UNIT - I

First order ODE

Topics

→ Exact, Linear and Bernouli's equations, Applications

→ Newton's law of cooling, Law of natural growth and decay

→ Equation not of first degree: equations solvable for p, equations solvable for y, equations solvable for x and ~~Clairaut's~~ Clairaut's method

Exact D.E:– Let $M(x,y)\,dx + N(x,y)\,dy = 0$ be a first order & first degree D.E.

where M, N are functions in terms of x, y

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then $Mdx + Ndy = 0$ is said to be exact D.E.

Working rule to solve exact D.E.

* The given equation is of the form $Mdx + Ndy = 0$
* Find $\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial x}$
* If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then given equation is an exact D.E.
* The solution can be obtained as

$$\underset{(y \text{ as constant})}{\int M\,dx} + \underset{(\text{the terms not containing } x)}{\int N\,dy} = c$$

problems

① Solve $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

Sol:– The equation is of the form $Mdx + Ndy = 0$

where $M = (e^y+1)\cos x$ $\qquad N = e^y \sin x$

$\frac{\partial M}{\partial y} = e^y \cos x$ $\qquad \frac{\partial N}{\partial x} = e^y \cos x$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so the given eq^n is an exact D.E.

The general soln is

$$\underset{(y \text{ as Constant})}{\int M dx} + \underset{(\text{terms not Containing } x)}{\int N dy} = c$$

$$\int (e^y+1)\cos x \, dx + \int 0 \, dy = c$$

$$(e^y+1)\sin x = c$$

② Solve $2xy\,dy - (x^2-y^2+1)dx = 0$

Sol:- The given equ is of the form $M dx + N dy = 0$

where $M = -(x^2-y^2+1)$ $\qquad N = 2xy$

$M = -x^2+y^2-1$

$\frac{\partial M}{\partial y} = 2y$ $\qquad \frac{\partial N}{\partial x} = 2y$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the given equ is an exact D.E.

The general soln is

$$\underset{(y \text{ as Constant})}{\int M dx} + \underset{(\text{terms not Containg } x)}{\int N dy} = c$$

$$\int (y^2-x^2-1)dx + \int 0 \, dy = c$$

$$y^2x - \frac{x^3}{3} - x = c$$

$$3xy^2 - x^3 - 3x = c$$

③ Solve $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

Sol:- The given equ is of the form $M dx + N dy = 0$

where $M = 1+e^{x/y}$ $\qquad N = e^{x/y}\left(1-\frac{x}{y}\right)$

$\frac{\partial M}{\partial y} = e^{x/y}\left(\frac{-x}{y^2}\right)$ $\qquad \frac{\partial N}{\partial x} = e^{x/y}\left(0-\frac{1}{y}\right) + \left(1-\frac{x}{y}\right)e^{x/y}\left(\frac{1}{y}\right)$

$= \frac{-xe^{x/y}}{y^2}$ $\qquad = \frac{-xe^{x/y}}{y^2}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the given equ is an exact D.E. (2)

The general soln is

$$\int M dx + \int N dy = c$$
(y as constant) (terms not containg x)

$$\int \left(1 + e^{x/y}\right) dx + \int 0 dy = c$$

$$x + \frac{e^{x/y}}{\frac{1}{y}} = c$$

$$x + y e^{x/y} = c$$

(4) Solve $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$

The given equ is of the form $Mdx + Ndy = 0$

where $M = 5x^4 + 3x^2y^2 - 2xy^3$ $N = 2x^3y - 3x^2y^2 - 5y^4$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2 \qquad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the given equ is an exact D.E.

The general soln is

$$\int M dx + \int N dy = c$$
(y as constant) (terms independent of x)

$$\int (5x^4 + 3x^2y^2 - 2xy^3) dx + \int -5y^4 dy = c$$

$$\frac{\cancel{5}x^5}{\cancel{5}} + \frac{\cancel{3}x^3}{\cancel{3}}y^2 - \frac{\cancel{2}x^2}{\cancel{2}}y^3 - \frac{\cancel{5}y^5}{\cancel{5}} = c$$

$$\Rightarrow x^5 + x^3y^2 - x^2y^3 - y^5 = c$$

H.W

(5) Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

(6) Solve $(r + \sin\theta - \cos\theta)dr + r(\sin\theta + \cos\theta)d\theta = 0$

Non exact D.E.

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then $Mdx + Ndy = 0$ is said to be non exact D.E.

Note

We can convert non exact D.E. into exact D.E. by multiplying with Integrating Factors.

To find an Integrating Factors (I.F.) of $Mdx + Ndy = 0$

Method (I)

If $M(x,y)dx + N(x,y)dy = 0$ is a homogeneous D.E. and $Mx + Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is an I.F. of $Mdx + Ndy = 0$

(1) Solve $x^2y\,dx - (x^3+y^3)dy = 0$ — (1)

Sol:- Here, given eqn is an homogeneous D.E. and comparing $Mdx + Ndy = 0$, $M = x^2y$ $\quad N = -x^3-y^3$

$\frac{\partial M}{\partial y} = x^2 \qquad \frac{\partial N}{\partial x} = -3x^2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Given eqn is not an exact D.E.

In this case,

$$I.F. = \frac{1}{Mx+Ny}$$

$$= \frac{1}{\cancel{x^3y} - \cancel{x^3y} - y^4}$$

$$= -\frac{1}{y^4}$$

multiplying eq (1) with I.F, then

$$-\frac{1}{y^4}(x^2y)\,dx + \frac{1}{y^4}(x^3+y^3)\,dy = 0$$

$$\frac{x^2}{y^3}dx - \frac{(x^3+y^3)}{y^4}dy = 0$$

Comparing with $M_1 dx + N_1 dy = 0$ then

$$M_1 = \frac{x^2}{y^3} \qquad N_1 = -\frac{x^3-y^3}{+y^4} = \frac{x^3}{y^4} + \frac{1}{y}$$

$$\frac{\partial M_1}{\partial y} = -\frac{3x^2}{y^4} \qquad \frac{\partial N_1}{\partial x} = -\frac{3x^2}{y^4}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ so the ~~giv~~ eqn is in exact.

The general soln is

$$\int M_1 dx + \int N_1 dy = c$$

(y as constant) (terms independent of x)

$$\int \frac{x^2}{y^3}dx + \int \frac{1}{y}dy = c$$

$$\boxed{\frac{x^3}{3y^3} + \log y = c}$$

② Solve $xy\,dx - (x^2+2y^2)dy = 0$ ①

$$M = xy \qquad N = -x^2 - 2y^2$$

$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = -2x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so it is not an exact.

$$I.F = \frac{1}{Mx+Ny} = \frac{1}{\cancel{x^2y} - \cancel{x^2y} - 2y^3} = -\frac{1}{2y^3}$$

multiplying with I.F. to eq①

$$\frac{x\cancel{y}}{-2y^{\cancel{3}}}dx - \frac{(x^2+2y^2)}{-2y^3}dy = 0$$

$$-\frac{x}{2y^2}dx + \left(\frac{x^2}{2y^3} + \frac{1}{y}\right)dy = 0$$

$$M_1 dx + N_1 dy = 0$$

$$M_1 = \frac{-x}{2y^2} \qquad N_1 = \frac{x^2}{2y^3} + \frac{1}{y}$$

$$\frac{\partial M_1}{\partial y} = -\frac{x}{\cancel{2}}\left(\frac{-\cancel{2}}{y^3}\right) \qquad \frac{\partial N_1}{\partial x} = \frac{2x}{2y^3} + 0$$

$$= \frac{x}{y^3} \qquad\qquad = \frac{x}{y^3}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ so the eqn is in exact.

The general soln is

$$\int M_1\,dx + \int N_1\,dy = c$$

(y as constant) (terms independent of x)

$$\int\left(\frac{-x}{2y^2}\right)dx + \int\left(\frac{x^2}{2y^3} + \frac{1}{y}\right)dy = c$$

$$\Rightarrow -\frac{x^2}{4y^2} + \frac{x^2}{2}\left(\frac{-3}{y^2}\right) + \log y = c$$

$$\Rightarrow \frac{-x^2}{4y^2} - \frac{3x^2}{2y^2} + \log y = c$$

HW

3) Solve $(3xy^2 - y^3)\,dx - (2x^2y - xy^2)\,dy = 0$

4) Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

Method (II)

If the equation $Mdx + Ndy = 0$ is of the form $yf(xy)\,dx + xg(xy)\,dy = 0$

and $Mx - Ny \neq 0$ then $\dfrac{1}{Mx - Ny}$ is an I.F. of $Mdx + Ndy = 0$

(1) Solve $y(xy\sin xy + \cos xy)\,dx + (xy\sin xy - \cos xy)x\,dy = 0$ — (1)

Sol:- eq (1) is of the form $y\,f(xy)\,dx + x\,g(xy)\,dy = 0$

eq (1) comparing with $Mdx + Ndy = 0$

$M = xy^2\sin xy + y\cos xy$ $\qquad N = x^2y\sin xy - x\cos xy$

Here $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x} \Rightarrow$ It is ~~an~~ not an exact D.E.

In this case

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{\cancel{x^2y^2\sin xy} + xy\cos xy - \cancel{x^2y^2\sin xy} + xy\cos xy}$$

$$= \frac{1}{2xy\cos xy}$$

multiplying eq (1) with I.F

$$\frac{1}{2xy\cos xy}\,y(xy\sin xy + \cos xy)\,dx + \frac{1}{2xy\cos xy}\,x(xy\sin xy - \cos xy)\,dy = 0$$

$$\frac{1}{2}\left(y\tan xy + \frac{1}{x}\right)dx + \frac{1}{2}\left(x\tan xy - \frac{1}{y}\right)dy = 0$$

which is of the form $M_1dx + N_1dy = 0$

$M_1 = \frac{1}{2}\left(y\tan xy + \frac{1}{x}\right)$ $\qquad N_1 = \frac{1}{2}\left(x\tan xy - \frac{1}{y}\right)$

$\dfrac{\partial M_1}{\partial y} = \frac{1}{2}(y\sec^2 xy \cdot x + \tan xy)$ $\qquad \dfrac{\partial N_1}{\partial x} = \frac{1}{2}(x\sec^2 xy \cdot y + \tan y)$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$ so the equation is an exact.

The general soln is

$$\underset{(y \text{ as Constant})}{\int M_1\,dx} + \underset{(\text{terms independent of } x)}{\int N_1\,dy} = C$$

$$\frac{1}{2}\int\left(y\tan xy+\frac{1}{x}\right)dx+\frac{1}{2}\int\left(\cancel{x\tan xy}-\frac{1}{y}\right)dy = c$$

$$\frac{1}{2}\,\frac{\cancel{y}\log(\sec xy)}{\cancel{y}}+\frac{1}{2}\log x+\frac{1}{2}(-\log y) = c$$

$$\Rightarrow \frac{1}{2}\left[\log\sec xy+\log x-\log y\right] = \log c$$

$$\Rightarrow \log|\sec xy|+\log\frac{x}{y} = \log c$$

$$\Rightarrow \boxed{\frac{x\sec xy}{y} = c}$$

HW
② Solve $(2xy+1)y\,dx+(1+2xy-x^3y^3)x\,dy = 0$

HW
③ Solve $y(1+xy)dx+x(1-xy)dy = 0$

Method III

If $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ & a continuous single variable function $f(x)$ such that $f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]$ then $e^{\int f(x)dx}$ is an I.F. of $Mdx+Ndy=0$

① Solve $2xy\,dy-(x^2+y^2+1)dx = 0$ —①

eq ① is of the form $Mdx+Ndy=0$

$M = -x^2-y^2-1$ $\quad N = 2xy$

$\frac{\partial M}{\partial y} = -2y$ $\quad \frac{\partial N}{\partial x} = 2y$

$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \Rightarrow$ not an exact.

$$f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right] = \frac{1}{2xy}\left[-2y-2y\right] = -\frac{\cancel{4}^{2}\cancel{y}}{\cancel{2}x\cancel{y}} = \frac{-2}{x}$$

$$I.F = e^{\int f(x)dx} = e^{-2\int\frac{1}{x}dx} = e^{-2\log x} = e^{\log\frac{1}{x^2}} = \frac{1}{x^2}$$

$$I.F = \frac{1}{x^2}$$

multiplying eq ① with I.F.

$$\frac{2xy}{x^2}dy - \frac{(x^2+y^2+1)}{x^2}dx = 0$$

$$\frac{2y}{x}dy - \left[1+\frac{y^2}{x^2}+\frac{1}{x^2}\right]dx = 0$$

It is in the form of $M_1 dx + N_1 dy = 0$

$$M_1 = -1-\frac{y^2}{x^2}-\frac{1}{x^2} \qquad N_1 = \frac{2y}{x}$$

$$\frac{\partial M_1}{\partial y} = -\frac{2y}{x^2} \qquad \frac{\partial N_1}{\partial x} = -\frac{2y}{x^2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \Rightarrow$ is an exact.

The general soln is

$$\underset{(y \text{ as constant})}{\int M_1 dx} + \underset{(\text{terms not containg } x)}{\int N_1 dy} = c$$

$$-\int\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx + 0 = c$$

$$\Rightarrow -\left[x+\frac{y^2}{x}-\frac{1}{x}\right] = c$$

$$\Rightarrow x^2-(y^2+1) = cx$$

HW
② Solve $(3xy-2ay^2)dx + (x^2-2axy)dy = 0$

HW
③ Solve $(x^3-2y^2)dx + 2xydy = 0$

Method (IV)

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ & a Continuous single variable function $g(y)$ such that $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = g(y)$ then $e^{\int g(y)dy}$ is an I.F. of $Mdx + Ndy = 0$

① Solve the D.E. $y(xy + e^x)dx - e^x dy = 0$ — ①

Sol:- $M = y(xy + e^x) = xy^2 + e^x y$ $\quad N = -e^x$

$\frac{\partial M}{\partial y} = 2xy + e^x$ $\quad \frac{\partial N}{\partial x} = -e^x$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Now $f(x) = \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{xy^2 + e^x y}\left[-e^x - 2xy - e^x\right]$

$= \frac{-2\left[\cancel{e^x + xy}\right]}{y\left[\cancel{e^x + xy}\right]} = -\frac{2}{y} = g(y)$

I.F $= e^{\int g(y)dy}$

$= e^{-2\int \frac{1}{y}dy} = e^{-2\log y} = e^{\log \frac{1}{y^2}} = \frac{1}{y^2}$

multplyng eq ① with I.F

$\frac{1}{y^2}\left[y(xy + e^x)\right]dx - \frac{e^x}{y^2}dy = 0$

$\frac{xy + e^x}{y}dx - \frac{e^x}{y^2}dy = 0$

It is in the form of $M_1 dx + N_1 dy = 0$

$M_1 = x + \frac{e^x}{y}$ $\quad N_1 = -\frac{e^x}{y^2}$

$\frac{\partial M_1}{\partial y} = -\frac{e^x}{y^2}$ $\quad \frac{\partial N_1}{\partial x} = -\frac{e^x}{y^2}$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ ⟹ which is exact.

