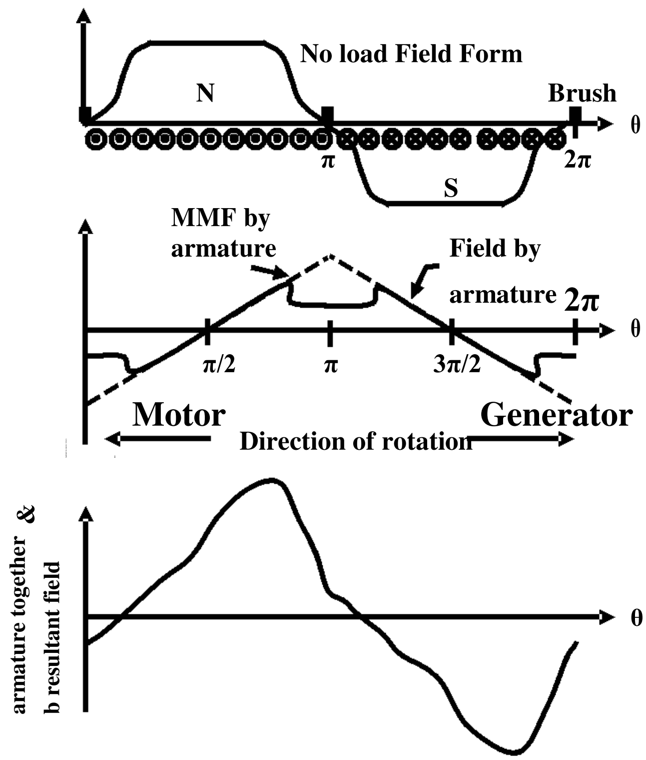


Figure 37.7: Effect of Armature Reaction

In this figure note that the armature mmf distribution is triangular in nature and the flux density distribution due to armature current is obtained by dividing armature mmf with the reluctance of the air gap. The reluctance is constant and small at any point under the pole. This means that the armature flux density will simply follow the armature mmf pattern. However, the reluctance in the q-axis region is quite large giving rise to small resultant flux of polarity same as the main pole behind in the q-axis.

In the same way one can explain the effect of loading a d.c motor by referring to Figures 37.6 and 37.7. Point to be noted here is that the lines of forces gets concentrated near the *leading* pole tip and rarefied near the *trailing* pole thereby producing torque along the direction of rotation. Also note the presence of some flux in the q-axis with a polarity same as main pole ahead.

37.6 Cross magnetising & Demagnetising AT

Usually the brushes in a d.c machine are along the GNP. The armature mmf $\vec{M}_a$ which acts always along the direction of the brush axis also acts along GNP. It may also be noted that $\vec{M}_a$ is at right angles to the field mmf $\vec{M}_f$ when brushes are not shifted. Thus $\vec{M}_a$ has cross magnetising effect on $\vec{M}_f$. Apparently $\vec{M}_a$ does not have any component opposing $\vec{M}_f$ directly.

The presence of cross magnetising armature mmf M_a distorts the no load field pattern caused by $\vec{M}_f$.

The cross magnetising armature AT can be calculated as shown below.

Let, P = Number of poles
z = Total number of armature conductors
a = Number of parallel paths

$$\text{Armature current} = I_a$$

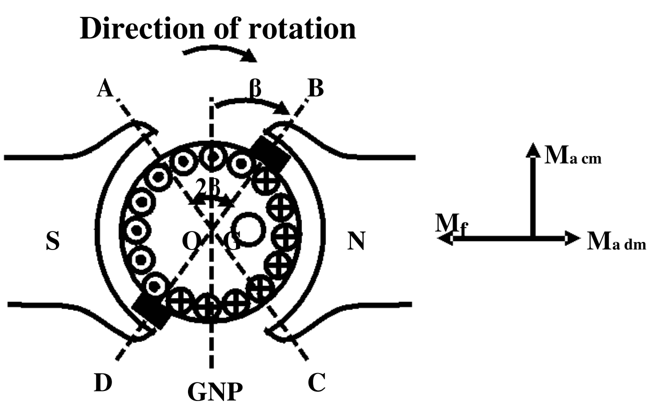
$$\text{Current through armature conductor} = I_a / a$$

$$\text{Total Ampere conductors} = \frac{I_a}{a} z$$

$$\text{Total AT} = \frac{I_a}{a}\frac{z}{2}$$

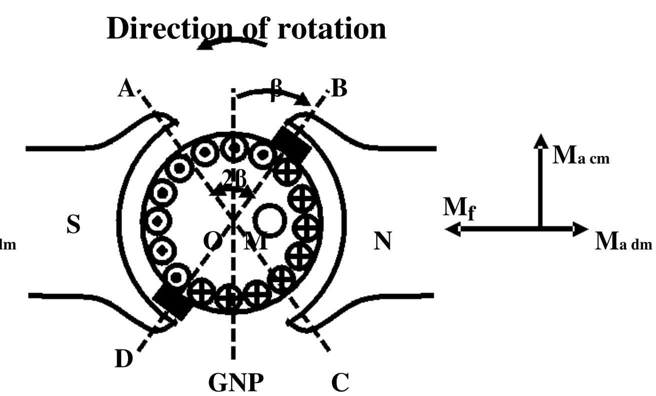
$$\therefore \text{Armature AT/pole} = \frac{I_a z}{2aP}$$

Demagnetising by armature mmf can occur when a *deliberate* brush shift is introduced. Small brush shift is sometimes given to improve commutation. For generator brush shift is given in the forward direction (in the direction of rotation) while for motor mode the brush shift is given in the backward direction (opposite to the direction of rotation) as shown in figures 37.8 and 37.9.



Brush shift in forward direction for Generator

Figure 37.8:



Brush shift in backward direction for Motor

Figure 37.9:

Let the brush shift be $\beta°$ (mechanical) for all the brushes. Then as depicted in the figure 37.8 the conductors present within the angle 2 $\beta°$ (i.e., ∠AOB and ∠COD) will be responsible for demagnetization and conductors present within the angle (180° - 2 $\beta°$) (i.e., ∠AOD and ∠BOC) will be responsible for crossmagnetisation for a 2 polar machine.

Ampere turns for demagnetization can be calculated as follows:

$$\text{Number of conductors spread over } 360° = z$$

$$\text{Number of conductors spread over } 2\beta° = \frac{z}{360°} 2\beta°$$

$$\text{Demagnetizing Ampere conductors contributed by } 2\beta^\circ = \frac{I_a}{a}\frac{z}{360^\circ}2\beta^\circ$$

Since brushes are placed in the inter polar regions and there are P number of brush positions,

$$\therefore \text{Total number of conductors responsible for demagnetization} = \frac{z}{360^\circ}(P\,2\beta^\circ)$$

$$\text{Total Demagnetising Ampere conductors} = P\frac{I_a}{a}\frac{z}{360^\circ}2\beta^\circ$$

$$\text{Total Demagnetising Ampere turns} = \frac{PI_a}{2a}\frac{z}{360^\circ}2\beta^\circ$$

$$\text{Total Demagnetising Ampere turns per pole} = \frac{I_a z}{2a}\frac{2\beta^\circ}{360^\circ}$$

To find expression for the cross magnetising, replace 2 $\beta°$ by (180° - 2 $\beta°$) in the above expression to get:

$$\text{Number of conductors responsible for cross magnetization} = \frac{z(360^\circ - P2\beta^\circ)}{360^\circ}$$

$$\text{Total cross Ampere turns} = \frac{I_a}{2a}\frac{z(360^\circ - P2\beta^\circ)}{360^\circ}$$

$$\text{Total cross Ampere turns per pole} = \frac{I_a}{2aP}\frac{z(360^\circ - P2\beta^\circ)}{360^\circ}$$

It may easily be verified that the sum of demagnetizing AT/pole and cross magnetising AT/pole is equal to total AT/pole as shown below:

$$\frac{I_a z}{2a}\frac{2\beta^\circ}{360^\circ} + \frac{I_a}{2aP}\frac{z(360^\circ - P2\beta^\circ)}{360^\circ} = \frac{I_a z}{2aP}$$

37.6.1 Commutation & Armature reaction

If we concentrate our attention to a single conductor, we immediately recognize that the direction of current reverses as it moves from the influence of one pole to the influence of the next opposite pole. This reversal of current in the conductor is called *commutation*. During no load operation when the conductor reaches the *magnetic neutral axis* or the q-axis, the induced voltage in it is zero as there is no flux is present in the q-axis. Also any coil present in this position and under going commutation, will get short circuited by the commutator segments and brushes. In other words we see that every coil will be short circuited whenever it undergoes commutation and fortunately at that time induced emf in the coil being zero, no circulating current will be present at least during no load condition. But as discussed earlier, flux in the quadrature axis will never be zero when the machine is loaded. Hence coil undergoing commutation will have circulating current causing problem so far as smooth commutation is concerned.

For small machines (up to few kilo watts) no special care is taken to avoid the armature reaction effects. However for large machines, to get rid of the ill effects of armature reaction one can use *compensating winding, inter poles* or both.

The basic idea of nullifying armature mmf is based on a very simple fact. We know that a magnetic field is produced in the vicinity when a conductor carries current. Naturally another conductor carrying same current but in the opposite direction if placed in close proximity of the first conductor, the resultant field in the vicinity will be close to zero. Additional winding called *compensating winding* is placed on the *pole faces* of the machine and connected in series with the armature circuit in such a way that the direction of current in compensating winding is opposite to that in the armature conductor as shown in Figure 37.10. It may be noted that compensating winding can not nullify the quadrature axis armature flux completely. Additional small poles called *inter poles* are provided in between the main poles in large machines to get rid of the commutation problem arising out of armature reaction.