The general soln is

$$\int M_1\,dx + \int N_1\,dy = c$$

(y as constant) (terms independent of x)

$$\int\left(x + \frac{e^x}{y}\right)dx + \int 0\,dy = c$$

$$\boxed{\frac{x^2}{2} + \frac{e^x}{y} = c}$$

H.W

(2) Solve $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3)\,dy = 0$

H.W

(3) Solve $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3)\,dy = 0$

Linear Differential equations of first order

The general form of first order Linear D.E. is

$$\frac{dy}{dx} + P(x)y = Q(x)$$

procedure

* The Linear D.E. is $\frac{dy}{dx} + P(x)y = Q(x)$
* Write down $I.F = e^{\int P(x)\,dx}$
* The general soln is given by

$$y \times I.F = \int (Q(x) \times I.F)\,dx + c$$

(OR)

* The Linear D.E. is of the type $\frac{dx}{dy} + P(y)x = Q(y)$
* Write down $I.F = e^{\int P(y)\,dy}$
* The general soln is given by

$$x \times I.F = \int (Q(y) \times I.F)\,dy + c$$

problems

(1) Solve $x\frac{dy}{dx} + y = \log x$

Given eqn can be written as

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{\log x}{x} \quad \text{--- (1)}$$

eq (1) is a linear form ie; $\frac{dy}{dx} + P(x)y = Q(x)$

$P(x) = \frac{1}{x}$, $Q(x) = \frac{\log x}{x}$

$$I.F = e^{\int P(x)\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\log x} = x$$

The general soln is

$$y \times I.F = \int (Q(x) \times I.F)\,dx$$

$$y \times x = \int \left(\frac{\log x}{\cancel{x}} \times \cancel{x}\right) dx$$

$$yx = x\log x - x + c$$

$$\boxed{xy = x(\log x - 1) + c}$$

② Solve $(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$

Given equ can be written as

$$\frac{dy}{dx} + \frac{2x}{1-x^2}y = \frac{x\sqrt{1-x^2}}{1-x^2} = \frac{x}{\sqrt{1-x^2}}$$

Comparing with Linear equation

$$P(x) = \frac{2x}{1-x^2} \qquad Q(x) = \frac{x}{\sqrt{1-x^2}}$$

$$I.F = e^{\int P(x)dx} = e^{\int \frac{2x}{1-x^2}dx} = e^{-\log(1-x^2)} = \frac{1}{1-x^2}$$

The general soln is

$$y \times I.F = \int (Q(x) \times I.F)dx$$

$$y.\frac{1}{1-x^2} = \int \frac{x}{\sqrt{1-x^2}} \cdot \frac{1}{1-x^2}dx$$

$$= \int \frac{x}{(1-x^2)^{3/2}}dx$$

put $1-x^2 = t$

$-2xdx = dt$

$xdx = -\frac{dt}{2}$

$$= -\frac{1}{2}\int \frac{1}{t^{3/2}}dt$$

$$= -\frac{1}{2}\left[\frac{t^{-3/2+1}}{-\frac{3}{2}+1}\right] + c$$

$$= \frac{1}{\sqrt{t}} + c = \frac{1}{\sqrt{1-x^2}} + c \Rightarrow \boxed{\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + c}$$

③ Solve $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{\sin 2x}{\log x}$

Here $P(x) = \frac{1}{x\log x}$ $\quad Q(x) = \frac{\sin 2x}{\log x}$

$$I.F = e^{\int P(x)\,dx} = e^{\int \frac{1}{x\log x}dx}$$

$$= e^{\log(\log x)} = \log x$$

The general soln is

$$y \times I.F = \int (Q(x) \times I.F)\,dx$$

$$y\log x = \int \left(\frac{\sin 2x}{\cancel{\log x}} \cancel{\log x}\right) dx$$

$$\boxed{y\log x = -\frac{\cos 2x}{2} + c}$$

④ Solve $(x + 2y^3)\frac{dy}{dx} = y$

The given eqn can be written as

$$\frac{dy}{dx} = \frac{y}{x+2y^3}$$

$$\frac{dx}{dy} = \frac{x+2y^3}{y} = \frac{x}{y} + 2y^2$$

$$\frac{dx}{dy} - \frac{x}{y} = 2y^2$$

Comparing with ~~$\frac{dy}{dx} + P(x)y = Q$~~ $\frac{dx}{dy} + P(y)x = Q(y)$

$P(y) = -\frac{1}{y}$ $\quad Q(y) = 2y^2$

$$I.F = e^{\int P(y)dy} = e^{-\int \frac{1}{y}dy} = e^{-\log y} = e^{\log \frac{1}{y}} = \frac{1}{y}$$

The general soln is

$$x \times I.F = \int (Q(y) \times I.F)\, dy$$

$$x \times \frac{1}{y} = \int 2y^2 \times \frac{1}{y}\, dy$$

$$= \cancel{2}\frac{y^2}{\cancel{2}} + c$$

$$\frac{x}{y} = y^2 + c$$

(5) Solve $(1+y^2)dx = (\tan^{-1}y - x)dy$

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

Here $P(y) = \frac{1}{1+y^2}$, $Q(y) = \frac{\tan^{-1}y}{1+y^2}$

$$I.F = e^{\int P(y)dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

The general soln is

$$x \times I.F = \int (Q(y) \times I.F)\, dy$$

$$x e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} e^{\tan^{-1}y}\, dy$$

put $\tan^{-1}y = t$

$\frac{1}{1+y^2}dy = dt$

$$= \int e^t \cdot t\, dt$$

$$= e^t(t-1) + c$$

$$\boxed{x e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c}$$

HW

⑥ Solve $(x+1)\frac{dy}{dx} - xy = e^x(x+1)^{n+1}$

HW

⑦ Solve $\sin 2x \frac{dy}{dx} - y = \tan x$

HW

⑧ Solve $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2}$ given $y=0$ when $x=1$

Bernouli's equation

The general form of Bernouli's equation is

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

procedure

The given eqn can be written as

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^n}P(x)y = Q(x)$$

$$y^{-n}\frac{dy}{dx} + y^{1-n}P(x) = Q(x)$$

put $y^{1-n} = t$

$(1-n)y^{\cancel{1}-n-\cancel{1}}\frac{dy}{dx} = \frac{dt}{dx}$

$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dt}{dx}$

$$\frac{1}{1-n}\frac{dt}{dx} + tP(x) = Q(x)$$

$$\frac{dt}{dx} + (1-n)tP(x) = Q(x)(1-n)$$

which is in Linear equation of first order.

problems

① Solve $x\frac{dy}{dx} + y = x^3y^6$

Given eqn can be written as

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{\cancel{y}}{xy^{\cancel{6}5}} = \frac{x^{\cancel{3}2}\cancel{y^6}}{\cancel{x}\cancel{y^6}}$$

$$\bar{y}^{6}\frac{dy}{dx} + \bar{y}^{5}\frac{1}{x} = x^{2}$$

It is a bernouli's equation. put $\bar{y}^{5} = t$

$$-5\bar{y}^{-6}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\bar{y}^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dt}{dx}$$

$$-\frac{1}{5}\frac{dt}{dx} + t.\frac{1}{x} = x^{2}$$

$$\Rightarrow \frac{dt}{dx} - \frac{5}{x}t = -5x^{2}$$

It is a Linear equation

$$I.F = e^{-\int\frac{5}{x}dx} = e^{-5\log x} = e^{\log\frac{1}{x^5}} = \frac{1}{x^{5}}$$

The general soln is

$$t \times I.F = \int(Q(x) \times I.F)\,dx$$

$$t.\frac{1}{x^{5}} = \int(-5x^{2})\left(\frac{1}{x^{5}}\right)dx$$

$$\frac{1}{x^{5}y^{5}} = -5\int x^{-3}dx$$

$$= -5\frac{x^{-2}}{-2} + c$$

$$\frac{1}{x^{5}y^{5}} = \frac{5}{2}\cdot\frac{1}{x^{2}} + c$$

$$\boxed{\frac{1}{y^{5}} = \frac{5}{2}x^{3} + cx^{5}}$$

② Solve $\frac{dy}{dx} + x\sin 2y = x^{3}\cos^{2}y$

$$\frac{1}{\cos^{2}y}\frac{dy}{dx} + \frac{x\,2\sin y\cancel{\cos y}}{\cos^{2}y} = x^{3}$$

$$\sec^{2}y\frac{dy}{dx} + 2\tan y\,x = x^{3}$$

put $\tan y = t$

$$\sec^{2}y\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + 2tx = x^{3}$$

$$I.F = e^{2\int x\,dx} = e^{\frac{2x^{2}}{2}} = e^{x^{2}}$$

The general soln is

$$t \times I.F = \int (Q(x) \times I.F)\, dx$$

$$t e^{x^2} = \int x^3 e^{x^2} dx$$

$$= \int x^2 . x e^{x^2} dx$$

$x^2 = u$

$2x\,dx = du$

$x\,dx = \frac{du}{2}$

$$= \int u e^u \frac{du}{2}$$

$$t e^{x^2} = \frac{1}{2} e^u (u-1) + c$$

$$\boxed{\tan y\, e^{x^2} = \frac{1}{2} e^{x^2}(x^2-1) + c}$$

HW

③ Solve $3\frac{dy}{dx} - y\cos x = y^4(\sin 2x - \cos x)$

HW

④ Solve $e^x \frac{dy}{dx} = 2xy^2 + ye^x$

HW

⑤ Solve $\frac{dy}{dx} + y\cot x = y^2 \sin^2 x \cos^2 x$

Applications of D.E

Newton's law of cooling

The rate of change of the temperature of a body is proportional to the difference of the temperature of the body and that of the surrounding medium.

Let θ be the temperature of the body at time 't' & θ_0 be the temperature of surrounding medium, By the Newton's law of cooling, we have

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -K(\theta - \theta_0)$$

$$\int \frac{d\theta}{\theta - \theta_0} = -k\int dt$$

$$\Rightarrow \log(\theta - \theta_0) = -kt + \log c$$

$$\log \frac{(\theta - \theta_0)}{c} = -kt$$

$$\Rightarrow \boxed{\theta - \theta_0 = c e^{-kt}}$$

problems

① The temperature of the body drops from $100°c$ to $75°c$ in 10mins when the surrounding air is at $20°c$ temperature. What will be ~~the~~ it's temperature after an half an hour? When will the temperature be $25°c$.

Sol:- Given $\theta_0 = 20°c$

We have $\theta - \theta_0 = c e^{-kt}$

$$\theta - 20 = c e^{-kt} \quad \text{—— ①}$$

The temperature drops from $100°c$ to $75°c$ in 10mis.

i.e; When $t = 0$ $\quad \theta = 100°c \Rightarrow ① \Rightarrow 100 - 20 = c e^{0(-k)} \Rightarrow c = 80$

When $t = 10$ $\quad \theta = 75°c \overset{①}{\Rightarrow} 75 - 20 = 80 e^{-K(10)}$

$$\Rightarrow K = -\frac{1}{10}\log\frac{55}{80}$$

~~K = 0.016~~

After half an hour (30mins) the temperature becomes (from ①)

$$\theta - 20 = 80 \times \cancel{e^{-0.016(30)}} e^{-30K}$$

$$= 80 \times e^{-30\left(-\frac{1}{10}\log\frac{55}{80}\right)}$$

$$\boxed{\theta = 46°c}$$

When $\theta = 25°c \Rightarrow t = ?$

eq① becomes $\quad 25 - 20 = 80 e^{-\left(-\frac{1}{10}\log\frac{55}{80}\right)t} \Rightarrow t = 74.86\text{mins}$

② If the air is maintained at $15^\circ C$ and the temperature of the body drops from $70^\circ C$ to $40^\circ C$ in 10 mins. What will be it's temperature after 30 mins.

Sol:- Given $\theta_0 = 15^\circ C$

We have $\theta - \theta_0 = c e^{-kt}$ ——— ①

When $t = 0$ $\quad \theta = 70^\circ C$

$70 - 15 = c e^{-k(0)} \Rightarrow c = 55$

eq ① becomes

$\theta - \theta_0 = 50 e^{-kt}$ ——— ②

When $t = 10$ mins $\quad \theta = 40^\circ C$

② becomes

$40 - 15 = 55 e^{-k(10)}$

$\Rightarrow e^{-10k} = \frac{\cancel{25}^{5}}{\cancel{55}_{11}} = \frac{5}{11}$

When $t = 30$ mins, the temperature becomes

$\theta = 15 + 55 \left(e^{-10k}\right)^3$

$= 15 + 55 \left(\frac{5}{11}\right)^3$

$\boxed{\theta = 20.16^\circ C}$

③ A body kept in air with temperature $25^\circ C$ cools from $140^\circ C$ to $80^\circ C$ in 20 mins. Find when the body cools down to $35^\circ C$.

Sol:- Given $\theta_0 = 25^\circ C$ $\quad$ we have $\theta - \theta_0 = c e^{-kt}$

When $t = 0$ $\quad \theta = 140^\circ C$ $\quad \theta - 25 = c e^{-kt}$ ——— ①

① becomes $140 - 25 = c(1) \Rightarrow c = 115$

① becomes

$\theta - 25 = 115 e^{-kt}$ ——— ②

When $t = 20$, $\theta = 80^\circ C$

$$80 - 25 = 115 e^{-k(20)}$$

$$\frac{\cancel{55}^{\,11}}{\cancel{115}_{\,23}} = e^{-20k} \Rightarrow e^{-20k} = \frac{11}{23}$$

When $\theta = 35^\circ c$ $\quad t = ?$

$$35 - 25 = 115 e^{-kt}$$

$$10 = 115\left(e^{-20k}\right)^{\frac{t}{20}}$$

$$10 = 115\left(\frac{11}{23}\right)^{t/20} \Rightarrow t = 66.2$$

After 66.2 mins, the ~~temper.~~ body cools down to $35^\circ c$

Hw
(4) An object cools from $120^\circ c$ to $95^\circ c$ in half an hour when surrounded by air whose temperature is $70^\circ F$. Find its temperat at the end of another half an hour.