Sectional view of a machine provided with both compensating and inter poles is shown in Figure 37.11 and the schematic representation of such a machine is shown in Figure 37.12.

Careful inspection of the figures mentioned reveal that the polarity of the inter pole should be same as that of the main pole ahead in case of generator and should be same as that of main pole behind in case of motor.

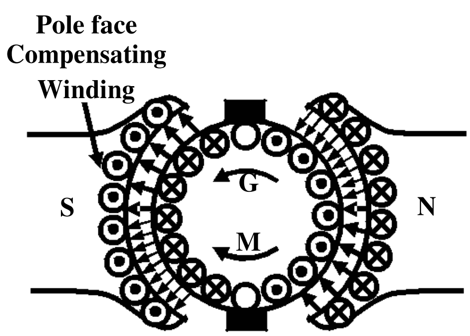


Figure 37.10: Position of compensating winding.

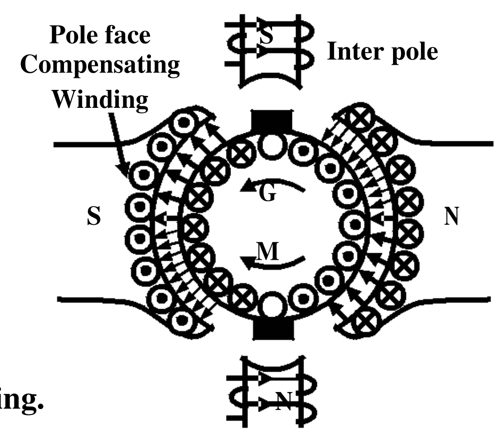


Figure 37.11: Inter pole coil.

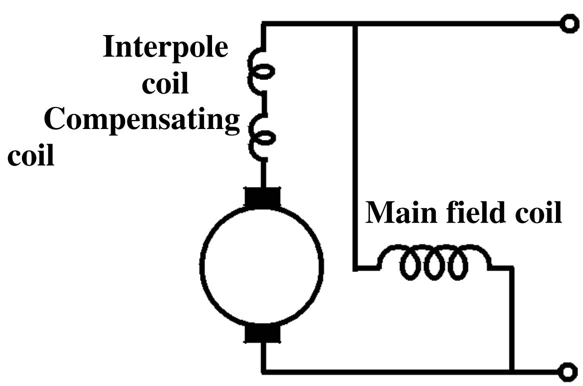


Figure 37.12: Interpole & compensating coil connection.

37.7 Tick the correct answer

1. A d.c generator is found to develop an armature voltage of 200 V. If the flux is reduced by 25% and speed is increased by 40%, the armature generated voltage will become:

   (A) 20 V (B) 107 V (C) 210 V (D) 373 V

2. A d.c motor runs steadily drawing an armature current of 15 A. To develop the same amount of torque at 20 A armature current, flux should be:

   (A) reduced by 25% (B) increased by 25%
   (C) reduced by 33% (D) increased by 33%

3. A d.c generator develops 200 V across its armature terminals with a certain polarity. To reverse the polarity of the armature voltage:

   (A) direction of field current should be reversed
   (B) direction of rotation should be reversed.
   (C) either of (A) and (B)
   (D) direction of both field current and speed should be reversed.

4. In a d.c shunt machine, the inter pole winding should be connected in

   (A) series with the armature.
   (B) series with the field winding.
   (C) parallel with the armature.
   (D) parallel with the field winding.

5. In a d.c shunt machine, compensating winding should be connected in

   (A) series with the armature.
   (B) series with the field winding.
   (C) parallel with the armature.
   (D) parallel with the field winding.

6. In a d.c generator, interpole coil should be connected in such a fashion that the polarity of the interpole is

   (A) same as that of main pole ahead.
   (B) same as that main pole behind.
   (C) either of (A) and (B).
   (D) dependent on armature current.

7. In a d.c motor, interpole coil should be connected in such a fashion that the polarity of the interpole ir

(A) same as that of main pole ahead.
(B) same as that of main pole behind.
(C) either of (A) and (B).
(D) dependent on feild current.

37.8 Answer the following

1. Write down the expression for electromagnetic torque in a d.c motor. Now comment how the direction of rotation can be reversed.

2. Write down the expression for the generated voltage in a d.c generator. Now comment how can you reverse the polarity of the generated voltage.

3. Comment on the direction of electromagnetic torque in a d.c motor if both armature current and field current are reversed.

4. A 4-pole, lap wound, d.c machine has total number of 800 armature conductors and produces 0.03 Wb flux per pole when field is excited. If the machine is driven by a prime mover at 1000 rpm, calculate the generated emf across the armature. If the generator is loaded to deliver an armature current of 50 A, Calculate the prime mover and electromagnetic torques developed at this load current. Neglect frictional torque.

5. A 4-pole, lap wound, d.c machine has a total number of 800 armature conductors and an armature resistance of 0.4 Ω. If the machine is found to run steadily as motor at 1000 rpm and drawing an armature current of 10 A from a 220 V D.C supply, calculate the back emf, electromagnetic torque and the load torque.

6. Clearly mention the purpose of providing interpoles in large d.c machines.

7. Comment on the polarity of the interpole for motor and generator modes.

8. Why and what for, is the compensating winding provided in large d.c machines?

9. How are interpole coil and compensating windings connected in d.c machine?

UNIT-4

DC Machines

D.C Motors

Contents

39.1 Goals of the lesson

In this lesson aspects of starting and speed control of d.c motors are discussed and explained. At the end principles of electric braking of d.c. shunt motor is discussed. After going through the lesson, the reader is expected to have clear ideas of the following.

1. The problems of starting d.c motors with full rated voltage.
2. Use and selection of variable resistance as a simple starter in the armature circuit of a d.c motor.
3. Superiority of commercial starter (3-point starter) over resistance starter. Various protective features incorporated in a commercial starter.
4. Various strategies (namely-armature resistance control, armature voltage control and field current control) adopted for controlling speed of d.c motors.
5. Importance of characteristics such as (i) speed vs. armature current and (ii) speed vs. torque which are relevant for clear understanding of speed control technique.
6. Principle of electric braking – qualitative explanation.

39.2 Introduction

Although in this section we shall mainly discuss *shunt* motor, however, a brief descriptions of (i) D.C shunt, (ii) separately excited and (iii) series motor widely used are given at the beginning. The armature and field coils are connected in parallel in a d.c shunt motor as shown in figure 39.1 and the parallel combination is supplied with voltage V. I_L, I_a and I_f are respectively the current drawn from supply, the armature current and the field current respectively. The following equations can be written by applying KCL, and KVL in the field circuit and KVL in the armature circuit.

$$I_L = I_a + I_f \text{ applying KCL}$$

$$I_f = \frac{V}{R_f} \text{ from KVL in field circuit}$$

$$I_a = \frac{V - E_b}{r_a} \text{ from KVL in the armature circuit}$$

$$= \frac{V - k\varphi n}{r_a}$$

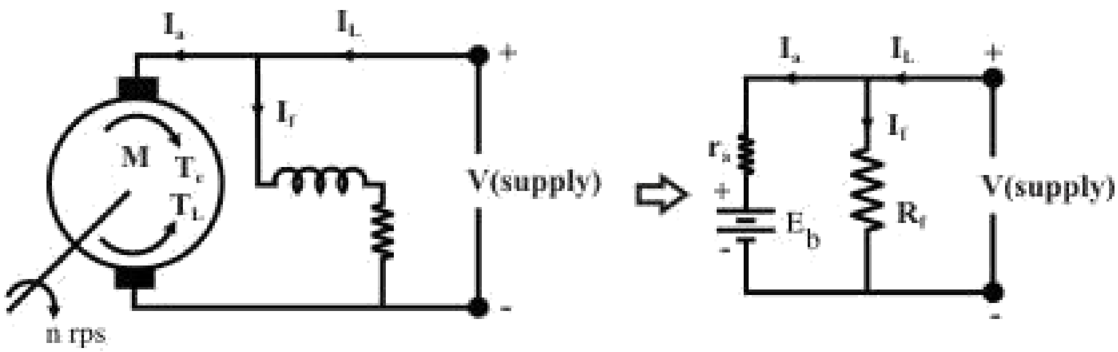


Figure 39.1: D.C shunt motor connection & its circuit representation.

39.3 Important Ideas

We have learnt in the previous lecture (37), that for motor operation:

1. Electromagnetic torque $T_e = k\varphi I_a$ developed by the motor acts along the direction of rotation.
2. The load torque T_L acts in the opposite direction of rotation or in opposition to T_e.
3. If $T_e = T_L$, motor operates with *constant* speed. .
4. If at any time $T_e > T_L$, the motor will *accelerate*.
5. If at any time $T_e < T_L$, the motor will *decelerate*.

Although our main focus of study will be the operation of motor under steady state condition, a knowledge of "how motor moves from one steady state operating point to another steady operating point" is important to note. To begin with let us study, how a motor from rest condition settles to the final operating point. Let us assume the motor is absolutely under no-load condition which essentially means $T_L = 0$ and there is friction present. Thus when supply is switched on, both $I_a = V/r_a$ and φ will be established developing T_e. As $T_L = 0$, motor should pick up speed due to acceleration. As motor speed increases, armature current decreases since back emf E_b rises. The value of T_e also progressively decreases. But so long T_e is present, acceleration will continue, increasing speed and back emf. A time will come when supply voltage and E_b will be same making armature current I_a zero. Now T_e becomes zero and acceleration stops and motor continues to run steadily at constant speed given by $n = V/(k\varphi)$ and drawing no armature current. Note that input power to the armature is zero and mechanical output power is zero as well.

Let us bring a little reality to the previous discussion. Let us not neglect frictional torque during acceleration period from rest. Let us also assume frictional torque to be constant and equal to T_{fric}. How the final operating point will be decided in this case? When supply will be switched on T_e will be developed and machine will accelerate if $T_e > T_{fric}$. With time T_e will decrease as I_a decreases. Eventually, a time will come when T_e becomes equal to T_L and motor will continue to run at constant steady no load speed n_0. The motor in the final steady state however will continue to draw a definite amount of armature current which will produce T_e just enough to balance T_{fric}.

Suppose, the motor is running steadily at no load speed n_0, drawing no load armature I_{a0} and producing torque T_{e0} (= T_{fric}). Now imagine, a constant load torque is suddenly imposed on the shaft of the motor at $t = 0$. Since speed can not change instantaneously, at $t = 0^+$, $I_a(t = 0^+) = I_{a0}$ and $T_e(t = 0^+) = T_{e0}$. Thus, at $t = 0^+$, opposing torque is $(T_L + T_{fric}) < T_{e0}$. Therefore, the motor should start decelerating drawing more armature current and developing more T_e. Final steady operating point will be reached when, $T_e = T_{fric} + T_L$ and motor will run at a new speed lower than no load speed n_0 but drawing I_a greater than the no load current I_{a0}.

In this section, we have learnt the mechanism of how a D.C motor gets loaded. To find out steady state operating point, one should only deal with steady state equations involving torque and current. For a shunt motor, operating point may change due to (i) change in field current or φ, (ii) change in load torque or (iii) change in both. Let us assume the initial operating point to be:

Armature current = I_{a1}

Field current = I_{f1}

Flux per pole = φ_1

Speed in rps = n_1

Load torque = T_{L1}

$$T_{e1} = k\varphi_1 I_{a1} = T_{L1} \tag{39.1}$$

$$E_{b1} = k\varphi_2 n_2 = V - I_{a1}r_a \tag{39.2}$$

Now suppose, we have changed field current and load torque to new values I_{f2} and T_{L2} respectively. Our problem is to find out the new steady state armature current and speed. Let,

New armature current = I_{a2}

New field current = I_{f2}

New flux per pole = φ_2

Speed in rps = n_2

New load torque = T_{L2}

$$T_{e2} = k\varphi_2 I_{a2} = T_{L2} \tag{39.3}$$

$$E_{b2} = k\varphi_2 n_2 = V - I_{a2}r_a \tag{39.4}$$

Now from equations 39.1 and 39.3 we get:

$$\frac{T_{e2}}{T_{e1}} = \frac{T_{L2}}{T_{L1}} = \frac{k\varphi_2 I_{a2}}{k\varphi_1 I_{a1}} \tag{39.5}$$

From equation 39.5, one can calculate the new armature current I_{a2}, the other things being known. Similarly using equations 39.2 and 39.4 we get:

$$\frac{E_{b1}}{E_{b2}} = \frac{k\varphi_2 n_2}{k\varphi_1 n_1} = \frac{V - I_{a1} r_a}{V - I_{a2} r_a} \tag{39.6}$$

Now we can calculate new steady state speed n_2 from equation 39.6.

39.4 Starting of D.C shunt motor

39.4.1 Problems of starting with full voltage

We know armature current in a d.c motor is given by

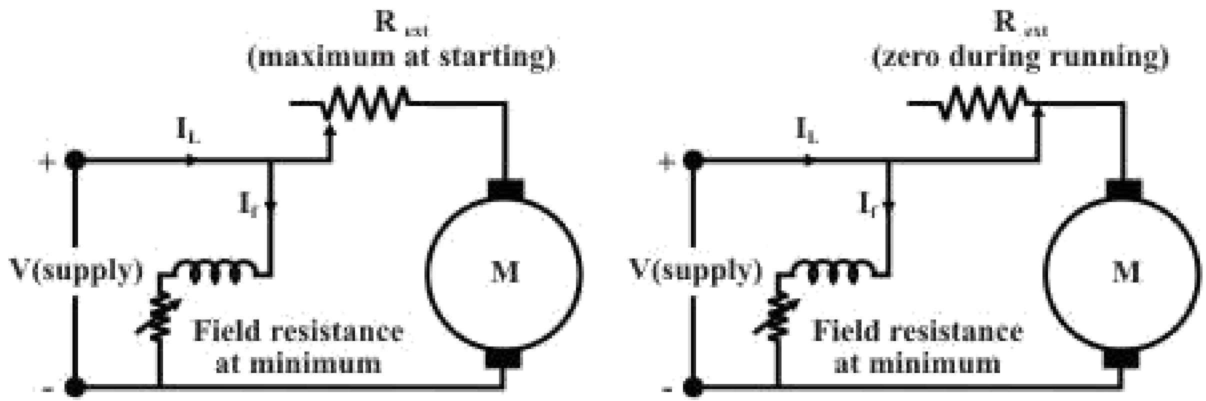
$$I_a = \frac{V - E_b}{r_a} = \frac{V - k\varphi n}{r_a}$$

At the instant of starting, rotor speed $n = 0$, hence starting armature current is $I_{ast} = \frac{V}{r_a}$. Since, armature resistance is quite small, starting current may be quite high (many times larger than the rated current) . A large machine, characterized by large rotor inertia (J), will pick up speed rather slowly. Thus the level of high starting current may be maintained for quite some time so as to cause serious damage to the brush/commutator and to the armature winding. Also the source should be capable of supplying this burst of large current. The other loads already connected to the same source, would experience a dip in the terminal voltage, every time a D.C motor is attempted to start with full voltage. This dip in supply voltage is caused due to sudden rise in voltage drop in the source's internal resistance. The duration for which this drop in voltage will persist once again depends on inertia (size) of the motor.

Hence, for small D.C motors extra precaution may not be necessary during starting as large starting current will very quickly die down because of fast rise in the back emf. However, for large motor, a *starter* is to be used during starting.

39.4.2 A simple starter

To limit the starting current, a suitable external resistance R_{ext} is connected in series (Figure 39.2(a)) with the armature so that $I_{ast} = \frac{V}{R_{ext} + r_a}$. At the time of starting, to have sufficient starting torque, field current is maximized by keeping the external field resistance R_f, to zero value. As the motor picks up speed, the value of R_{ext} is gradually decreased to zero so that during running no external resistance remains in the armature circuit. But each time one has to *restart* the motor, the external armature resistance must be set to maximum value by moving the jockey manually. Imagine, the motor to be running with $R_{ext} = 0$ (Figure 39.2(b)).



(a) External armature resistance maximum at starting

(b) External armature resistance zero during running.

Figure 39.2: A simple starter in the form of external armature resistance.

Now if the supply goes off (due to some problem in the supply side or due to load shedding), motor will come to a stop. All on a sudden, let us imagine, supply is restored. This is then nothing but full voltage starting. In other words, one should be constantly alert to set the resistance to maximum value whenever the motor comes to a stop. This is one major limitation of a simple rheostatic starter.

39.4.3 3-point starter

A "3-point starter" is extensively used to start a D.C shunt motor. It not only overcomes the difficulty of a plain resistance starter, but also provides additional *protective features* such as *over load protection* and *no volt protection*. The diagram of a 3 -point starter connected to a shunt motor is shown in figure 39.3. Although, the circuit looks a bit clumsy at a first glance, the basic working principle is same as that of plain resistance starter.

The starter is shown enclosed within the dotted rectangular box having three terminals marked as A, L and F for external connections. Terminal A is connected to one armature terminal Al of the motor. Terminal F is connected to one field terminal F1 of the motor and terminal L is connected to one supply terminal as shown. F2 terminal of field coil is connected to A2 through an external variable field resistance and the common point connected to supply (-ve). The external armatures resistances consist of several resistances connected in series and are shown in the form of an arc. The junctions of the resistances are brought out as terminals (called studs) and marked as 1,2,.. .12. Just beneath the resistances, a continuous *copper strip* also in the form of an arc is present.

There is a *handle* which can be moved in the clockwise direction against the spring tension. The spring tension keeps the handle in the OFF position when no one attempts to move it. Now let us trace the circuit from terminal L (supply + ve) . The wire from L passes through a small electro magnet called OLRC, (the function of which we shall discuss a little later) and enters through the handle shown by dashed lines. Near the end of the handle two copper strips are firmly connected with the wire. The furthest strip is shown circular shaped and the other strip is shown to be rectangular. When the handle is moved to the right, the *circular strip* of the handle will make contacts with resistance terminals 1, 2 etc. progressively. On the other hand, the rectangular strip will make contact with the continuous *arc copper strip*. The other end of this strip is brought as terminal F after going through an electromagnet coil (called NVRC). Terminal F is finally connected to motor field terminal Fl.