Hw
(5) A copper ball is heated to a temperature of $80^\circ c$. Then at time $t = 0$ it is placed in water which is maintained at $30^\circ C$. If at $t = 3$ min, the temperature of the ball is reduced to $50^\circ c$. Find the time at which the temperature of the ball is $40^\circ C$

Hw
(6) Suppose that an object is heated to $300^\circ F$ and allowed to cool in a room whose air temperature is $80^\circ F$, if after 10 mins the temperature of the object is $250^\circ F$, what will be it's temperature after 20 mins.

Law of natural growth and decay

Let $x(t)$ be the amount of substance at time 't'. A law of chemical conversion states that the rate of change of amount $x(t)$ of a chemically changing substance is proportional to the amount of substance available at that time.

i.e: $\frac{dx}{dt} \propto x$

$\frac{dx}{dt} = kx$ (growth)

$\frac{1}{x}dx = k\,dt$

$\int \frac{1}{x}dx = k\int dt$

$\log x = kt + \log c$

$\log \frac{x}{c} = kt$

$\frac{x}{c} = e^{kt}$

$$\boxed{x = ce^{kt}}$$

$\frac{dx}{dt} = -kx$ (decay)

$\frac{1}{x}dx = -k\,dt$

$\int \frac{1}{x}dx = -k\int dt$

$\log x = -kt + \log c$

$\log \frac{x}{c} = -kt$

$\frac{x}{c} = e^{-kt}$

$$\boxed{x = ce^{-kt}}$$

Note:- * If t increases and x increases we can take $\frac{dx}{dt} = kx$ $(k>0)$

* If t increases and x decreases we can take $\frac{dx}{dt} = -kx$ $(k>0)$

Problems

1. A bacterial culture, growing exponentially increases from 200 to 500 gms in the period of 6am to 9am. How many grams will be present at noon (12pm)

Sol:- Given $x = 200$ $t = 0$

$x = 500$ $t = 3$ hrs (6am to 9am)

$x = ?$ $t = 6$ hrs (12pm (noon))

By law of natural growth

$\frac{dx}{dt} \propto x \Rightarrow x = ce^{kt}$ —— (1)

$t=0, \quad \cancel{200}\; x = 200$

$200 = ce^{k(0)} \Rightarrow c = 200$

① becomes

$$x = 200e^{kt} \quad \text{—— ②}$$

When $x = 500, \quad t = 3\,hrs$

$500 = 200e^{3k}$

$\Rightarrow e^{3k} = \frac{500}{200} = \frac{5}{2}$

$\Rightarrow 3k = \log\frac{5}{2}$

$\Rightarrow k = \frac{1}{3}\log\frac{5}{2}$

When $t = 6\,hrs \quad x = ?$

From ②

$\cancel{500} \quad x = 200e^{\frac{1}{\cancel{3}}\log\frac{5}{2}\cdot\cancel{6}^{2}}$

$\boxed{x = 1250\,gms}$

② A bacterial culture, growing exponentially, increases from 100 to 400gms in 10hrs. How much was present after 3hrs, from the initial instant?

Sol: Given $x = 100 \quad t = 0$

$x = 400 \quad t = 10\,hrs$

$x = ? \quad t = 13\,hrs$

By law of natural growth

$\frac{dx}{dt} \propto x \Rightarrow x = ce^{kt}$ —— ①

When $t=0, \quad x = 100$

$\Rightarrow 100 = ce^{k(0)} \Rightarrow \boxed{c = 100}$

① becomes $x = 100e^{kt}$ —— ②

When $x = 400 \quad t = 10\,hrs$

$400 = 100e^{10k}$

$\Rightarrow e^{10k} = \frac{400}{100} = 4$

$\Rightarrow 10k = \log 4$

$\Rightarrow \boxed{k = \frac{1}{10}\log 4}$

When $\cancel{t} = 13\,hrs \quad x = ?$

From ②

$x = 100e^{(\frac{1}{10}\log 4)13}$

$\boxed{x = 606.2\,gms}$

Equations of first order But not of the first degree

The general form of the first order D.E. of degrees $n>1$ is

$$P_0\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \cdots + P_{n-1}\left(\frac{dy}{dx}\right) + P_n = 0 \quad (1)$$

where $P_0\ P_1\ P_2\ \cdots\ P_{n-1}\ P_n$ are functions of x and y.

If we denotes $\frac{dy}{dx}$ as p then eq (1) becomes

$$P_0 p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots P_{n-1}p + P_n = 0$$

Equations solvable for p

Let $P_0 p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_{n-1}p + P_n = 0$ — (1)

By taking into n linear factors, eq (1) can be written as

$$[p - f_1(x,y)][p - f_2(x,y)] \cdots [p - f_n(x,y)] = 0 \quad (2)$$

equating each factor equal to zero, we obtain n equations of the first order and first degree, then

$$p = \frac{dy}{dx} = f_1(x,y),\ p = \frac{dy}{dx} = f_2(x,y)\ \cdots,\ p = \frac{dy}{dx} = f_n(x,y)$$

The solutions can be

$$F_1(x,y,c_1) = 0,\ F_2(x,y,c_2) = 0, \cdots F_n(x,y,c_n) = 0$$

where $c_1\ c_2\ c_3\ \cdots\ c_n$ are arbitrary Constants,

If we replace all the Constants $c_1\ c_2\ c_3 \cdots c_n$ with a single Constant 'c' then 'n' solutions becomes

$$F_1(x,y,c) = 0,\ F_2(x,y,c) = 0\ \cdots\ F_n(x,y,c) = 0$$

Combining the above equations, the solution of (1) is

$$F_1(x,y,c) \cdot F_2(x,y,c) \cdots F_n(x,y,c) = 0$$

problems

① Solve the following D.E's

i) $p^2 = ax^3$ where $p = \frac{dy}{dx}$

Sol:- Given $p^2 = ax^3$

$p = \pm (ax^3)^{1/2}$

$p = \pm a^{1/2} x^{3/2}$

$\frac{dy}{dx} = \pm \sqrt{a}\, x^{3/2}$

$\Rightarrow dy = \pm \sqrt{a}\, x^{3/2} dx$

$\Rightarrow \int dy = \pm \sqrt{a} \int x^{3/2} dx$

$\Rightarrow y = \pm \sqrt{a} \left[\frac{x^{3/2+1}}{3/2+1} \right] + c$

$\Rightarrow y = \pm \sqrt{a}\, \frac{2}{5}\, x^{5/2} + c$

$\Rightarrow 5y = \pm 2\sqrt{a}\, x^{5/2} + 5c$

$\Rightarrow 5y - 5c = \pm 2\sqrt{a}\, x^{5/2}$

$\Rightarrow$ squaring on both sides

$25(y-c)^2 = \left(\pm 2\sqrt{a}\, x^{5/2}\right)^2 = 4ax^5.$

$\boxed{25(y-c)^2 = 4ax^5}$ which is the required soln.

ii) $\left(\frac{dy}{dx}\right)^3 - ax^4 = 0$

$\left(\frac{dy}{dx}\right)^3 = ax^4$

$\frac{dy}{dx} = a^{1/3} x^{4/3}$

$$dy = a^{1/3} x^{4/3} dx$$

$$\int dy = a^{1/3} \int x^{4/3} dx$$

$$y = a^{1/3} \left[\frac{x^{4/3+1}}{\frac{4}{3}+1} \right] + c$$

$$y = \frac{3}{7} a^{1/3} x^{7/3} + c$$

$$y - c = \frac{3}{7} a^{1/3} x^{7/3}$$

$$7(y-c) = 3 a^{1/3} x^{7/3}$$

cubing on bothsides

$$343(y-c)^3 = (3a^{1/3} x^{7/3})^3 = 27 a x^7.$$

(iii) $p^2 - 5p + 6 = 0$

Sol:- $(p-2)(p-3) = 0$

$p - 2 = 0$, $p - 3 = 0$

$p = 2$ $\qquad$ $p = 3$

$\frac{dy}{dx} = 2$ $\qquad$ $\frac{dy}{dx} = 3$

$dy = 2dx$ $\qquad$ $dy = 3dx$

$\int dy = 2\int dx$ $\qquad$ $\int dy = 3\int dx$

$\Rightarrow y = 2x + c$ $\qquad$ $y = 3x + c$

$y - 2x - c = 0$, $\quad y - 3x - c = 0$

$\therefore$ The required soln is

$$\boxed{(y - 2x - c)(y - 3x - c) = 0}$$

iv) $p^2 - 7p + 12 = 0$

$(p-3)(p-4) = 0$

$p = 3 \qquad p = 4$

$\frac{dy}{dx} = 3 \qquad \frac{dy}{dx} = 4$

$\int dy = 3\int dx \qquad \int dy = 4\int dx$

$y = 3x + c \qquad y = 4x + c$

$\Rightarrow y - 3x - c = 0, \quad y - 4x - c = 0$

$\therefore$ The combined soln is

$(y - 3x - c)(y - 4x - c) = 0$

HW

v) $p^2 - 2p\cosh x + 1 = 0$

② Solve

i) $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$

Given eqn can be written as

$yp^2 + (x-y)p - x = 0$

$yp^2 + xp - yp - x = 0$

$yp^2 - yp + xp - x = 0$

$yp(p-1) + x(p-1) = 0$

$(yp + x)(p-1) = 0$

$p - 1 = 0 \qquad yp + x = 0$

$p = 1 \qquad y\frac{dy}{dx} + x = 0$

$\frac{dy}{dx} = 1 \qquad y\,dy + x\,dx = 0$

$y\,dy = -x\,dx$

$dy = 1\,dx$

$\int dy = \int 1 dx$ $\qquad\qquad$ $\int y\,dy = -\int x\,dx$

$y = x + c$ $\qquad\qquad$ $\frac{y^2}{2} = -\frac{x^2}{2} + c$

$y - x - c = 0$ $\qquad\qquad$ $\frac{x^2}{2} + \frac{y^2}{2} - c = 0$

$\Rightarrow x^2 + y^2 - 2c = 0$

$\therefore$ The Combined soln is

$$(y - x - c)(x^2 + y^2 - 2c) = 0$$

ii) Solve $x^2p^2 + xyp - 6y^2 = 0$

which is a quadratic eqn in 'p'.

$x^2p^2 + 3xyp - 2xyp - 6y^2 = 0$

$\Rightarrow xp(xp + 3y) - 2y(xp + 3y) = 0$

$\Rightarrow (xp + 3y)(xp - 2y) = 0$

$\Rightarrow$

$xp + 3y = 0$ $\qquad\qquad$ $xp - 2y = 0$

$x\frac{dy}{dx} = -3y$ $\qquad\qquad$ $x\frac{dy}{dx} - 2y = 0$

$\int \frac{1}{y} dy = \int \frac{-3}{x} dx$ $\qquad\qquad$ $x\frac{dy}{dx} = 2y$

$\Rightarrow \int \frac{1}{y} dy = \int \frac{2}{x} dx$

$\Rightarrow \log y = -3\log x + \log c$

$\Rightarrow \log y + 3\log x = \log c$ $\qquad\qquad$ $\Rightarrow \log y = 2\log x + \log c$

$\Rightarrow yx^3 = c$ $\qquad\qquad$ $\Rightarrow \log y - \log x^2 = \log c$

$\Rightarrow x^3y - c = 0$ $\qquad\qquad$ $\Rightarrow \log \frac{y}{x^2} = \log c$

$\Rightarrow \frac{y}{x^2} = c$

$\frac{y}{x^2} = c$

$\therefore$ The combined soln is

$$\left(x^3y - c\right)\left(\frac{y}{x^2} - c\right) = 0$$

iii) $p(p+y) = x(x+y)$

$p^2 + py = x^2 + xy$

$\Rightarrow p^2 + py - x^2 - xy = 0$

$\Rightarrow p^2 - x^2 + y(p-x) = 0$

$\Rightarrow (p-x)(p+x) + y(p-x) = 0$

$\Rightarrow (p-x)(p+x+y) = 0$

$\Rightarrow p - x = 0$

$\Rightarrow \frac{dy}{dx} - x = 0$

$\Rightarrow dy = x\,dx$

$\Rightarrow y = \frac{x^2}{2} + c$

$\Rightarrow y - \frac{x^2}{2} - c = 0$

$p + y = -x$

$\frac{dy}{dx} + y = -x$

$I.F = e^{\int 1 dx} = e^x$

$ye^x = -\int xe^x dx$

$ye^x = -e^x(x-1) + c$

$y = -(x-1) + ce^{-x}$

$y + x - 1 - ce^{-x} = 0$

$\therefore$ The Combined soln is

$$\left(y - \frac{x^2}{2} - c\right)\left(y + x - 1 - ce^{-x}\right) = 0$$

(3) Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

Given eqn can be written as

$p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$

$\frac{p^2 - 1}{p} = \frac{x}{y} - \frac{y}{x}$

$$p^2 - 1 = p\left(\frac{x}{y} - \frac{y}{x}\right)$$

$$p^2 + p\left(\frac{y}{x} - \frac{x}{y}\right) - 1 = 0$$

$$p = \frac{x}{y} \quad \text{and} \quad p = \frac{-y}{x}$$

$$\frac{dy}{dx} = \frac{x}{y} \qquad \frac{dy}{dx} = \frac{-y}{x}$$

$$\int y\,dy = \int x\,dx \qquad \int \frac{1}{y}\,dy = -\int \frac{1}{x}\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c \qquad \log y = -\log x + \log c$$

$$\log y + \log x = \log c$$

$$\frac{y^2}{2} - \frac{x^2}{2} - c = 0 \qquad xy = c$$

$\therefore$ The combined soln is

$$\boxed{\left(\frac{y^2}{2} - \frac{x^2}{2} - c\right)(xy - c) = 0}$$

$$p^2 + p\left(\frac{y}{x} - \frac{x}{y}\right) - 1 = 0$$

$$\Rightarrow p = \frac{-\left(\frac{y}{x} - \frac{x}{y}\right) \pm \sqrt{\left(\frac{x}{y} - \frac{y}{x}\right)^2 + 4\frac{y}{x}\cdot\frac{x}{y}}}{2}$$

$$= \frac{\left(\frac{x}{y} - \frac{y}{x}\right) \pm \left(\frac{x}{y} + \frac{y}{x}\right)}{2}$$

$$= \frac{x}{y} \text{ (or) } -\frac{y}{x}$$

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H.W.