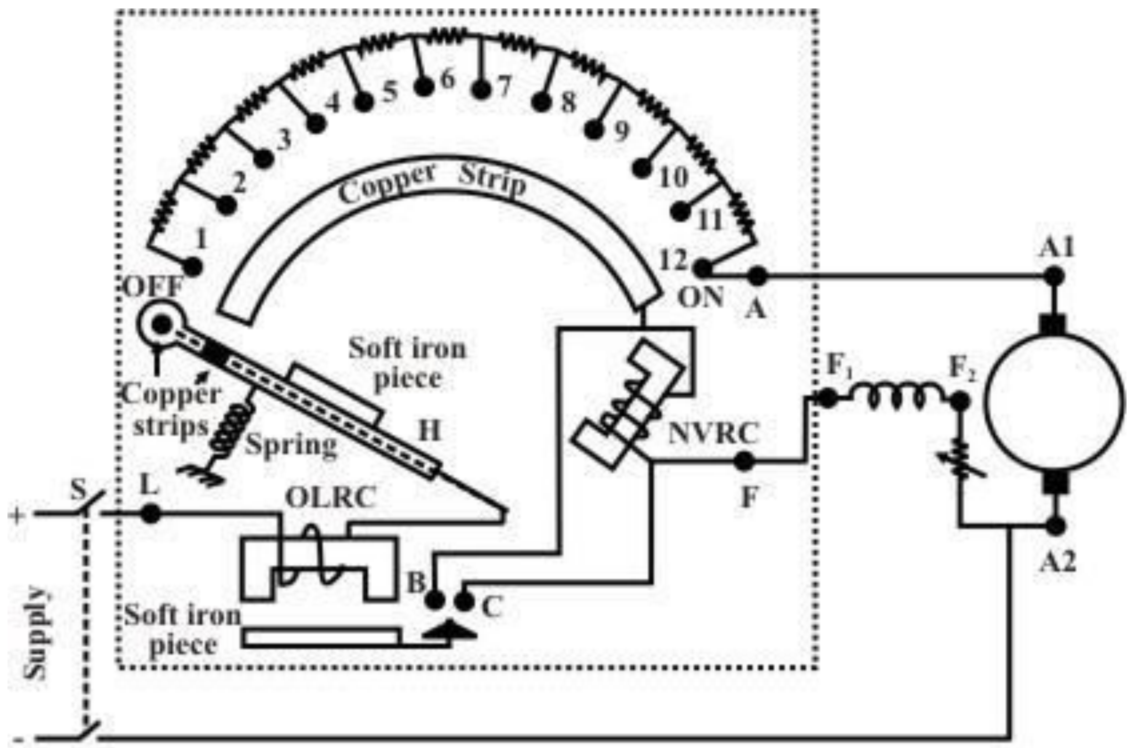


Figure 39.3: A 3-point starter.

Working principle

Let us explain the operation of the starter. Initially the handle is in the OFF position. Neither armature nor the field of the motor gets supply. Now the handle is moved to stud number 1. In this position armature and all the resistances in series gets connected to the supply. Field coil gets full supply as the rectangular strip makes contact with arc copper strip. As the machine picks up speed handle is moved further to stud number 2. In this position the external resistance in the armature circuit is less as the first resistance is left out. Field however, continues to get full voltage by virtue of the continuous arc strip. Continuing in this way, all resistances will be left out when stud number 12 (ON) is reached. In this position, the electromagnet (NVRC) will attract the soft iron piece attached to the handle. Even if the operator removes his hand from the handle, it will still remain in the ON position as spring restoring force will be balanced by the force of attraction between NVRC and the soft iron piece of the handle. The *no volt release coil* (NVRC) carries same current as that of the field coil. In case supply voltage goes off, field coil current will decrease to zero. Hence NVRC will be deenergised and will not be able to exert any force on the soft iron piece of the handle. Restoring force of the spring will bring the handle back in the OFF position.

The starter also provides *over load* protection for the motor. The other electromagnet, OLRC *overload release coil* along with a soft iron piece kept under it, is used to achieve this. The current flowing through OLRC is the line current I_L drawn by the motor. As the motor is loaded, I_a hence I_L increases. Therefore, I_L is a measure of loading of the motor. Suppose we want that the motor should not be over loaded beyond rated current. Now gap between the electromagnet and the soft iron piece is so adjusted that for $I_L \leq I_{rated}$, the iron piece will not be pulled up. However, if $I_L \leq I_{rated}$ force of attraction will be sufficient to pull up iron piece. This upward movement of the iron piece of OLRC is utilized to de-energize NVRC. To the iron a copper strip (shaped in figure) is attached. During over loading condition, this copper strip will also move up and put a *short circuit* between two terminals B and C. Carefully note that B and C are nothing but the two ends of the NVRC. In other words, when over load occurs a short circuit path is created across the NVRC. Hence NVRC will not carry any current now and gets deenergised. The moment it gets deenergised, spring action will bring the handle in the OFF position thereby disconnecting the motor from the supply.

Three point starter has one disadvantage. If we want to run the machine at higher speed (above rated speed) by *field weakening* (i.e., by reducing field current), the strength of NVRC magnet may become so weak that it will fail to hold the handle in the ON position and the spring action will bring it back in the OFF position. Thus we find that a false disconnection of the motor takes place even when there is neither *over load* nor any *sudden disruption of supply*.

39.5 Speed control of shunt motor

We know that the speed of shunt motor is given by:

$$n = \frac{V_a - I_a r_a}{k\varphi}$$

where, V_a is the voltage applied across the armature and φ is the flux per pole and is proportional to the field current I_f. As explained earlier, armature current I_a is decided by the mechanical load present on the shaft. Therefore, by varying V_a and I_f we can vary n. For fixed supply voltage and the motor connected as shunt we can vary V_a by controlling an external resistance connected in series with the armature. I_f of course can be varied by controlling external field resistance R_f connected with the field circuit. Thus for .shunt motor we have essentially two methods for controlling speed, namely by:

1. varying armature resistance.
2. varying field resistance.

39.5.1 Speed control by varying armature resistance

The inherent armature resistance r_a being small, speed n versus armature current I_a characteristic will be a straight line with a small negative slope as shown in figure 39.4. In the discussion to follow we shall not disturb the field current from its rated value. At no load (i.e., $I_a = 0$) speed is highest and $n_0 = \frac{V_a}{k\varphi} = \frac{V}{k\varphi}$. Note that for shunt motor voltage applied to the field and armature circuit are same and equal to the supply voltage V. However, as the motor is loaded, $I_a r_a$ drop increases making speed a little less than the no load speed n_0 . For a well designed shunt motor this drop in speed is small and about 3 to 5% with respect to no load speed. This drop in speed from no load to full load condition expressed as a percentage of no load speed is called the *inherent speed regulation* of the motor.

$$\text{Inherent \% speed regulation} = \frac{n_0 - n}{n_0} \times 100$$

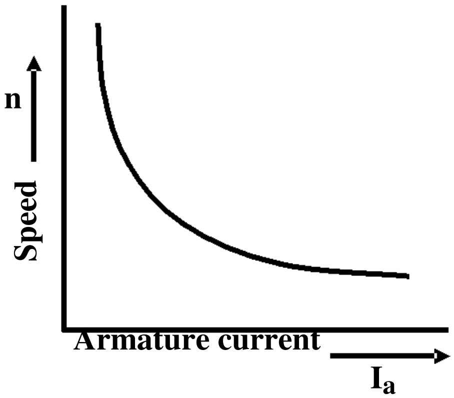


Figure 39.4: Speed vs. armature current characteristic.

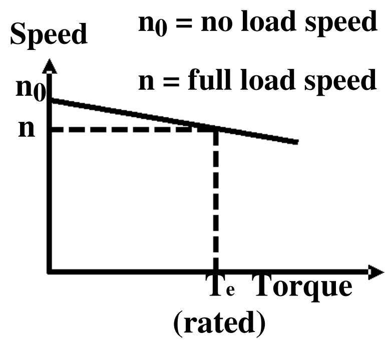


Figure 39.5: Speed vs. torque characteristic.

It is for this reason, a d.c shunt motor is said to be practically a constant speed motor (with no external armature resistance connected) since speed drops by a small amount from no load to full load condition.

Since $T_e = k\varphi I_a$, for constant φ operation, T_e becomes simply proportional to I_a. Therefore, speed vs. torque characteristic is also similar to speed vs. armature current characteristic as shown in figure 39.5.

The slope of the n vs I_a or n vs T_e characteristic can be modified by deliberately connecting external resistance r_{ext} in the armature circuit. One can get a family of speed vs. armature curves as shown in figures 39.6 and 39.7 for various values of r_{ext}. From these characteristic it can be explained how speed control is achieved. Let us assume that the load torque T_L is constant and field current is also kept constant. Therefore, since steady state operation demands $T_e = T_L$, $T_e = k\varphi I_a$ too will remain constant; which means I_a will not change. Suppose $r_{ext} = 0$, then at rated load torque, operating point will be at C and motor speed will be n. If additional resistance r_{ext1} is introduced in the armature circuit, new steady state operating speed will be n_1 corresponding to the operating point D. In this way one can get a speed of n_2 corresponding to the operating point E, when r_{ext2} is introduced in the armature circuit. This same load torque is supplied at various speed. Variation of the speed is smooth and speed will decrease smoothly if r_{ext} is increased. Obviously, this method is suitable for controlling speed below the *base* speed and for supplying constant rated load torque which ensures rated armature current always. Although, this method provides smooth wide range speed control (from base speed down to zero speed), has a serious draw back since energy loss takes place in the external resistance r_{ext} reducing the efficiency of the motor.

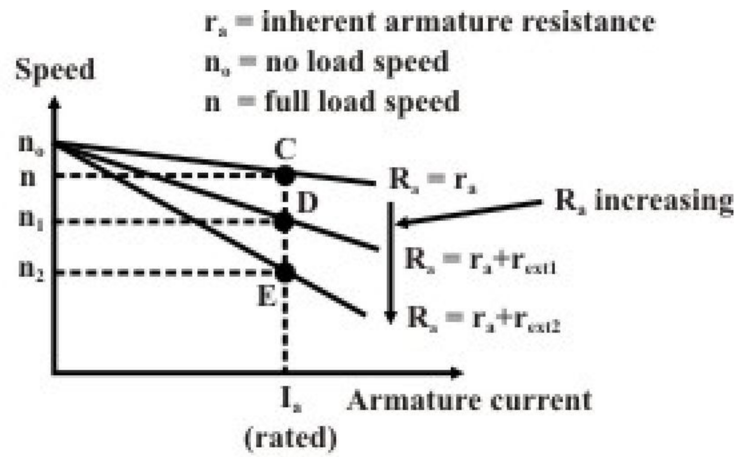


Figure 39.6: Family of speed vs. armature current characteristic.

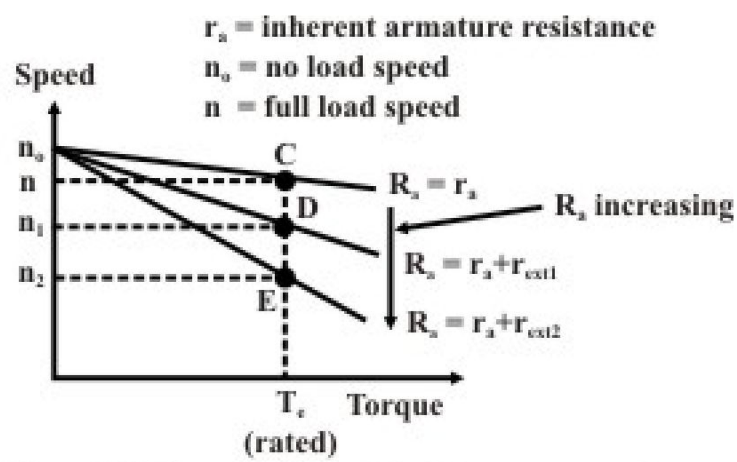


Figure 39.7: Family of speed vs. Torque current characteristic.

39.5.2 Speed control by varying field current

In this method field circuit resistance is varied to control the speed of a d.c shunt motor. Let us rewrite .the basic equation to understand the method.

$$n = \frac{V - I_a r_a}{k\varphi}$$

If we vary I_f, flux φ will change, hence speed will vary. To change I_f an external resistance is connected in series with the field windings. The field coil produces rated flux when no external resistance is connected and rated voltage is applied across field coil. It should be understood that we can only decrease flux from its rated value by adding external resistance. Thus the speed of the motor will rise as we decrease the field current and speed control above the *base* speed will be achieved. Speed versus armature current characteristic is shown in figure 39.8 for two flux values φ and φ_1 . Since $\varphi_1 < \varphi$, the no load speed n'_o for flux value φ_1 is more than the no load speed n_o corresponding to φ . However, this method will not be suitable for constant load torque.

To make this point clear, let us assume that the load torque is constant at rated value. So from the initial steady condition, we have $T_{L\ rated} = T_{e1} = k\varphi\, I_{a\ rated}$. If load torque remains constant and flux is reduced to φ_1 , new armature current in the steady state is obtained from $k\varphi_1 I_{a1} = T_{L\ rated}$. Therefore new armature current is

$$I_{a1} = \frac{\varphi}{\varphi_1} I_{a\ rated}$$

But the fraction, $\frac{\varphi}{\varphi_1} > 1$; hence new armature current will be greater than the rated armature current and the motor will be overloaded. This method therefore, will be suitable for a load whose torque demand decreases with the rise in speed keeping the output power constant as shown in figure 39.9. Obviously this method is based on *flux weakening* of the main field. Therefore at higher speed main flux may become so weakened, that armature reaction effect will be more pronounced causing problem in commutation.

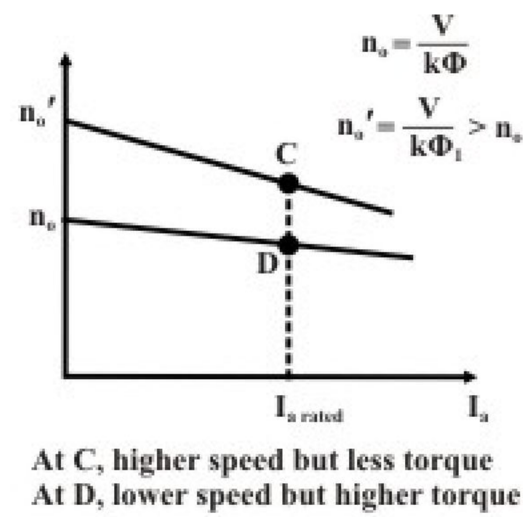


Figure 39.8: Family of speed vs. armature current characteristic.

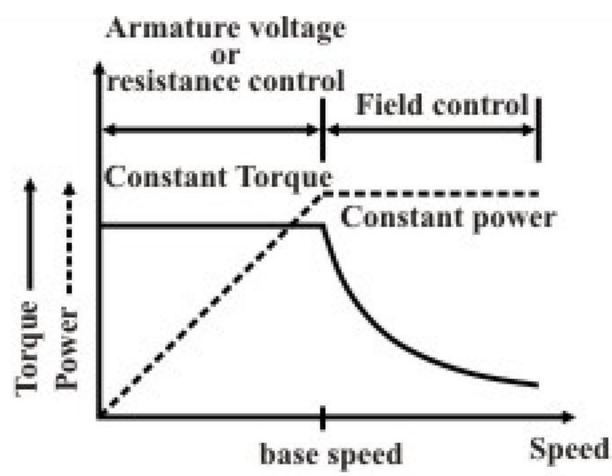


Figure 39.9: Constant torque & power operation.

39.5.3 Speed control by armature voltage variation

In this method of speed control, armature is supplied from a separate variable d.c voltage source, while the field is separately excited with fixed rated voltage as shown in figure 39.10. Here the armature resistance and field current are not varied. Since the no load speed $n_0 = \frac{V}{k\phi_a}$, the speed versus I_a characteristic will shift parallely as shown in figure 39.11 for different values of V_a.

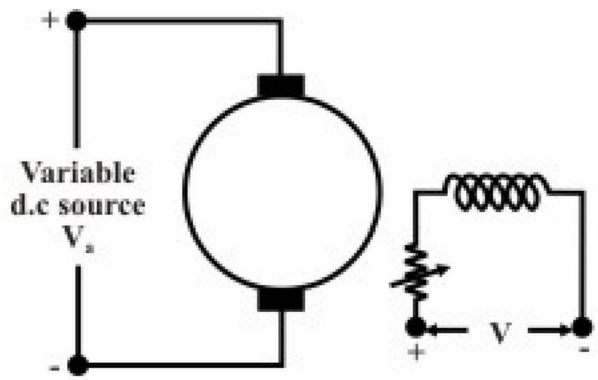


Figure 39.10: Speed control by controlling armature voltage.

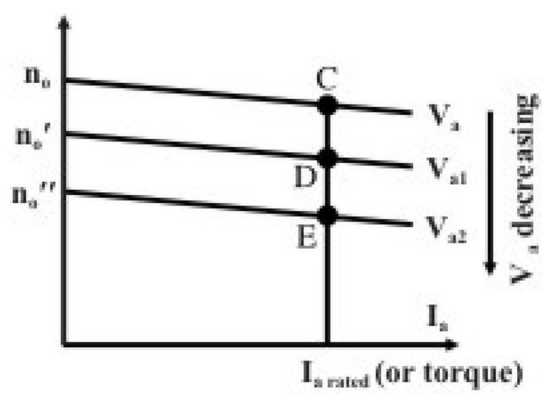


Figure 39.11: Family of *n* VS. I_a characteristics.

As flux remains constant, this method is suitable for constant torque loads. In a way armature voltage control method is similar to that of armature resistance control method except that the former one is much superior as no extra power loss takes place in the armature circuit. Armature voltage control method is adopted for controlling speed from base speed down to very small speed as one should not apply across the armature a voltage which is higher than the rated voltage.

39.5.4 Ward Leonard method: combination of V_a and I_f control

In this scheme, both field and armature control are integrated as shown in figure 39.12. Arrangement for field control is rather simple. One has to simply connect an appropriate rheostat in the field circuit for this purpose. However, in the pre power electronic era, obtaining a *variable* d.c supply was not easy and a separately excited d.c generator was used to supply the motor armature. Obviously to run this generator, a *prime mover* is required. A 3-phase induction motor is used as the prime mover which is supplied from a 3-phase supply. By controlling t field current of the generator, the generated emf, hence V_a can be varied. The potential divider connection uses two rheostats in parallel to facilitate reversal of generator field current.