(4) i) Solve $p^2 + 2py\cot x = y^2$

ii) If the curve whose D.E. is $p^2 + 2py\cot x = y^2$ passes through $\left(\frac{\pi}{2}, 1\right)$

Equations solvable for y

The general form of the solvable for y is given by

$$y = f(x, p) \quad \text{--- (1)}$$

Diff (1) w.r.to x

$$\frac{dy}{dx} = F\left(x, p, \frac{dp}{dx}\right)$$

$$p = F\left(x, p, \frac{dp}{dx}\right) \quad \text{--- (2)}$$

It is a D.E. of first order in two variables.

The soln of (2) is

$$\phi(x, p, c) = 0 \quad \text{--- (3)}$$

eliminating p from (1) & (3), we will get a relation between x, y and c which is the required soln of the given system.

Note: (1) In some cases, the elimination of p between (1) & (3) is not possible, then (1) and (3) together constitute the solution.

i.e., $x = F_1(p, c)$, $y = F_2(p, c)$

Note: (2) In some cases, eq (2) can be expressed as $f_1(x, p) \cdot f_2\left(x, p, \frac{dp}{dy}\right) = 0$

In such cases, we ignore the first factor $f_1(x, p)$ which doesnot involve $\frac{dp}{dx}$ and proceed with $f_2\left(x, p, \frac{dp}{dx}\right) = 0$

Problems

1. Solve the following D.E.'s

(i) $y=(x-a)p-p^2$

Sol:- Given eqn is $y=(x-a)p-p^2$ — ①

Diff ① w.r.to x

$$\frac{dy}{dx} = (x-a)\frac{dp}{dx} + p - 2p\frac{dp}{dx}$$

$$\cancel{p} = ((x-a)-2p)\frac{dp}{dx} + \cancel{p}$$

$$\frac{dp}{dx}(x-a-2p) = 0$$

$$\Rightarrow \frac{dp}{dx} = 0$$

Integrating we get

$p = c$ — ②

Eliminating p from ① and ②

$$\boxed{y = (x-a)c - c^2}$$

which is the required soln.

(ii) $y = (1+p)x + p^2$ — ①

Diff ① w.r.to x

$$\frac{dy}{dx} = (1+p).1 + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

$$\cancel{p} = 1+\cancel{p}+(x+2p)\frac{dp}{dx}$$

$$\Rightarrow (x+2p)\frac{dp}{dx} = -1$$

$$\frac{dp}{dx} = \frac{-1}{x+2p}$$

$$\frac{dx}{dp} = -x-2p$$

$$\frac{dx}{dp} + x = -2p$$

which is a linear eqn in x.

Here $P=1$, $Q=-2p$

$$I.F = e^{\int p\,dp} = e^p$$

The soln is

$$x \times e^p = \int(-2p)e^p dp$$

$$xe^p = -2e^p(p-1)+c$$

$$x = 2(1-p)+ce^{-p}$$ — ②

sub ② in ①

$$\boxed{y = (1+p)(2(1-p)+ce^{-p}) + p^2}$$

iii) $y = x + a\tan^{-1}p$ — ①

Diff ① w.r.to x

$$\frac{dy}{dx} = 1 + \frac{a}{1+p^2}\frac{dp}{dx}$$

$$\Rightarrow \frac{a}{1+p^2}\frac{dp}{dx} = p-1$$

$$\Rightarrow \frac{dp}{dx} = \frac{(p-1)(1+p^2)}{a}$$

$$\Rightarrow \frac{dx}{a} = \frac{1}{(p-1)(p^2+1)}dp$$

$$\frac{1}{(p-1)(p^2+1)} = \frac{A}{p-1} + \frac{Bp+C}{p^2+1}$$

$A = \frac{1}{2}$ $\quad B = -\frac{1}{2}$ $\quad C = -\frac{1}{2}$

$$\frac{1}{(p-1)(p^2+1)} = \frac{1}{2}\left[\frac{1}{p-1} - \frac{p}{p^2+1} - \frac{1}{p^2+1}\right]$$

$$\Rightarrow dx = \frac{a}{2}\left[\frac{1}{p-1} - \frac{p}{p^2+1} - \frac{1}{p^2+1}\right]dp$$

$$\Rightarrow \int dx = \frac{a}{2}\int\left(\frac{1}{p-1} - \frac{p}{p^2+1} - \frac{1}{p^2+1}\right)dp$$

$$x = \frac{a}{2}\left[\log(p-1) - \frac{1}{2}\log(p^2+1) - \tan^{-1}p + \log c\right]$$

$$x = \frac{a}{2}\left[\log\left[\frac{c(p-1)}{\sqrt{p^2+1}}\right] - \tan^{-1}p\right] \quad \text{— ②}$$

sub ② in ①

$$\boxed{y = \frac{a}{2}\left[\log\left[\frac{c(p-1)}{\sqrt{p^2+1}}\right] - \tan^{-1}p\right] + a\tan^{-1}p}$$

iv) $y = 2px - p^2$ — ①

Diff ① w.r.to x

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$p - 2p = (2x - 2p)\frac{dp}{dx}$$

$$\cancel{2(x-p)\frac{dp}{dx} + p = 0}$$

$$\frac{-p}{2(x-p)} = \frac{dp}{dx}$$

$$\frac{dx}{dp} = \frac{-2(x-p)}{p}$$

$$= -\frac{2}{p}x + \frac{2\cancel{p}}{\cancel{p}}$$

$$\frac{dx}{dp} + \frac{2}{p}x = 2 \quad \text{— ②}$$

which is a linear D.E.

$P(x) = \frac{2}{p}$, $Q(x) = 2$

$$I.F = e^{\int \frac{2}{p}dp} = e^{2\log p} = p^2$$

The soln is

$$x \cdot p^2 = \int 2p^2 \, dp$$

$$xp^2 = \frac{2p^3}{3} + c$$

$$x = \frac{2p^3}{3p^2} + \frac{c}{p^2}$$

$$x = \frac{2p}{3} + \frac{c}{p^2} \quad \text{--- (3)}$$

sub (3) in (1)

$$y = 2p\left[\frac{2p}{3} + \frac{c}{p^2}\right] - p^2$$

$$\boxed{y = \frac{4}{3}p^2 + \frac{2c}{p} - p^2}$$

(2) Solve the following D.E's

i) $y + px = x^4 p^2$

Sol:- Given eqn can be written as

$$y = -px + x^4 p^2 \quad \text{--- (1)}$$

Diff (1) w.r.to x

$$\frac{dy}{dx} = -p - x\frac{dp}{dx} + x^4 2p\frac{dp}{dx} + p^2 4x^3$$

$$p + p + x\frac{dp}{dx} = 2px^3\left(x\frac{dp}{dx} + 2p\right)$$

$$\left(2p + x\frac{dp}{dx}\right) - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0$$

$$\left(2p + x\frac{dp}{dx}\right)\left(1 - 2px^3\right) = 0$$

Neglecting the factor, which does not involve $\frac{dp}{dx}$, we have

$$2p + x\frac{dp}{dx} = 0$$

$$x\frac{dp}{dx} = -2p$$

$$-\frac{1}{2p}\,dp = \frac{1}{x}\,dx$$

~~$-\frac{1}{2}\log p = \log x + \log c = \log cx$~~

$$\frac{1}{p}\,dp = -\frac{2}{x}\,dx$$

$$\log p = -2\log x + \log c = \log\frac{c}{x^2}$$

$$(p = c/x^2) \quad \text{--- (2)}$$

sub ② in ①

$$y + \frac{c}{\cancel{x^4}}x = \cancel{x^4}\cdot\frac{c^2}{\cancel{x^4}}$$

$$\boxed{xy + c = c^2 x}$$

ii) $y = 2px + p^4x^2$ — ①

Diff ① w.r.to x

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + p^4\cdot 2x + x^2 4p^3\frac{dp}{dx}$$

$$p - 2p - 2xp^4 = (2x + 4x^2p^3)\frac{dp}{dx}$$

$$-p - 2xp^4 = 2x(1 + 2xp^3)\frac{dp}{dx}$$

$$-p(1 + 2xp^3) = 2x(1 + 2xp^3)\frac{dp}{dx}$$

$$(1 + 2p^3x)\left(p + 2x\frac{dp}{dx}\right) = 0$$

$$p + 2x\frac{dp}{dx} = 0$$

$$\Rightarrow 2x\frac{dp}{dx} = -p$$

$$\Rightarrow \int\frac{1}{p}dp = -\int\frac{1}{2x}dx$$

$$\log p = -\frac{1}{2}\log x + \log c$$

~~$2\log p$~~ =

$$p = \frac{c}{\sqrt{x}}$$

$$p = \frac{c}{\sqrt{x}} \quad \text{—— ②}$$

sub ② in ①

$$\boxed{y = 2\frac{c}{\sqrt{x}}x + \left(\frac{c}{\sqrt{x}}\right)^4 x^2}$$

③ i) $y = p\tan p + \log\cos p$ where $p = \frac{dy}{dx}$

Sol:- Given. $y = p\tan p + \log\cos p$ —— ①

Diff ① w.r.to x

$$\frac{dy}{dx} = p\sec^2 p\frac{dp}{dx} + \tan p\frac{dp}{dx} + \frac{1}{\cos p}(-\sin p)\frac{dp}{dx}$$

$$\cancel{p} = \cancel{p}\sec^2 p\frac{dp}{dx} + \cancel{\tan p\frac{dp}{dx}} - \cancel{\tan p\frac{dp}{dx}}$$

$$\sec^2 p\frac{dp}{dx} = 1$$

$$\Rightarrow dx = \sec^2 p\, dp$$

$$\int dx = \int \sec^2 p\, dp$$

$$x = \tan p + c \quad —— ②$$

eq ① & ② together form the required soln in parametric form.

ii) $y = p\sin p + \cos p$ —— ①

Diff ① w.r.to x

$$\frac{dy}{dx} = \cancel{p\cos p + \sin p\frac{dp}{dx}}\ p\cos p\frac{dp}{dx} + \cancel{\sin p\frac{dp}{dx}} - \cancel{\sin p\frac{dp}{dx}}$$

$$\cancel{p} = \cancel{p}\cos p\frac{dp}{dx}$$

$$\cos p\frac{dp}{dx} = 1$$

$$\Rightarrow \int \cos p\, dp = \int 1\, dx$$

$$\sin p = x + c$$

$$x = \sin p + c \quad —— ②$$

eq ① & ② together form the required soln in parametric form.

H.W

① $y = \sin p - p\cos p$

② $4y = x^2 + p^2$

③ $yp^2 - 2xp + y = 0$

Equations solvable for x

The general form of the solvable for x is given by

$x = f(y, p)$ —— ①

Diff w.r.to y and write $\frac{dx}{dy}$ with $\frac{1}{p}$

$$\frac{dx}{dy} = F\left(y, p, \frac{dp}{dy}\right)$$

$$\frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right)$$

The soln is

$\phi(y, p, c) = 0$ —— ②

Eliminating p between ① & ②, we get the required soln of ①

i.e.; $\phi(x, y, c) = 0$

Note:- ① If the elimination of p between ① & ② is not possible, then we solve ① & ② to express x and y in terms of p and c.

i.e.; $x = f_1(p, c)$, $y = f_2(p, c)$ —— ③

These two equations together will be the soln of ①

Note: ②:- Sometime the soln ③ is also not possible, in those case ① and ② together constitute the soln giving x and y in terms of p.

Note: ③:- In some problems eq $F\left(y, p, \frac{dp}{dy}\right)$ can be expressed as

$$F_1(y, p)\, F_2\left(y, p, \frac{dp}{dy}\right) = 0$$

In such case we ignore the first factor $F_1(y, p)$ which does not involve $\frac{dp}{dy}$ and proceed with $F_2\left(y, p, \frac{dp}{dy}\right) = 0$

Note: ④:- If instead of ignoring the factor $F_1(y, p)$, we eliminate p between ① and $F_1(y, p) = 0$ we obtain an equation involving no constant 'c'. This is known as singular soln of ①

problems

(1) Solve the following D.E's

i) $x = 3y - \log p$

Sol:- Given $x = 3y - \log p$ —— (1)

Diff (1) w.r.to y and write $\frac{dx}{dy}$ with $\frac{1}{p}$

$$\frac{dx}{dy} = 3 - \frac{1}{p}\frac{dp}{dy}$$

$$\frac{1}{p} - 3 = -\frac{1}{p}\frac{dp}{dy}$$

$$\frac{1-3p}{\cancel{p}} = -\frac{1}{\cancel{p}}\frac{dp}{dy}$$

$$\Rightarrow dy = \frac{1}{3p-1}dp$$

$$\int dy = \int \frac{1}{3p-1}dp$$

$$y = \frac{1}{3}\log(3p-1) + \log c$$

$$y = \log(3p-1)^{1/3}c.$$

$$e^y = c(3p-1)^{1/3}$$

$$(3p-1)^{1/3} = ce^{y}$$

$$3p-1 = ce^{3y}$$

$$p = \frac{1+ce^{3y}}{3} \text{ —— (2)}$$

eliminating p from (1) & (2)

$$\boxed{x = 3y - \log\left(\frac{1+ce^{3y}}{3}\right)}$$

which is the required soln.

iii) $x = 4(p+p^3)$ — ①

Diff ① w.r.to y and write $\frac{dx}{dy}$ with $\frac{1}{p}$

$$\frac{dx}{dy} = 4\left(\frac{dp}{dy} + 3p^2\frac{dp}{dy}\right)$$

$$\frac{1}{p} = 4\left(\frac{dp}{dy} + 3p^2\frac{dp}{dy}\right)$$

$$\frac{1}{p} = 4(1+3p^2)\frac{dp}{dy}$$

$$dy = 4p(1+3p^2)\,dp$$

$$\int dy = \int 4p\,dp + 12\int p^3\,dp$$

$$y = \frac{\cancel{4}^{2}p^2}{\cancel{2}} + \frac{\cancel{12}^{3}p^4}{\cancel{4}} + c$$

$$\therefore y = 2p^2 + 3p^4 + c \quad \text{—— ②}$$

Both ① & ② constitutes the soln of ①.

Hw

iv) $p^3 - 4xyp + 8y^2 = 0$

② Solve the following D.E's

i) $y = 2px + p^3y^2$

Given eqn can be written as

$$2px = y - p^3y^2$$

$$2x = \frac{y}{p} - p^2y^2 \quad \text{—— ①}$$

Diff ① w.r.to y and write $\frac{dx}{dy}$ with $\frac{1}{p}$

$$2\frac{dx}{dy} = \frac{p\cdot 1 - y\frac{dp}{dy}}{p^2} - \left(p^2 2y + y^2 2p\frac{dp}{dy}\right)$$

$$\frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2p^2y - 2y^2p\frac{dp}{dy}$$

$$\frac{2}{p}-\frac{1}{p}+2p^2y = -\frac{dp}{dy}\left(\frac{y}{p^2}+2y^2p\right)$$

$$\left(\frac{1}{p}+2p^2y\right)+\frac{y}{p}\left(\frac{1}{p}+2p^2y\right)\frac{dp}{dy} = 0$$

$$\left(\frac{1}{p}+2p^2y\right)\left(1+\frac{y}{p}\frac{dp}{dy}\right) = 0$$

We can neglect first factor, because it is not involving $\frac{dp}{dy}$, so

$$1+\frac{y}{p}\frac{dp}{dy} = 0$$

$$\frac{y}{p}\frac{dp}{dy} = -1$$

$$\frac{dp}{dy} = \frac{-p}{y}$$

$$\int\frac{dp}{p} = -\int\frac{1}{y}\,dy$$

$$\log p = -\log y+\log c$$

$$\log p = \log\frac{c}{y}$$

$$p = \frac{c}{y} \quad \text{--- (2)}$$

Sub (2) in given eqn

$$y = 2\left(\frac{c}{y}\right)x+\left(\frac{c^3}{y^3}\right)y^2$$

$$y = \frac{2cx}{y}+\frac{c^3}{y}$$

$$\boxed{y^2 = 2cx+c^3}$$

ii) $$x = \frac{y\log y}{p}-\frac{p}{y} \quad \text{--- (1)}$$

Diff (1) w.r.to y and write $\frac{dx}{dy}$ with $\frac{1}{p}$

$$\frac{dx}{dy} = \frac{p\left(y\cdot\frac{1}{y}+\log y\cdot 1\right)-y\log y\frac{dp}{dy}}{p^2}-\frac{y\frac{dp}{dy}-p\cdot 1}{y^2}$$

$$= \frac{p(1+\log y) - y\log y \frac{dp}{dy}}{p^2} - \frac{y\frac{dp}{dy} - p}{y^2}$$

$$\frac{1}{p} = \frac{1}{p}(1+\log y) - \frac{y}{p^2}\log y \frac{dp}{dy} - \frac{1}{y}\frac{dp}{dy} + \frac{p}{y^2}$$

$$\cancel{\frac{1}{p}} = \cancel{\frac{1}{p}} + \frac{1}{p}\log y - \frac{y}{p^2}\log y \frac{dp}{dy} - \frac{1}{y}\frac{dp}{dy} + \frac{p}{y^2}$$

$$\Rightarrow \frac{1}{p}\log y + \frac{p}{y^2} = \left(\frac{y}{p^2}\log y + \frac{1}{y}\right)\frac{dp}{dy}$$

$$\Rightarrow \frac{p}{y^{\cancel{2}}}\left(\frac{y^2}{p^2}\log y + 1\right) = \cancel{\frac{1}{y}}\left(\frac{y^2}{p^2}\log y + 1\right)\frac{dp}{dy}$$

$$\left(\frac{y^2}{p^2}\log y + 1\right)\left(\frac{p}{y} - (1)\frac{dp}{dy}\right) = 0$$

Neglecting first factor

$$\frac{p}{y} - \frac{dp}{dy} = 0$$

$$\Rightarrow \frac{p}{y} = \frac{dp}{dy}$$

$$\Rightarrow \int \frac{dp}{p} = \int \frac{1}{y}dy$$

$$\Rightarrow \log p = \log y + \log c$$

$$p = cy \quad \text{—} ②$$

sub ② in ①

$$x = \frac{y\log y}{cy} - \frac{cy}{y}$$

$$cx = \log y - c^2$$

$$\boxed{\log y = cx + c^2}$$

iii) $p = \tan\left(x - \frac{p}{1+p^2}\right)$

H.W

Clairaut's equation

An equation of the form $y = px + f(p)$ is known as Clairaut's form.
The general soln is obtained by replacing "p" with "c".

procedure for singular soln

→ Given eqn is of the form $y = px + f(p)$ —— ①

→ The general soln of clairaut's equation is obtained by replacing p with c

ie; $y = cx + f(c)$ —— ①

→ Diff ① w.r.to c

$x + f'(c) = 0$ —— ②

Diff ① w.r.to x $\frac{dy}{dx} = p.1 + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$

$p = p + \frac{dp}{dx}(x + f'(p))$

$\Rightarrow \frac{dp}{dx}(x + f'(p)) = 0$

$\Rightarrow \frac{dp}{dx} = 0 \Rightarrow x = c \quad x + f'(p) = 0$

Eliminate 'c' from ① & ② which will gives the singular soln.

problems

① Solve the following D.E's

i) $y = px + p^n$ —— ①

~~The general~~ eq ① is the clairaut's equation.
The general soln is obtained replacing 'p' with c

$y = cx + c^n$

ii) $xp^2 - yp + 2 = 0$

Given eqn can be written as

$yp = xp^2 + 2$

$y = \frac{xp^2}{p} + \frac{2}{p}$

$y = xp + \frac{2}{p}$

soln is

$\boxed{y = xc + \frac{2}{c}}$

iii) $y = px + \log p$

$y = cx + \log c$

iv) $p = \log(px - y)$

$e^p = ~~\log~~\ px - y$

$y = px - e^p$

$\boxed{y = cx - e^c}$

v) $p = \tan(px - y)$

$px - y = \tan^{-1}p$

$y = px - \tan^{-1}p$

$\boxed{y = cx - \tan^{-1}c}$

vi) $(y - px)(p - 1) = p$

$y - px = \frac{p}{1-p}$

$y = px + \frac{p}{1-p}$

$\boxed{y = cx + \frac{c}{1-c}}$

vii) $\cos y \cos px + \sin y \sin px = p$

$\cos(y - px) = p$

$y - px = \cos^{-1}p$

$y = px + \cos^{-1}p$

$\boxed{y = cx + \cos^{-1}c}$

viii) $\sin px \cos y = \cos px \sin y + p$

HW

② Solve $y = px + p^2$ — ① and obtain singular soln

eq ① is a clairaut's form

soln of ① is

$y = cx + c^2$ — ②

Diff ① w.r.to x

$\frac{dy}{dx} = p \cdot 1 + x\frac{dp}{dx} + 2p\frac{dp}{dx}$

$p = p + x\frac{dp}{dx} + 2p\frac{dp}{dx}$

$\Rightarrow x\frac{dp}{dx} + 2p\frac{dp}{dx} = 0$

$\Rightarrow (x + 2p)\frac{dp}{dx} = 0$

Taking $x + 2p = 0$ & $\frac{dp}{dx} = 0$

$p = -\frac{x}{2}$ $\qquad$ $p = c$

substituting $p = -\frac{x}{2}$ in ①

$y = -\frac{x}{2} \cdot x + \left(-\frac{x}{2}\right)^2$

$y = -\frac{x^2}{2} + \frac{x^2}{4} = -\frac{x^2}{4}$

$\boxed{x^2 + 4y = 0}$

which is the required singular soln.

③ Find the general and singular of

i) $\sin(px - y) = p$

$px - y = \sin^{-1} p$

$y = px - \sin^{-1} p$ — ①

Diff ① w.r.to x

$\frac{dy}{dx} = p \cdot 1 + x\frac{dp}{dx} - \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$

$p = p + \left(x - \frac{1}{\sqrt{1-p^2}}\right)\frac{dp}{dx} = 0$

$\Rightarrow \left(x - \frac{1}{\sqrt{1-p^2}}\right)\frac{dp}{dx} = 0$

$\Rightarrow x - \frac{1}{\sqrt{1-p^2}} = 0$ & $\frac{dp}{dx} = 0$

$x = \frac{1}{\sqrt{1-p^2}}$

$\sqrt{1-p^2} = \frac{1}{x}$

$1 - p^2 = \frac{1}{x^2}$

$p^2 = 1 - \frac{1}{x^2}$

$p = \sqrt{1 - \frac{1}{x^2}}$

substituting p in ①

$y = \sqrt{1 - \frac{1}{x^2}} \cdot x - \sin^{-1}\left(\sqrt{1 - \frac{1}{x^2}}\right)$

$= \frac{\sqrt{x^2-1}}{x} \cdot x - \sin^{-1}\left(\sqrt{1 - \frac{1}{x^2}}\right)$

$\boxed{y = \sqrt{x^2-1} - \sin^{-1}\left(\sqrt{1 - \frac{1}{x^2}}\right)}$

which is required singular soln.

$\frac{dp}{dx} = 0$

$p = c$

sub $p = c$ in ①

$y = cx + c^2$

which is general soln

H.W. ii) $y = px - \sqrt{1+p^2}$

H.W. iii) $y = px + p - p^2$

UNIT-II

ODE of Higher order

* Second order Linear D.E. with constant coefficients:
  Non homogeneous terms of the type e^{ax}, $\sin ax$, $\cos ax$, polynomials in x, $e^{ax}V(x)$ and $xV(x)$,
* method of variation of parameters.
* Equations reducible to linear ODE with constant coefficients:
  Legendre's equation, Cauchy-Euler equation,

Second order Linear D.E. with constant coefficients

The general form of second order Linear D.E. is

$$f(D)y = Q(x) \quad \text{——} \quad (1)$$

where $f(D)$ be the D.E. of second order

If $Q(x) = 0$ then $f(D)y = 0$ is called homogeneous Linear D.E.

If $Q(x) \neq 0$ then $f(D)y = Q(x)$ is called non homogeneous Linear D.E.

The complete solution of eq (1) is

$$y = C.F. + P.I.$$

where C.F = Complementary Function
P.I. = Perticular Integral.

(OR)

$$y = y_c + y_p$$

y_c = complementary soln
y_p = perticular soln

To find the Complementary Function

$f(D)y = 0$ be the given equation — (1)

Auxillary Equation of (1) is

$f(m) = 0$

Case (i)

If the roots are real and distinct

ie; $m = a, b, c, d$ then

Complementary function is

$$y_c = c_1 e^{ax} + c_2 e^{bx} + c_3 e^{cx} + c_4 e^{dx}$$

Case (ii)

If the roots are real and repeated.

ie; $m = a, a, b, c$ then

Complementary function is

$$y_c = (c_1 + c_2 x) e^{ax} + c_3 e^{bx} + c_4 e^{cx}$$

Case (iii)

If the roots are real and repeated

ie; $m = a, a, a, b$ then

Complementary function is

$$y_c = (c_1 + c_2 x + c_3 x^2) e^{ax} + c_4 e^{bx}$$

Case (iv)

If the roots are imaginary

ie; $m = \pm bi, \pm di$ then

$$y_c = c_1 \cos bx + c_2 \sin bx + c_3 \cos dx + c_4 \sin dx$$

Case (v) If the roots are complex

ie; $m = a \pm bi, \; c \pm di$ then

$y_c = e^{ax}(c_1\cos bx + c_2\sin bx) + e^{cx}(c_3\cos dx + c_4\sin dx)$ (2)

Case (vi)

If the roots are complex & repeated

$m = a \pm bi,\ a \pm bi$ then

$y_c = e^{ax}((c_1 + c_2 x)\cos bx + (c_3 + c_4 x)\sin bx)$

problems

(1) Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

The given eqn can be written as

$(D^2 + 5D + 6)y = 0$

which is of the form $f(D)y = 0$

A.E. is $f(m) = 0$

$\Rightarrow m^2 + 5m + 6 = 0$

$\Rightarrow (m+2)(m+3) = 0$

$\Rightarrow m = -2, -3$

Here the roots are real and distinct.

The solution is

$$\boxed{y_c = c_1 e^{-2x} + c_2 e^{-3x}}$$

(2) Solve $(D^2 + 6D + 9)y = 0$

A.E is $f(m) = 0$

$\Rightarrow m^2 + 6m + 9 = 0$

$\Rightarrow (m+3)(m+3) = 0$

$\Rightarrow m = -3, -3$

Here the roots are repeated

The solution is $\boxed{y_c = (c_1 + c_2 x)e^{-3x}}$

③ Solve $(D^2+4)y=0$

A.E. is $f(m)=0$

$\Rightarrow m^2+4=0$

$\Rightarrow m=\pm 2i$

Roots are Imaginary

The soln is

$$\boxed{y_c = c_1\cos 2x + c_2\sin 2x}$$

④ Solve $(D^2+2D+2)y=0$

A.E. is $f(m)=0$

$\Rightarrow m^2+2m+2=0$

$\Rightarrow m = \frac{-2\pm 2i}{2}$

$\Rightarrow m=-1\pm i$

roots are imaginary & complex

$$\boxed{y_c = e^{-x}(c_1\cos x + c_2\sin x)}$$

⑤ Solve $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$

Sol:- Given eqn can be written as

$(D^2+D+1)y=0$

A.E. is $f(m)=0$

$\Rightarrow m^2+m+1=0$

$\Rightarrow m = \frac{-1\pm\sqrt{1-4}}{2}$

$m = \frac{-1\pm i\sqrt{3}}{2}$

$\Rightarrow m = \frac{-1}{2}\pm\frac{\sqrt{3}}{2}i$

The roots are complex and conjugate.