First the induction motor is started with generator field current zero (by adjusting the jockey positions of the rheostats). Field supply of the motor is switched on with motor field rheostat set to zero. The applied voltage to the motor V_a, can now be gradually increased to the rated value by slowly increasing the generator field current. In this scheme, no starter is required for the d.c motor as the applied voltage to the armature is gradually increased. To control the speed of the d.c motor below base speed by armature voltage, excitation of the d.c generator is varied, while to control the speed above base speed field current of the d.c motor is varied maintaining constant V_a. Reversal of direction of rotation of the motor can be obtained by adjusting jockeys of the generator field rheostats. Although, wide range smooth speed control is achieved, the cost involved is rather high as we require one additional d.c generator and a 3-phase induction motor of simialr rating as that of the d.c motor whose speed is intended to be controlled.

In present day, variable d.c supply can easily be obtained from a.c supply by using controlled rectifiers thus avoiding the use of additional induction motor and generator set to implement Ward leonard method.

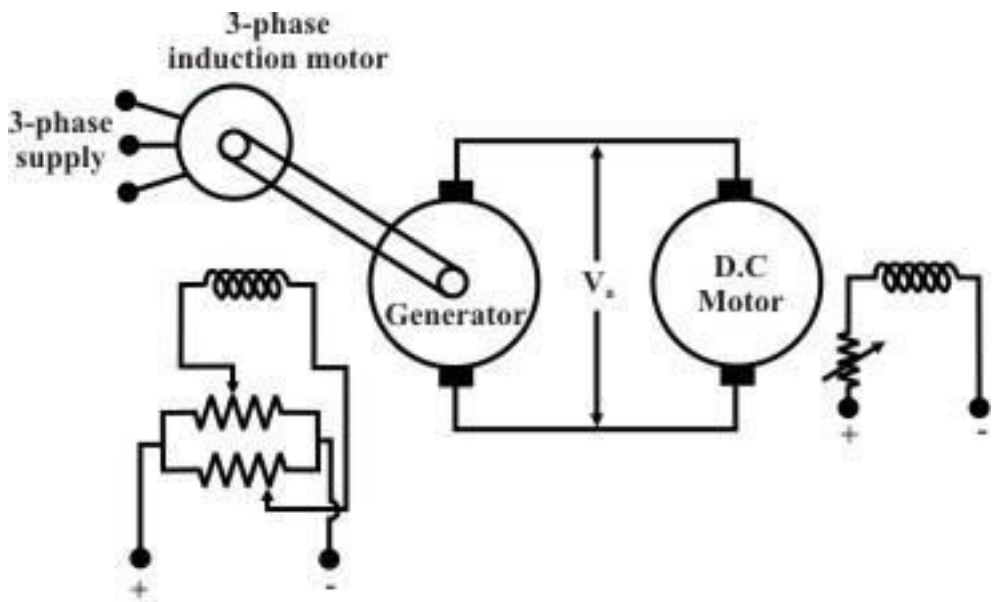


Figure 39.12: Scheme for Ward Leonard method.

39.6 Series motor

In this motor the field winding is connected in series with the armature and the combination is supplied with d.c voltage as depicted in figure 39.13. Unlike a shunt motor, here field current is not independent of armature current. In fact, field and armature currents are equal i.e., $I_f = I_a$. Now torque produced in a d.c motor is:

$$T \propto \varphi I_a$$

$$\propto I_f I_a$$

$$\propto I_a^2 \text{ before saturation sets in i.e., } \varphi \propto I_a \propto$$

$$I_a \text{ after saturation sets in at large } I_a$$

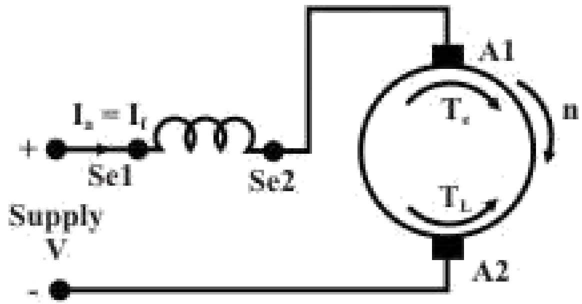


Figure 39.13: Series motor.

Since torque is proportional to the square of the armature current, starting torque of a series motor is quite high compared to a similarly rated d.c shunt motor.

39.6.1 Characteristics of series motor

Torque vs. armature current characteristic

Since $T \propto I_a^2$ in the linear zone and $T \propto I_a$ in the saturation zone, the T vs. I_a characteristic is as shown in figure 39.14

speed vs. armature current

From the KVL equation of the motor, the relation between speed and armature current can be obtained as follows:

$$V = I_a(r_a + r_{se}) + E_b$$

$$= I_a r + k\varphi n$$

$$\text{or, } n = \frac{V - I_a r}{k\varphi}$$

$$\text{In the linear zone } n = \frac{V - I_a r}{k' I_a}$$

$$= \frac{V}{k' I_a} - \frac{r}{k'}$$

$$\text{In the saturation zone } n = \frac{V - I_a r}{k' \varphi_{sat}}$$

The relationship is inverse in nature making speed dangerously high as $I_a \to 0$. Remember that the value of I_a, is a measure of degree of loading. Therefore, a series motor should never be operated under no load condition. Unlike a shunt motor, a series motor has no finite no load speed. Speed versus armature current characteristic is shown in figure nvsia:side: b.

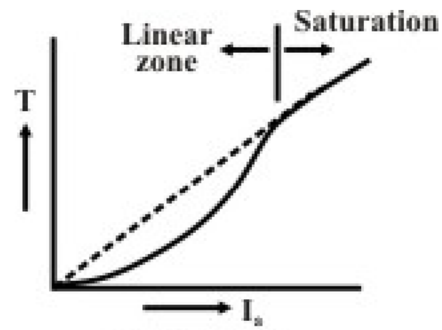


Figure 39.14

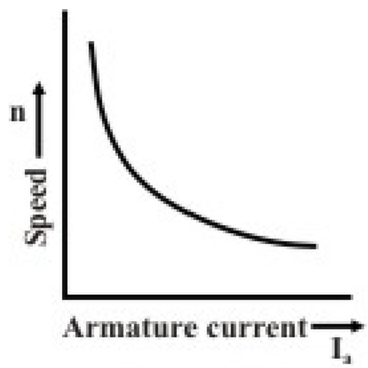


Figure 39.15

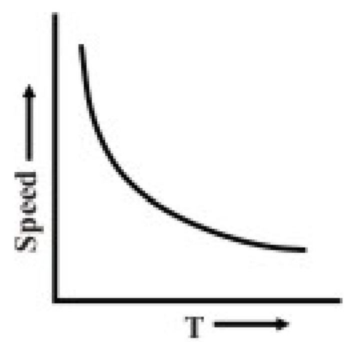


Figure 39.16

speed vs. torque characteristic

Since $I_a \propto \sqrt{T}$ in the linear zone, the relationship between speed and torque is

$$\frac{V}{k''\sqrt{T}} - \frac{r}{k'}$$

k'' and k' represent appropriate constants to take into account the proportionality that exist between current, torque and flux in the linear zone. This relation is also inverse in nature indicating once again that at light load or no load $(T \to 0)$ condition; series motor speed approaches a dangerously high value. The characteristic is shown in figure 39.16. For this reason, a series motor is never connected to mechanical load through belt drive. If belt snaps, the motor becomes unloaded and as a consequence speed goes up unrestricted causing mechanical damages to the motor.

39.7 Speed control of series motor

39.7.1 Speed control below base speed

For constant load torque, steady armature current remains constant, hence flux also remains constant. Since the machine resistance $r_a + r_{se}$ is quite small, the back emf E_b is approximately equal to the armature terminal voltage V_a. Therefore, speed is proportional to V_a. If V_a is reduced, speed too will be reduced. This V_a can be controlled either by connecting external resistance in series or by changing the supply voltage.

Series-parallel connection of motors

If for a drive two or more (even number) of identical motors are used (as in traction), the motors may be suitably connected to have different applied voltages across the motors for controlling speed. In series connection of the motors shown in figure 39.17, the applied voltage across each motor is $V/2$ while in parallel connection shown in figure 39.18, the applied voltage across each motor is V. The back emf in the former case will be approximately half than that in the latter case. For same armature current in both the cases (which means flux per pole is same), speed will be half in series connection compared to parallel connection.

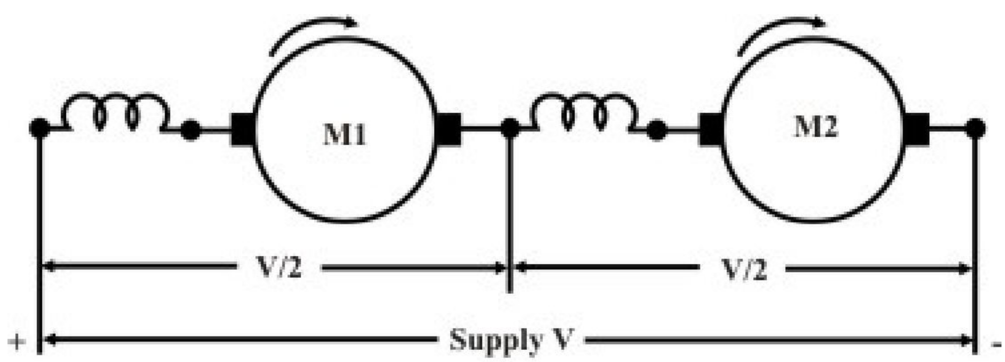


Figure 39.17: Motors connected in series.

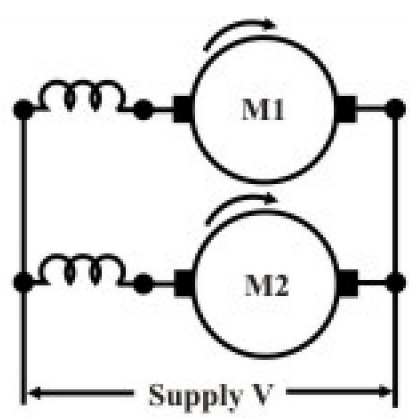


Figure 39.18: Motors connected in parallel.

39.7.2 Speed control above base speed

Flux or field current control is adopted to control speed above the base speed. In a series motor, independent control of field current is not so obvious as armature and field coils are in series. However, this can be achieved by the following methods:

1. **Using a *diverter* resistance connected across the field coil.**
   In this method shown in figure 39.19, a portion of the armature current is diverted through the diverter resistance. So field current is now not equal to the armature current; in fact it is less than the armature current. Flux weakening thus caused, raises the speed of the motor.

2. **Changing number of turns of field coil provided with tapings.**
   In this case shown figure 39.20, armature and field currents are same. However provision is kept to change the number of turns of the field coil. When number of turns changes, field mmf $N_{se} I_f$ changes, changing the flux hence speed of the motor.

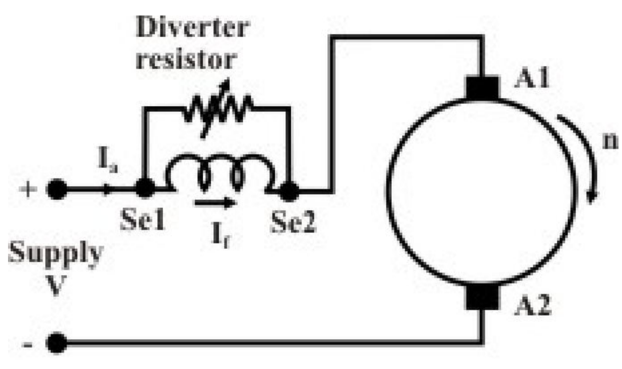


Figure 39.19: Field control with diverter.

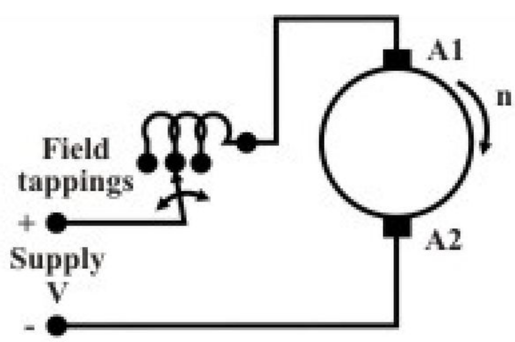


Figure 39.20: Field control with tappings.

3. **Connecting field coils wound over each pole in series or in. parallel.**
   Generally the field terminals of a d.c machine are brought out after connecting the field coils (wound over each pole) in series. Consider a 4 pole series motor where there will be 4 individual coils placed over the poles. If the terminals of the individual coils are brought out, then there exist several options for connecting them. The four coils could be connected in series as in figure 39.21; the 4 coils could be connected in parallel or parallel combination of 2 in series and other 2 in series as shown in figure 39.22. n figure For series connection of the coils (figure 39.21) flux produced is proportional to I_a and

for series-parallel connection (figure 39.22) flux produced is proportional to $\frac{I_a}{2}$.

Therefore, for same armature current I_a, flux will be doubled in the second case and naturally speed will be approximately doubled as back emf in both the cases is close to supply voltage V. Thus control of speed in the ratio of 1:2 is possible for series parallel connection.

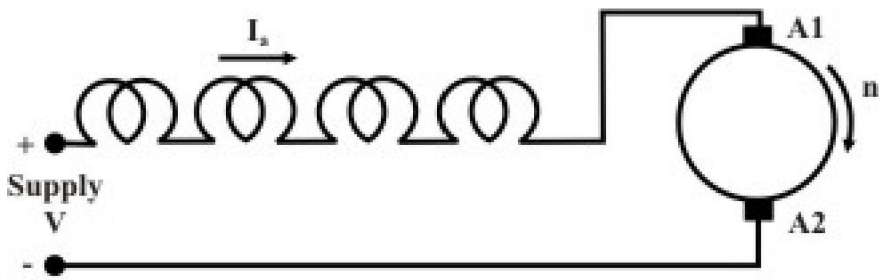


Figure 39.21: coils in series.

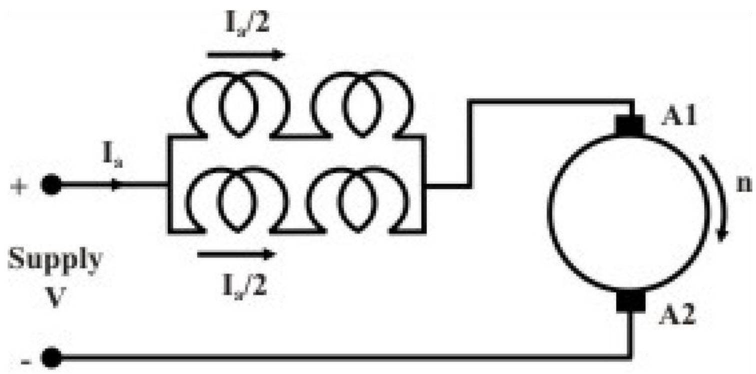


Figure 39.22: Series-parallel connection of coils.

In a similar way, reader can work out the variation of speed possible between (i) all coils connected in series and (ii) all coils connected in parallel.

39.8 Braking of d.c shunt motor: basic idea

It is often necessary in many applications to stop a running motor rather quickly. We know that any moving or rotating object acquires kinetic energy. Therefore, how fast we can bring the object to rest will depend essentially upon how quickly we can extract its kinetic energy and make arrangement to dissipate that energy somewhere else. If you stop pedaling your bicycle, it will eventually come to a stop eventually after moving quite some distance. The initial kinetic energy stored, in this case dissipates as heat in the friction of the road. However, to make the stopping faster, brake is applied with the help of rubber brake shoes on the rim of the wheels. Thus stored K.E now gets two ways of getting dissipated, one at the wheel-brake shoe interface (where most of the energy is dissipated) and the other at the road-tier interface. This is a good method no doubt, but regular maintenance of brake shoes due to wear and tear is necessary.

If a motor is simply disconnected from supply it will eventually come to stop no doubt, but will take longer time particularly for large motors having high rotational inertia. Because here the stored energy has to dissipate mainly through bearing friction and wind friction. The situation can be improved, by forcing the motor to operate as a generator during braking. The idea can be understood remembering that in motor mode electromagnetic torque acts along the direction of rotation while in generator the electromagnetic torque acts in the opposite direction of rotation. Thus by forcing the machine to operate as generator during the braking period, a torque opposite to the direction of rotation will be imposed on the shaft, thereby helping the machine to come to stop quickly. During braking action, the initial K.E stored in the rotor is either dissipated in an external resistance or fed back to the supply or both.

39.8.1 Rheostatic braking

Consider a d.c shunt motor operating from a d.c supply with the switch S connected to position 1 as shown in figure 39.23. S is a *single pole double throw switch* and can be connected either to position 1 or to position 2. One end of an external resistance R_b is connected to position 2 of the switch S as shown.

Let with S in position 1, motor runs at n rpm, drawing an armature current I_a and the back emf is $E_b = k\varphi n$. Note the polarity of E_b which, as usual for motor mode in opposition with the supply voltage. Also note T_e and n have same clock wise direction.

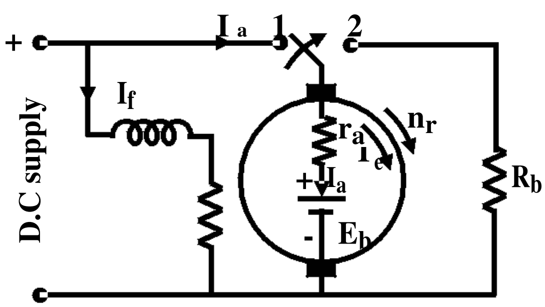


Figure 39.23: Machine operates as motor

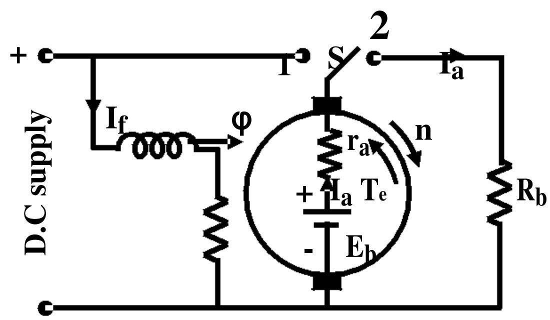


Figure 39.24: Machine operates as generator during braking

Now if S is suddenly thrown to position 2 at $t = 0$, the armature gets disconnected from the supply and terminated by R_b with field coil remains energized from the supply. Since speed of the rotor can not change instantaneously, the back emf value E_b is still maintained with same polarity prevailing at $t = 0_-$. Thus at $t = 0_+$, armature current will be $I_a = E_b/(r_a + R_b)$ and with reversed direction compared to direction prevailing during motor mode at $t = 0_-$.

Obviously for $t > 0$, the machine is operating as generator dissipating power to R_b and now the electromagnetic torque T_e must act in the opposite direction to that of n since I_a has changed direction but φ has not (recall $T_e \propto \varphi I_a$). As time passes after switching, n decreases reducing K.E and as a consequence both E_b and I_a decrease. In other words value of braking torque will be highest at $t = 0_+$, and it decreases progressively and becoming zero when the machine finally come to a stop.