The general soln is

$$y = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

6) Solve $y'' + y' - 2y = 0, \quad y(0) = 4, \quad y'(0) = 1$

Sol:- Given eqn can be written as $(D^2 + D - 2)y = 0$

A.E is $f(m) = 0$

$\Rightarrow m^2 + m - 2 = 0$

$\Rightarrow (m-1)(m+2) = 0$

$\Rightarrow m = 1, -2$

The soln is $y = c_1 e^{x} + c_2 e^{-2x}$ —— (1)

Given $y(0) = 4$

$\Rightarrow y(0) = c_1 e^{0} + c_2 e^{-2(0)} = 4$

$\Rightarrow c_1 + c_2 = 4$ —— (2)

Diff (1) w.r.to x

$y' = c_1 e^{x} - 2c_2 e^{-2x}$

$y'(0) = c_1 e^{0} - 2c_2 e^{-2(0)} = 1$

$\Rightarrow c_1 - 2c_2 = 1$ —— (3)

From (2) & (3)

$\Rightarrow c_1 = 3, \ c_2 = 1$

The general soln is

$$\boxed{y = 3e^{x} + 1 \cdot e^{-2x}}$$

H.W

7) Solve $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$ $\quad m = -1, -1, 2, 2$

$y = (c_1 + c_2 x)e^{-x} + (c_3 + c_4 x)e^{2x}$

H.W

8) Solve $(D^4 + 8D^2 + 16)y = 0$

$m = 2i, 2i, -2i, -2i$

$y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$

To find the soln of $f(D)y = Q(x)$

Inverse operator

The operator D^{-1} is called the inverse of the differential operator D.

$f(D)y = Q(x)$

P.I. of $y = \frac{1}{f(D)} Q(x)$

If $f(D) = D - \alpha$

$$y_p = \frac{1}{D-\alpha} Q(x) = e^{\alpha x}\int Q(x)\, e^{-\alpha x}\, dx$$

If $f(D) = D + \alpha$

$$y_p = \frac{1}{D+\alpha} Q(x) = e^{-\alpha x}\int Q(x)\, e^{\alpha x}\, dx$$

problems

① Find i) $\frac{1}{D} x^2$ ii) $\frac{1}{D^3} \cos x$

i) $$\frac{1}{D} x^2 = \frac{1}{(D-0)} x^2 = e^{-0.x}\int x^2 e^{0.x}\, dx = \int x^2 dx = \frac{x^3}{3}$$

ii) $$\frac{1}{D^3}\cos x = \frac{1}{D^2}\left(\frac{1}{D}\cos x\right) = \frac{1}{D^2}(\sin x) = \frac{1}{D}\left(\frac{1}{D}\sin x\right) = \frac{1}{D}(-\cos x)$$

$$= -\frac{1}{D}(\cos x)$$

$$= -\sin x$$

② Solve $(D^2 - 5D + 6)y = xe^{4x}$

Given equ is of the form $f(D)y = Q(x)$

A.E. is $f(m) = 0$

$\Rightarrow m^2 - 5m + 6 = 0$

$\Rightarrow m = 2, 3$

$y_c = c_1 e^{2x} + c_2 e^{3x}$

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{(D^2-5D+6)} xe^{4x}$$

$$= \frac{1}{(D-2)(D-3)} xe^{4x}$$

$$= \left[\frac{1}{D-3} - \frac{1}{D-2}\right] xe^{4x}$$

$$= \frac{1}{D-3} xe^{4x} - \frac{1}{D-2} xe^{4x}$$

$$= e^{3x}\int xe^{4x}.e^{-3x}dx - e^{2x}\int xe^{4x}.e^{-2x}dx$$

$$= e^{3x}\int xe^{x}dx - e^{2x}\int xe^{2x}dx$$

$$= e^{3x}\left[xe^{x} - e^{x}\right] - e^{2x}\left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right]$$

$$y_p = e^{4x}\frac{(2x-3)}{4}$$

The general soln is $y = y_c + y_p$

$$\boxed{y = c_1e^{2x} + c_2e^{3x} + e^{4x}\frac{(2x-3)}{4}}$$

Hw
(3) Solve $(D^2+4)y = \tan 2x$

Hw
(4) Solve $(D^2+5D+6)y = e^x$

P.I. of $f(D)y = Q(x)$ when $Q(x) = e^{ax}$

$$f(D)y = e^{ax}$$

$$y_p = \frac{1}{f(D)} e^{ax}$$

put $D = a$

$$y_p = \frac{1}{f(a)} e^{ax} \quad \text{if } f(a) \neq 0$$

If $f(D) = (D-a)^k$

$$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^k} e^{ax} = \frac{e^{ax} x^k}{k!} \quad \text{if } f(a) = 0$$

problems

① Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{2x}$

Sol:- Given eqn can be written as

$$(D^2 + 4D + 3)y = e^{2x} \text{ — ①}$$

A.E. of ① is $f(m) = 0$

$$\Rightarrow m^2 + 4m + 3 = 0$$

$$\Rightarrow (m+1)(m+3) = 0$$

$$\Rightarrow m = -1, -3$$

Complementary soln is

$$\boxed{y_c = c_1 e^{-x} + c_2 e^{-3x}}$$

To find y_p

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{(D^2+4D+3)} e^{2x}$$

put $D = 2$

$$= \frac{1}{4+8+3} e^{2x}$$

$$= \frac{e^{2x}}{15}$$

$\therefore$ The general soln is

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-3x} + \frac{e^{2x}}{15}}$$

② Solve $(D^2+2D+1)y = e^{-x}$ —— ①

Sol: A.E. of ① is $f(m) = 0$

$$\Rightarrow m^2 + 2m + 1 = 0$$

$$\Rightarrow (m+1)^2 = 0$$

$$\Rightarrow m = -1, -1$$

$$y_c = (c_1 + c_2 x) e^{-x}$$

To find y_p

$$y_p = \frac{1}{D^2+2D+1} e^{-x}$$

$$= \frac{1}{(D+1)^2} e^{-x}$$

$f(D) = (D+1)^2$

$f(-1) = (-1+1)^2 = 0$

Here $f(D) = 0$

Since $f(a) = 0$

$$y_p = e^{-x} \cdot \frac{x^2}{2!}$$

$\therefore$ The general soln is $y = y_c + y_p$

$$\boxed{y = (c_1 + c_2 x) e^{-x} + \frac{x^2 e^{-x}}{2}}$$

③ Solve $(D^2+6D+9)y = 2e^{-3x}$ — ①

Sol: A.E. of ① is $f(m)=0$

$\Rightarrow m^2+6m+9=0$

$\Rightarrow (m+3)^2=0$

$\Rightarrow m=-3,-3.$

$y_c = (c_1+c_2x)e^{-3x}$

To find y_p

$$y_p = \frac{1}{D^2+6D+9} 2e^{-3x}$$

$$= 2\left[\frac{e^{-3x}}{(D+3)^2}\right]$$

Since $f(-3)=0$

$$y_p = \cancel{2}\, e^{-3x}\cdot \frac{x^2}{\cancel{2!}}$$

$$y_p = x^2 e^{-3x}$$

∴ The general soln is

$$y = y_c + y_p$$

$$\boxed{y = (c_1+c_2x)e^{-3x} + x^2e^{-3x}}$$

④ Solve $(D^3-1)y = (e^x+1)^2$ — ①

A.E. of ① is $f(m)=0$

$\Rightarrow m^3-1=0$

$\Rightarrow m = 1, \frac{-1\pm i\sqrt{3}}{2}$

$$y_c = c_1e^x + e^{-\frac{x}{2}}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$$

$$y_p = \frac{1}{D^3-1}(e^x+1)^2$$

$$= \frac{1}{D^3-1}\left(e^{2x}+2e^{x}+1\right)$$

$$= \frac{e^{2x}}{D^3-1} + \frac{2e^{x}}{D^3-1} + \frac{e^{0\cdot x}}{D^3-1}$$

$$= \frac{e^{2x}}{8-1} + \frac{2e^{x}}{(D-1)(D^2+D+1)} + \frac{1}{-1}$$

$$= \frac{e^{2x}}{7} - 1 + \frac{2}{3}\cdot\frac{1}{D-1}e^{x}$$

$$y_p = \frac{e^{2x}}{7} - \cancel{1} + \frac{2}{3}\frac{xe^{x}}{1!}$$

The general soln is

$$y = y_c + y_p$$

$$y = c_1e^{x} + e^{-\frac{x}{2}}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right] + \frac{e^{2x}}{7} - 1 + \frac{2}{3}xe^{x}$$

Hw
5. Solve $(D^3-3D^2+4)y = (1+e^{-x})^3$

Hw
6. Solve $(D^3+3D^2-4)y = \sinh 2x + 7$

To find $f(D)y = Q(x)$ when $Q(x) = \sin ax$ (or) $\cos ax$

$$P.I. = y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{f(D)} \sin ax$$

$$= \frac{1}{\phi(D^2)} \sin ax \qquad (\because \text{Convert } f(D) \text{ into } \phi(D^2))$$

put $D^2 = -a^2$

$$= \frac{1}{\phi(-a^2)} \sin ax \qquad (\because \phi(-a^2) \neq 0)$$

If $\phi(-a^2) = 0$ then

$$y_p = \frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax$$

~~and~~ (or)

$$y_p = \frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax$$

problems

(1) Solve $(D^2+3D+2)y = \sin 3x$ — (1)

Sol:- Comparing with $f(D)y = Q(x)$

$f(D) = D^2+3D+2 \qquad Q(x) = \sin 3x$

A.E. of (1) is

$f(m) = 0$

$\Rightarrow m^2+3m+2 = 0$

$\Rightarrow m = -1, -2$

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

$$y_p = \frac{1}{f(D)} \sin 3x$$

$= \frac{1}{D^2+3D+2} \sin 3x$ put $D^2 = -3^2 = -9$

$= \frac{1}{-9+3D+2} \sin 3x$

$= \frac{1}{3D-7} \sin 3x$

$= \frac{(3D+7)}{9D^2-49} \sin 3x$

$= \frac{(3D+7)}{-81-49} \sin 3x$

$y_p = \frac{-1}{130}\left[9\cos 3x + 7\sin 3x\right]$

The general soln is

$y = y_c + y_p$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{130}\left[9\cos 3x + 7\sin 3x\right]}$$

② Solve $(D^2-4)y = 2\cos^2 x$ — ①

Sol:- $f(D)y = Q(x)$

A.E. of ① is $f(m) = 0 \Rightarrow m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

$y_c = c_1 e^{-2x} + c_2 e^{2x}$

To find y_p

$y_p = \frac{1}{f(D)} Q(x)$

$= \frac{1}{D^2-4}(1+\cos 2x)$

$= \frac{e^{0x}}{D^2-4} + \frac{\cos 2x}{D^2-4}$

$$= \frac{e^{0.x}}{0-4} + \frac{\cos 2x}{-4-4}$$

$$= -\frac{1}{4} - \frac{1}{8}\cos 2x$$

$\therefore$ The general soln is

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{-2x} + c_2 e^{2x} - \frac{1}{4} - \frac{\cos 2x}{8}}$$

③ Solve $(D^2+1)y = \sin x \sin 2x \quad \text{—①}$

A.E of ① is $f(m) = 0$

$$m^2+1 = 0 \Rightarrow m = \pm i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

To find y_p

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2+1} \sin x \sin 2x$$

$$= \frac{1}{2}\frac{1}{D^2+1} 2\sin x \sin 2x$$

$$= \frac{1}{2}\frac{\cos x - \cos 3x}{D^2+1}$$

$$= \frac{1}{2}\frac{x}{2}\sin x - \frac{1}{2}\frac{\cos 3x}{-9+1}$$

$$y_p = \frac{x}{4}\sin x + \frac{1}{16}\cos 3x$$

The general soln is

$$y = y_c + y_p$$

$$\boxed{y = c_1 \cos x + c_2 \sin x + \frac{x}{4}\sin x + \frac{1}{16}\cos 3x}$$

Solve $(D^2+9)y = \cos 3x + \sin 2x$ — (1)

A.E. of (1) is $f(m) = 0$

$\Rightarrow m^2+9=0$

$\Rightarrow m = \pm 3i$

$y_c = c_1 \cos 3x + c_2 \sin 3x$

To find y_p

$$y_p = \frac{1}{D^2+9}(\cos 3x + \sin 2x)$$

$$= \frac{\cos 3x}{D^2+9} + \frac{\sin 2x}{D^2+9}$$

$$= \frac{x}{2(3)} \sin 3x + \frac{\sin 2x}{-4+9}$$

$$y_p = \frac{x}{6}\sin 3x + \frac{1}{5}\sin 2x$$

The general soln is

$$y = y_c + y_p$$

$$= (c_1 \cos 3x + c_2 \sin 3x) + \frac{x}{6}\sin 3x + \frac{1}{5}\sin 2x$$

HW

(5) Solve $(D^4 - 2D^3 + 2D^2 - 2D + 1)y = \cos x$

HW

(6) Solve $y'' + 4y' + 20y = 23\sin t - 15\cos t$, $y(0) = 0$, $y'(0) = -1$.

To find P.I. of $f(D)y = Q(x)$ where $Q(x) = x^k$

$$y_p = \frac{1}{f(D)} x^K$$

Now write $\frac{1}{f(D)}$ as $[1 \pm \phi(D)]^{-1}$ and expand in ascending powers of D using Binomial theorem upto the term containing D^K.

Important formulae's

$$\frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$$

$$\frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$$

$$\frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$$

$$\frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$$

$$\frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \cdots$$

$$\frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \cdots$$

problems

① Solve $(D^3 - 3D - 2)y = x^2$ —— ①

Sol:- A.E. of ① is $f(m) = 0$

$\Rightarrow m^3 - 3m - 2 = 0$

$\Rightarrow m = 1, 1, 2$

$$y_c = (c_1 + c_2 x)e^x + c_3 e^{2x}$$

$$y_p = \frac{1}{D^3 - 3D - 2} x^2$$

$$= \frac{1}{-2\left[1 - \left(\frac{D^3 - 3D}{2}\right)\right]} x^2$$

$$= -\frac{1}{2}\left[1 + \frac{D^3-3D}{2} + \left(\frac{D^3-3D}{2}\right)^2 + \cdots\right]x^2$$

power

$$y_p = -\frac{1}{2}\left[x^2 - 3x + 18\right]$$

$$\therefore y = y_c + y_p$$

$$y = (c_1 + c_2 x)e^x + c_3 e^{2x} - \frac{1}{2}(x^2 - 3x + 18)$$

2) Solve $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x + x$ — (1)

Sol:- A.E. of (1) is $f(m) = 0$

$$\Rightarrow m^3 - 3m^2 + 4m - 2 = 0$$

$$\Rightarrow m = 1,\ 1 \pm i$$

$$y_c = c_1 e^x + e^x(c_2\cos x + c_3 \sin x)$$

$$y_p = \frac{1}{D^3 - 3D^2 + 4D - 2}(e^x + \cos x + x)$$

$$= \frac{e^x}{D^3 - 3D^2 + 4D - 2} + \frac{\cos x}{D^3 - 3D^2 + 4D - 2} + \frac{x}{D^3 - 3D^2 + 4D - 2}$$

$$= \frac{1}{(D-1)(D^2-2D+2)}e^x + \frac{\cos x}{-D + 3 + 4D - 2} + \frac{1}{-2\left(1 - \frac{D^3 - 3D^2 + 4D}{2}\right)}x$$

$$= \frac{e^x}{(1-2+2)(D-1)} + \frac{\cos x}{3D+1} - \frac{1}{2}\left[1 - \left(\frac{D^3 - 3D^2 + 4D}{2}\right)\right]^{-1}$$

$$= xe^x + \frac{(3D-1)}{9D^2-1}\cos x - \frac{1}{2}\left[1 + \frac{D^3 - 3D^2 + 4D}{2} + \cdots\right]x$$

$$= xe^x + \frac{1}{10}(-3\sin x - \cos x) - \frac{1}{2}(x+2)$$

$$= xe^x + \frac{1}{10}(3\sin x + \cos x) - \frac{1}{2}(x+2)$$

$$\therefore y = y_c + y_p$$

H.W. ③ Solve $(D^3+3D^2+2)y = 2\cos(2x+3)+2e^x+x^2$

H.W. ④ Solve $(D^2-4D+4)y = 8x^2+e^{2x}$

H.W. ⑤ Solve $(D^3-3D^2-10D+24)y = x+3$

To find P.I. of $f(D)y = Q(x)$ when $Q(x) = e^{ax}v(x)$

$f(D)y = Q(x)$

P.I. of y is $y_p = \frac{1}{f(D)}Q(x)$

$= \frac{1}{f(D)}e^{ax}v(x)$

$= e^{ax}\frac{1}{f(D+a)}v(x)$

problems

① Solve $(D^2+2)y = e^x\cos x$ — ①

A.E. of ① is $f(m)=0$

$\Rightarrow m^2+2=0$

$\Rightarrow m = \pm\sqrt{2}i$

$y_c = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x$

To find y_p

$y_p = \frac{1}{D^2+2}e^x\cos x$

$= e^x\frac{1}{(D+1)^2+2}\cos x$

$= e^x\frac{1}{D^2+2D+3}\cos x$

$= e^x\frac{1}{-1+2D+3}\cos x$

$= e^x\frac{1}{2D+2}\cos x$

$= \frac{e^x}{2}\frac{(D-1)}{(D+1)(D-1)}\cos x$

$= \frac{e^x}{2}\frac{(D-1)}{D^2-1}\cos x$

$= \frac{e^x}{2}\frac{(D-1)\cos x}{-2}$

$= -\frac{e^x}{4}[-\sin x-\cos x]$

$= \frac{e^x}{4}[\sin x+\cos x]$

$\therefore y = y_c + y_p$

$$y = (c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + \frac{e^x}{4}[\sin x + \cos x]$$

(2) Solve $(D^3 - 3D^2 + 3D - 1)y = x^2 e^x$ — (1)

A.E. of (1) is $f(m) = 0$

$\Rightarrow m^3 - 3m^2 + 3m - 1 = 0$

$\Rightarrow m = 1, 1, 1$

$y_c = (c_1 + c_2 x + c_3 x^2)e^x$

To find y_p

$$y_p = \frac{1}{(D^3 - 3D^2 + 3D - 1)} x^2 e^x$$

$$= \frac{1}{(D-1)^3} x^2 e^x$$

$$= e^x \frac{1}{(D+1-1)^3} x^2$$

$$= e^x \frac{1}{D^3} x^2$$

$$= e^x \frac{1}{D^2}\left[\frac{1}{D} x^2\right]$$

$$= e^x \cdot \frac{1}{D^2} \frac{x^3}{3}$$

$$= \frac{e^x}{3} \frac{1}{D}\left(\frac{1}{D} x^3\right)$$

$$= \frac{e^x}{3} \frac{1}{D}\left(\frac{x^4}{4}\right)$$

$$= \frac{e^x}{12} \frac{x^5}{5}$$

$$= \frac{e^x x^5}{60}$$

$\therefore y = y_c + y_p$

$$= (c_1 + c_2 x + c_3 x^2)e^x + \frac{e^x x^5}{60}$$

(3) Solve $\frac{d^2y}{dx^2} + y = e^{-x} + x^3 + e^x \sin x$

Sol:-

$$(D^2+1)y = e^{-x} + x^3 + e^x \sin x \quad \text{—— (1)}$$

A.E. of (1) is $f(m) = 0$

$$\Rightarrow m^2 + 1 = 0$$

$$\Rightarrow m = \pm i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

To find y_p

$$y_p = \frac{1}{D^2+1}\left[e^{-x} + x^3 + e^x \sin x\right]$$

$$= \frac{e^{-x}}{D^2+1} + \frac{x^3}{D^2+1} + \frac{e^x \sin x}{D^2+1}$$

$$= \frac{e^{-x}}{1^2+1} + (1+D^2)^{-1} x^3 + e^x \left[\frac{1}{(D+1)^2+1}\right] \sin x$$

$$= \frac{e^{-x}}{2} + (1 - D^2 + D^4 - D^6 + \cdots)x^3 + e^x \left[\frac{1}{D^2+2D+2}\right] \sin x$$

$$= \frac{e^{-x}}{2} + x^3 - 6x + e^x \left[\frac{1}{-1+2D+2}\right] \sin x$$

$$= \frac{e^{-x}}{2} + x^3 - 6x + e^x \left(\frac{1}{2D+1} \sin x\right)$$

$$= \frac{e^{-x}}{2} + x^3 - 6x + e^x \left[\frac{(2D-1)}{4D^2-1} \sin x\right]$$

$$= \frac{e^{-x}}{2} + x^3 - 6x + e^x \left[\frac{(2D-1)\sin x}{-4-1}\right]$$

$$= \frac{e^{-x}}{2} + x^3 - 6x - \frac{e^x}{5}\left[2\cos x - \sin x\right]$$

$$\therefore y = y_c + y_p$$

$$y = c_1 \cos x + c_2 \sin x + \frac{e^{-x}}{2} + x^3 - 6x - \frac{e^x}{5}\left[2\cos x - \sin x\right]$$

④ Solve $(D^3-4D^2-D+4)y = e^{3x}\cos 2x$
HW

⑤ Solve $(D^3-7D^2+14D-8)y = e^x\cos 2x$
HW

HW
⑥ Solve $(D^2+1)y = x^2\cosh x$

HW
⑦ Solve $(D^2+1)y = \sin x \sin 2x + e^x x^2$

HW
⑧ Solve $(D^2-2D+2)y = e^x\tan x$

HW
⑨ Solve $(D^2+9)y = (x^2+1)e^{3x}$

To find P.I. of $f(D)y = Q(x)$ when $Q(x) = xV(x)$

P.I. is $y_p = \dfrac{1}{f(D)}Q(x)$

$= \dfrac{1}{f(D)} xV(x)$

$y_p = \left[x - \dfrac{f'(D)}{f(D)}\right]\dfrac{1}{f(D)}V(x)$

problems

① solve $(D^2+2D+1)y = x\cos x$ — ①

Sol:- A.E of ① is $f(m)=0$

$\Rightarrow m^2+2m+1=0$

$\Rightarrow m=-1,-1$

$y_c = (c_1+c_2x)e^{-x}$

To find y_p

$y_p = \dfrac{1}{f(D)}Q(x)$

$= \dfrac{1}{(D^2+2D+1)} x\cos x$

$= \left[x - \dfrac{2D+2}{D^2+2D+1}\right]\dfrac{1}{D^2+2D+1}\cos x$

$$= \left[x - \frac{2D+2}{D^2+2D+1}\right] \frac{\cos x}{\cancel{-1}+2D+\cancel{1}}$$

$$= \left[x - \frac{2D+2}{D^2+2D+1}\right] \frac{1}{2}\sin x$$

$$= \frac{1}{2}\left[x\sin x - \frac{2\cancel{(D+1)}}{(D+1)^{\cancel{2}}} \sin x\right]$$

$$= \frac{1}{2}\left[x\sin x - 2\frac{(D-1)}{D^2-1}\sin x\right]$$

$$= \frac{1}{2}\left[x\sin x - \cancel{2}\frac{(D-1)\sin x}{\cancel{-2}}\right]$$

$$= \frac{1}{2}\left[x\sin x + \cos x - \sin x\right]$$

$\therefore\ y = y_c + y_p$

$$y = (c_1 + c_2 x)e^{-x} + \frac{1}{2}\left[x\sin x + \cos x - \sin x\right]$$

② Solve $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + y = xe^x \sin x$

Given equ can be written as

$$(D^2 - 2D + 1)y = xe^x \sin x \quad \text{—— ①}$$

A.E. of ① is $f(m) = 0$

$\Rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow m = 1, 1$

$y_c = (c_1 + c_2 x)e^x$

To find y_p

$$y_p = \frac{1}{D^2-2D+1} xe^x \sin x = \frac{1}{(D-1)^2} xe^x \sin x$$

$$= e^x \left[\frac{1}{(D+\cancel{1}-\cancel{1})^2} x\sin x\right]$$

$$= e^x \left[\frac{1}{D^2} x \sin x\right]$$

$$= e^x \left[x - \frac{2D}{D^2}\right] \frac{1}{D^2} \sin x$$

$$= e^x \left[x - \frac{2}{D}\right](-\sin x)$$

$$= e^x \left[-x\sin x + \frac{2}{D}\sin x\right]$$

$$= e^x \left[-x\sin x \cancel{+} 2\cos x\right]$$

$$y_p = -e^x \left[x\sin x + 2\cos x\right]$$

∴ The general soln is

$$y = y_c + y_p$$

$$\boxed{y = (C_1 + C_2 x) e^x - e^x \left[x\sin x + 2\cos x\right]}$$

Hw
(3) Solve $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$

Hw
(4) Solve $\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = xe^x \sin x$

Hw
(5) Solve $\frac{d^2y}{dx^2} + 9y = x\sin 2x$

Hw
(6) Solve $(D^4 + 2D^2 + 1)y = x^2\cos^2 x$

Hw
(7) Solve $(D^2+1)y = t\cos 2t$ given $x=0$, $\frac{dx}{dt} = 0$ at $t=0$

Method of variation of parameters

General soln of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ by the method of variation of parameters

Working rule

→ Given equ is $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ —— (1)

→ Find the soln of (1)

Let the soln is $y_c = c_1 U(x) + c_2 V(x)$

→ Take $y_p = AU(x) + BV(x)$

where A & B are functions of x.

→ Find $W(x) = UV' - VU'$

(or)

$$W(x) = \begin{vmatrix} U & V \\ U' & V' \end{vmatrix}$$

→ Find $A = -\int \frac{VR}{W(x)}dx \quad B = \int \frac{UR}{W(x)}dx$

→ The general soln is $y = y_c + y_p$

Problem

(1) Apply the method of variation of parameters to solve

$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

Sol:- Given eqn ~~can be written~~ is $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ —— (1)

eq (1) comparing with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

$P = 0, \quad Q = 1, \quad R = \operatorname{cosec} x$

A.E. of (1) is $\Rightarrow f(m) = 0 \Rightarrow m^2 + 1 = 0$

$\Rightarrow m = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$ — ②

eq② Comparing with $y_c = c_1 U(x) + c_2 V(x)$

$U(x) = \cos x$, $V(x) = \sin x$

To find y_p

$y_p = A\cos x + B\sin x$

where $A = -\int \frac{VR}{U\frac{dV}{dx} - V\frac{dU}{dx}}$, $B = \int \frac{UR}{U\frac{dV}{dx} - V\frac{dU}{dx}}$

$= -\int \frac{\sin x \operatorname{cosec} x}{\cos x \cos x - \sin x(-\sin x)} dx$, $= \int \frac{\cos x \operatorname{cosec} x}{(1)} dx$

$= -\int \frac{\cancel{\sin x} \cdot \frac{1}{\cancel{\sin x}}}{\cos^2 x + \sin^2 x} dx$, $= \int \cot x \, dx$

$= \log|\sin x|$

$A = -x$

$y_c = -x\cos x + \log|\sin x| \sin x$

∴ The general soln is

$y = y_c + y_p$

$= (c_1 \cos x + c_2 \sin x) - x\cos x + \log|\sin x| \sin x$

② Solve by the method of variation of parameters

$(D^2 - 2D)y = e^x \sin x$ — ①

Sol: A.E of ① is

$f(m) = 0 \Rightarrow m^2 - 2m = 0$

$\Rightarrow m(m-2) = 0$

$\Rightarrow m = 0, 2$

$y_c = c_1 e^{0x} + c_2 e^{2x}$

$y_c = c_1 + c_2 e^{2x}$ —— (2)

eq (2) Comparing with $y_c = c_1 U(x) + c_2 V(x)$

$U(x) = 1, \quad V(x) = e^{2x}$

To find y_p

$y_p = A + B e^{2x}$

where $A = -\int \frac{VR}{U\frac{dv}{dx} - V\frac{du}{dx}} dx$

$A = -\int \frac{e^{2x} \cdot e^x \sin x}{1 \cdot 2e^{2x} - e^{2x}(0)} dx$

$= -\int \frac{\cancel{e^{2x}} \cdot e^x \sin x}{2\cancel{e^{2x}}} dx$

$= -\frac{1}{2}\int e^x \sin x \, dx$

$= -\frac{1}{2} \frac{e^x}{1^2+1^2}[\sin x - \cos x]$

$= -\frac{1}{4} e^x (\sin x - \cos x)$

$B = \int \frac{UR}{U\frac{dv}{dx} - V\frac{du}{dx}} dx$

$= \int \frac{1 \cdot e^x \sin x}{2 e^{2x}} dx$

$= \frac{1}{2}\int e^{-x} \sin x \, dx$

$= \frac{1}{2}\left[-\frac{1}{2} e^{-x}(\sin x + \cos x)\right]$

$= -\frac{1}{4} e^{-x}(\sin x + \cos x)$

$y_p = -\frac{1}{4} e^x(\sin x - \cos x) - \frac{e^{-x}}{4}(\sin x + \cos x) e^{2x}$

$= -\frac{1}{2} e^x \sin x$

$y = y_c + y_p$

$$\boxed{y = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x}$$

(3) $\frac{d^2y}{dx^2} + y = \tan x$

(4) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x \tan x$

(5) $\frac{d^2y}{dx^2} + a^2 y = \sec ax$

Equations reducible to linear ODE with Constant coefficients

Cauchy - Euler Equation

An equation is of the form $x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_{n-1} x \frac{dy}{dx}$ $+ P_n y = \phi(x)$ — ① is called Cauchy-Euler equation (or) Cauchy's homogeneous linear equation.