39.8.2 Plugging or dynamic braking

This method of braking can be understood by referring to figures 39.25 and 39.26. Here S is a double pole double throw switch. For usual motoring mode, S is connected to positions 1 and 1'. Across terminals 2 and 2', a series combination of an external resistance R_b and supply voltage with polarity as indicated is connected. However, during motor mode this part of the circuit remains inactive.

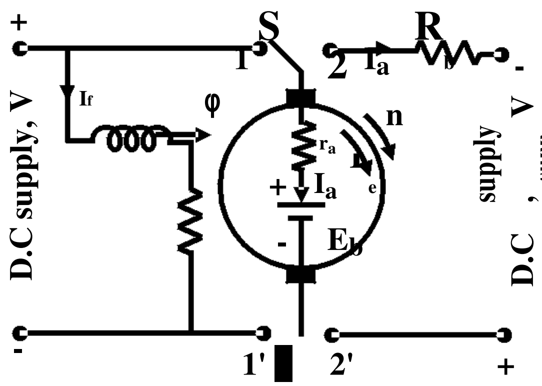


Figure 39.25: Machine operates as motor

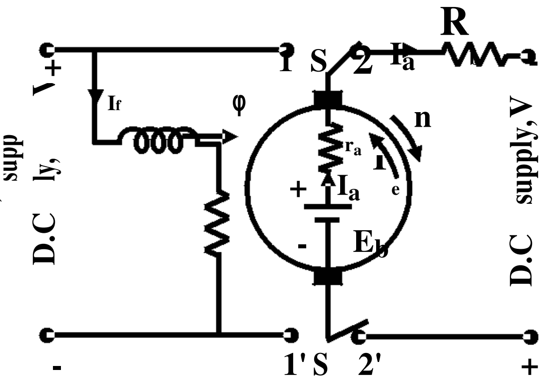


Figure 39.26: Machine operates as generator during braking (plugging).

To initiate braking, the switch is thrown to position 2 and 2' at t = 0, thereby disconnecting the armature from the left hand supply. Here at $t = 0_+$, the armature current will be $I_a = (E_b + V)/(r_a + R_b)$ as E_b and the right hand supply voltage have additive polarities by virtue of the connection. Here also I_a reverses direction producing T_e in opposite direction to n. I_a decreases as E_b decreases with time as speed decreases. However, I_a can not become zero at any time due to presence of supply V. So unlike rheostatic braking, substantial magnitude of braking torque prevails. Hence stopping of the motor is expected to be much faster then rheostatic breaking. But what happens, if S continuous to be in position 1' and 2' even after zero speed has been attained? The answer is rather simple, the machine will start picking up speed in the reverse direction operating as a motor. So care should be taken to disconnect the right hand supply, the moment armature speed becomes zero.

39.8.3 Regenerative braking

A machine operating as motor may go into regenerative braking mode if its speed becomes sufficiently high so as to make back emf greater than the supply voltage i.e., $E_b > V$. Obviously under this condition the direction of I_a will reverse imposing torque which is opposite to the direction of rotation. The situation is explained in figures 39.27 and 39.28. The normal motor operation is shown in figure 39.27 where armature motoring current I_a is drawn from the supply and as usual $E_b < V$. Since $E_b = k\varphi\, n_1$. The question is how speed on its own become large enough to make $E_b < V$ causing regenerative braking. Such a situation may occur in practice when the mechanical load itself becomes active. Imagine the d.c motor is coupled to the wheel of locomotive which is moving along a plain track without any gradient as shown in figure 39.27. Machine is running as a motor at a speed of n_1 rpm. However, when the track has a downward gradient (shown in figure 39.28), component of gravitational force along the track also appears which will try to accelerate the motor and may increase its speed to n_2 such that $E_b = k\varphi\, n_2 > V$. In such a scenario, direction of I_a reverses, feeding power back to supply. Regenerative braking here will not stop the motor but will help to arrest rise of dangerously high speed.

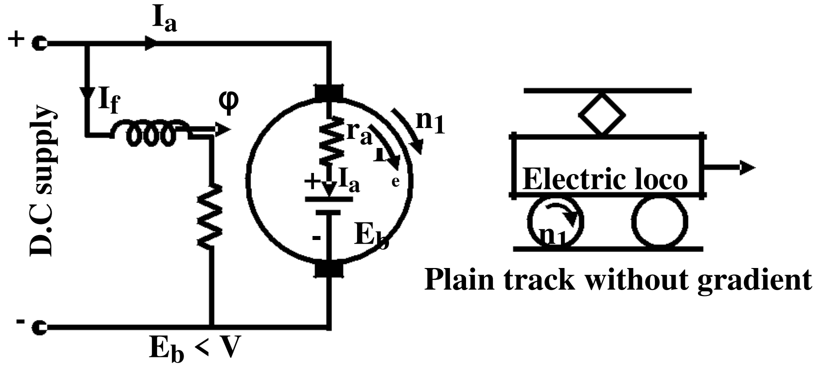


Figure 39.27: Machine operates as motor

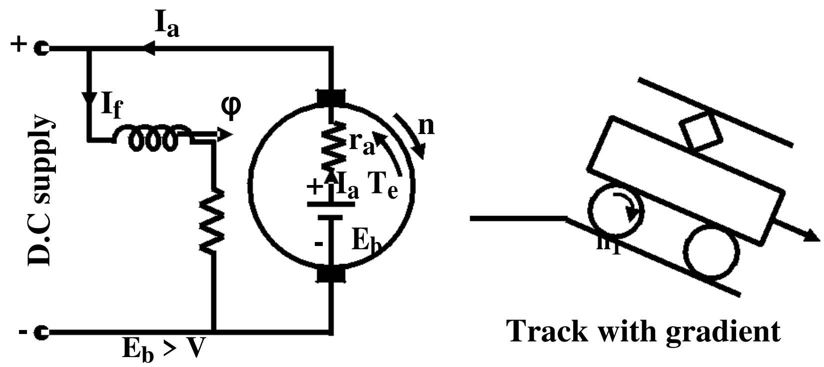


Figure 39.28: Machine enters regenerative braking mode.

39.9 Tick the correct answer

1. A 200 V, 1000 rpm, d.c shunt motor has an armature resistance of 0.8 Ω and its rated armature current is 20 A. Ratio of armature starting current to rated current with full voltage starting will be:

   (A) 1 V (B) 12.5 V (C) 25 V (D) 16 V

2. A 200 V, 1000 rpm, d.c shunt motor has an armature resistance of 0.8 Ω and found to run from a 200 V supply steadily at 950 rpm with a back emf of 190 V. The armature current is:

   (A) 237.5 A (B) 10 A (C) 250 A (D) 12.5 A

3. A d.c 220 V, shunt motor has an armature resistance of 1 Ω and a field circuit resistance of 150 Ω. While running steadily from 220 V supply, its back emf is found to be 209 V. The motor is drawing a line current of:

   (A) 11 A (B) 12.47 A (C) 221.47 A (D) 9.53 A

4. A 220 V, d.c shunt motor has r_a = 0.8 Ω and draws an armature current of 20 A while supplying a constant load torque. If flux is suddenly reduced by 10%, then immediately the armature current will become:

(A) 45.5 A and the new steady state armature current will be 22.2 A.
(B) 20 A and the new steady state armature current will be 22.2 A.
(C) 22.2 A and the new steady state armature current will be 45.5 A.
(D) 20 A and the new steady state armature current will be 25 A.

5. A 220 V, d.c shunt motor has r_a = 0.8 Ω and draws an armature current of 20 A while supplying a constant load torque. If a 4.2 Ω resistance is inserted in the armature circuit suddenly, then immediately the armature current will become:

   (A) 20 A and the new steady state armature current will be 3.2 A.
   (B) 3.2 A and the new steady state armature current will be 20 A.
   (C) 47.2 A and the new steady state armature current will be 3.2 A.
   (D) 3.2 A and the new steady state armature current will be 47.2 A.

6. A separately excited 220 V, d.c generator has r_a = 0.6 Ω and while supplying a constant load torque, draws an armature current of 30 A at rated voltage. If armature supply voltage is reduced by 20%, the new steady state armature current will be:

   (A) 24 A (B) 6 A (C) 30 A (D) 36 A

7. A 250 V, d.c shunt motor having negligible armature resistance runs at 1000 rpm at rated voltage. If the supply voltage is reduced by 25%, new steady state speed of the motor will be about:

   (A) 750 rpm (B) 250 rpm (C) 1000 rpm (D) 1250 rpm

39.10 Solve the following

1. A d.c motor takes an armature current of 50A at 220V. The resistance of the armature is 0.2Ω. The motor has 6 poles and the armature is lap wound with 430 conductors. The flux per pole is 0.03Wb. Calculate the speed at which the motor is running and the electromagnetic torque developed.

2. A 10KW, 250V, 1200 rpm d.c shunt motor has a full load efficiency of 80%, $r_a = 0.2\Omega$ and $R_f = 125\Omega$. The machine is initially operating at full load condition developing full load torque.

   i. What extra resistance should be inserted is the armature circuit if the motor speed is to be reduced to 960 rpm?
   ii. What additional resistance is to be inserted in the field circuit in order to raise the speed to 1300 rpm?

Note that for both parts (i) and (ii) the initial condition is the same full load condition as stated in the *first paragraph* and load torque remains constant throughout. Effect of saturation and armature reaction may be neglected.