Where $P_1\ P_2\ P_3 \ldots\ P_n$ are real Constants and $\phi(x)$ is a function of x

The operator form of eq ① is

$$(x^n D^n + P_1 x^{n-1} D^{n-1} + \cdots + P_{n-1} x D + P_n) y = \phi(x)$$

where $D = \frac{d}{dx}$

This equation can be transformed into a linear equation with Constant coefficients by the change of independent variable with the substitution $x = e^z$ (or) $z = \log x$, $xD = \theta$, $x^2D^2 = \theta(\theta-1)$

problems ①

① Solve $(x^2D^2 - 4xD + 6)y = x^2$

Sol:- This is a Cauchy-Euler equation.

Let $x = e^z \Rightarrow \log x = z$ $\quad \frac{d}{dx} = D, \frac{d}{dz} = \theta$

$xD = \theta, \; x^2D^2 = \theta(\theta-1)$

Given eq can be written as

$$(\theta(\theta-1) - 4\theta + 6)y = e^{2z}$$

$$(\theta^2 - 5\theta + 6)y = e^{2z} \quad \text{—①}$$

eq ① is a D.E. with Constant coefficients.

A.E. of ① is $f(m) = 0 \Rightarrow m^2 - 5m + 6 = 0$

$\Rightarrow m = 2, 3$

$$y_c = c_1 e^{2z} + c_2 e^{3z}$$

To find y_p

$$y_p = \frac{e^{2z}}{(\theta-3)(\theta-2)} = \frac{e^{2z}}{(2-3)}\frac{z}{1!} = -ze^{2z}$$

The general soln is

$$y = y_c + y_p$$

$$y = c_1 e^{2z} + c_2 e^{3z} - ze^{2z}$$

$$\boxed{y = c_1 x^2 + c_2 x^3 - \log x \, x^2}$$

(2) Solve $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \log x$

Given eqn can be written as

$$(x^2D^2 - xD + 1)y = \log x \quad \text{—— (1)}$$

Let $x = e^z \Rightarrow z = \log x$

$$xD = \theta, \quad x^2D^2 = \theta(\theta-1)$$

eq (1) becomes

$$(\theta(\theta-1) - \theta + 1)y = z$$

$$(\theta^2 - 2\theta + 1)y = z \quad \text{—— (2)}$$

This is a linear D.E. with constant coefficients

A.E. of (2) is $f(m) = 0$

$$\Rightarrow m^2 - 2m + 1 = 0$$

$$\Rightarrow m = 1, 1$$

$$y_c = (c_1 + c_2 z)e^z$$

To find y_p

$$y_p = \frac{z}{(\theta-1)^2} = (1-\theta)^{-2}(z)$$

$$= (1 + 2\theta + 3\theta^2 + \cdots)z$$

$$= z + 2$$

The general soln is

$y = y_c + y_p$

$= (c_1 + c_2 z)\, e^{z} + z + 2$

$$\boxed{y = (c_1 + c_2 \log x)\, x + \log x + 2}$$

(3) Solve $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^4$

Sol:- Given eqn can be written as

$(x^2 D^2 - xD - 4) y = x^4$

Let $x = e^{z} \Rightarrow \log x = z$

$xD = \theta, \quad x^2 D^2 = \theta(\theta - 1)$

$(\theta(\theta-1) - 2\theta - 4) y = e^{4z}$

$(\theta^2 - 3\theta - 4) y = e^{4z}$ —— (1)

A.E. of (1) is $m^2 - 3m - 4 = 0$

$\Rightarrow m = -1, 4$

$y_c = c_1 e^{-z} + c_2 e^{4z}$

To find y_p

$y_p = \frac{1}{\theta^2 - 3\theta - 4} e^{4z} = \frac{e^{4z}}{(\theta+1)(\theta-4)} = \frac{e^{4z} \cdot z}{5} = \frac{x^4 \log x}{5}$

∴ The general soln is

$y = y_c + y_p$

$= c_1 e^{-z} + c_2 e^{4z} + \frac{x^4 \log x}{5}$

$= \frac{c_1}{x} + c_2 x^4 + \frac{x^4 \log x}{5}$

④ Solve $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} + 2y = 10\left(x+\dfrac{1}{x}\right)$

Sol:— Given equ can be written as

$$\left(x^3D^3 + 2x^2D^2 + 2\right)y = 10\left(x+\frac{1}{x}\right) \quad — ①$$

Let $x = e^z \Rightarrow \log x = z$

$$xD = \theta,\quad x^2D^2 = \theta(\theta-1) \quad x^3D^3 = \theta(\theta-1)(\theta-2)$$

eq ① can be written as

$$\left(\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) + 2\right)y = 10\left(e^z + e^{-z}\right)$$

$$\left(\theta^3 - \theta^2 + 2\right)y = 10\left(e^z + e^{-z}\right) \quad — ②$$

This is a linear D.E with constant coefficients

A.E of ② is $m^3 - m^2 + 2 = 0$

$\Rightarrow m = -1,\ 1+i,\ 1-i$

$$y_c = c_1e^{-z} + e^z\left(c_2\cos z + c_3\sin z\right)$$

$$y_p = \frac{1}{\theta^3-\theta^2+2}\,10\left(e^z + e^{-z}\right)$$

$$= 10\,\frac{e^z}{\theta^3-\theta^2+2} + 10\,\frac{e^{-z}}{\theta^3-\theta^2+2}$$

$$= 10\,.\frac{e^z}{1-1+2} + 10\,.\frac{e^{-z}}{(\theta+1)(\theta^2-2\theta+2)}$$

$$= \cancel{10}\,5e^z + \frac{10\,z\,e^{-z}}{(1+2+2)}$$

$$y_p = 5e^z + 2z\,e^{-z}$$

The general soln is $y = y_c + y_p = c_1e^{-z} + e^z\left(c_2\cos z + c_2\sin z\right) + 5e^z + 2z\,e^{-z}$

where $z = \log x$

H.W ① Solve $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + 4y = (1+x)^2$

H.W ② Solve $\left(x^2D^2 - 3xD + 1\right)y = \dfrac{\log x \sin\log x + 1}{x}$

Legendre's Linear equation

An equation of the form $(a+bx)^n\frac{d^ny}{dx^n}+P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+P_ny=Q(x)$

is called Legendre's linear equation.

where $P_1\ P_2\ P_3 \ldots P_n$ are real constants and

$Q(x)$ is a function of x.

The soln can be obtained by substituting $a+bx=e^z$

$\Rightarrow z=\log(a+bx)$

Here $(a+bx)D=b\theta$, $(a+bx)^2D^2=b^2\theta(\theta-1)$,

$(a+bx)^3D^3=b^3\theta(\theta-1)(\theta-2)$ ......

problems

① Solve the D.E. $(2x-1)^3\frac{d^3y}{dx^3}+(2x-1)\frac{dy}{dx}-2y=x$

Sol: Given D.E. $(2x-1)^3\frac{d^3y}{dx^3}+(2x-1)\frac{dy}{dx}-2y=x$ — ①

Let $2x-1=e^z \Rightarrow z=\log(2x-1)$

$(2x-1)D=2\theta$, $(2x-1)^2D^2=2^2\theta(\theta-1)$

$(2x-1)^3D^3=2^3\theta(\theta-1)(\theta-2)$

eq① can be written as

$$\left[2^3\theta(\theta-1)(\theta-2)+2(\theta-1)\right]y=\frac{1}{2}(1+e^z)$$

$$(8\theta^3-24\theta^2+18\theta-2)y=\frac{e^z+1}{2} \quad — ②$$

A.E of ② is $8m^3-24m^2+18m-2=0$

$\Rightarrow m=1,\ 1\pm\frac{\sqrt{3}}{2}$

$$y_c=c_1e^z+c_2e^{\left(1+\frac{\sqrt{3}}{2}\right)z}+c_3e^{\left(1-\frac{\sqrt{3}}{2}\right)z}$$

To find y_p

$$\begin{array}{c|cccc} 1 & 8 & -24 & 18 & -2 \\ & 0 & 8 & -16 & 2 \\ \hline & 8 & -16 & 2 & 0 \end{array}$$

$$(\theta-1)(8\theta^2-16\theta+2)$$

$$y_p = \frac{1}{2}\left[\frac{e^z+1}{8\theta^3-24\theta^2+18\theta-2}\right]$$

$$= \frac{1}{2}\left[\frac{e^z}{8\theta^3-24\theta^2+18\theta-2}\right] + \frac{1}{2}\left[\frac{1}{8\theta^3-24\theta^2+18\theta-2}\right]$$

$$= \frac{1}{2}\,\frac{e^z \cdot z}{2(8-16+2)} + \frac{1}{2}\left[\frac{e^{0 \cdot z}}{8(0)-24(0)+18(0)-2}\right]$$

$$= \frac{-1}{12}\cdot z e^z + \frac{1}{2}\cdot\left(\frac{-1}{2}\right)$$

$$= \frac{-ze^z}{12} - \frac{1}{4}$$

$$= \frac{-\log(2x-1)(2x-1)}{12} - \frac{1}{4}$$

$\therefore$ The general soln is

$$y = y_c + y_p$$

$$= c_1 e^z + c_2 e^{\left(1+\frac{\sqrt{3}}{2}\right)z} + c_3 e^{\left(1-\frac{\sqrt{3}}{2}\right)z} - \frac{\log(2x-1)(2x-1)}{12} - \frac{1}{4}$$

where $z = \log(2x-1)$

UNIT - III

MULTIPLE INTEGRALS

multiple Integral: - A double (or) triple integral is known as multiple integral.

It is an extension of a definite integral of a function of single variable to a function of two or three variables.

Multiple integrals are useful in evaluating area, volume, mass, centroid and moments of inertia in plane and solid regions,

Double integral: - The definite integral can be extended to functions of more than one variable (two variables) is called double integral.

Let $z = f(x, y)$ be a function of two variables then double integral of $f(x, y)$ is denoted with $\iint_R f(x, y)\,dx\,dy$ over R.

Properties

i) $\iint_R (f + g)\,dx\,dy = \iint_R f\,dx\,dy + \iint_R g\,dx\,dy$

ii) $\iint_R kf\,dx\,dy = k\iint_R f\,dx\,dy$ where 'k' is constant.

iii) $\iint_R f\,dx\,dy = \iint_{R_1} f\,dx\,dy + \iint_{R_2} f\,dx\,dy$

Evaluation of Double integrals

Suppose R can be described by inequalities of the form $a \le x \le b$, $y_1(x) \le y \le y_2(x)$ represented the boundary of R then

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y)\,dy\,dx = \int_{x=a}^{b} \left[\int_{y=y_1(x)}^{y_2(x)} f(x, y)\,dy \right] dx$$

Note:— ① If all the four limits of integration are constants, then the double integral can be evaluated from the following way.

i.e.; We first integrate with respect to x and then w.r.to 'y' (OR) we first integrate w.r.to y and then w.r.to x'.

Note:— ② Suppose if the limits for one variable is a function of the other and constants for other variable, then first we have to integrate the variable w.r.to variable for which limits are function of the other.

Note:— ③ While evaluating a multiple integral w.r.to one variable then other variables ~~too~~ are treated as constants.

Problems

① Evaluate $\int_0^3\int_1^2 xy(1+x+y)\,dx\,dy$.

Sol:- $\int_0^3\int_1^2 xy(1+x+y)\,dx\,dy = \int_{x=0}^{3}\int_{y=1}^{2}(xy + x^2y + xy^2)\,dx\,dy$

$$= \int_{x=0}^{3}\left(\frac{xy^2}{2} + \frac{x^2y^2}{2} + \frac{xy^3}{3}\right)_{y=1}^{2} dx$$

$$= \int_{x=0}^{3}\left[\left(2x + 2x^2 + \frac{8}{3}x\right) - \left(\frac{x}{2} + \frac{x^2}{2} + \frac{x}{3}\right)\right]dx$$

$$= \int_0^3\left[\left(\frac{14x}{3} + 2x^2\right) - \left(\frac{5x}{6} + \frac{x^2}{2}\right)\right]dx$$

$$= \int_0^3 \left(\frac{23}{6}x + \frac{3}{2}x^2\right) dx = \left(\frac{23}{6}\frac{x^2}{2} + \frac{3}{2}\frac{x^3}{3}\right)_0^3 = \frac{123}{4}$$

② Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dx\, dy$

Sol:-

$$\int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} (x^2 + y^2)\, dx\, dy = \int_{x=0}^{1} \left(x^2 y + \frac{y^3}{3}\right)_{y=x}^{\sqrt{x}} dx$$

$$= \int_{x=0}^{1} \left(x^2\sqrt{x} + \frac{x\sqrt{x}}{3} - x^3 - \frac{x^3}{3}\right) dx$$

$$= \int_{x=0}^{1} \left(x^{5/2} + \frac{x^{3/2}}{3} - \frac{4}{3}x^3\right) dx$$

$$= \left[\frac{x^{7/2}}{7/2} + \frac{x^{5/2}}{5/2} - \frac{4}{3}\frac{x^4}{4}\right]_{x=0}^{1}$$

$$= \frac{9}{105} = \frac{3}{35}$$

③ Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$

Sol:-

$$\int_0^\infty \int_0^\infty e^{-x^2} \cdot e^{-y^2}\, dx\, dy = \int_0^\infty e^{-x^2} \left[\int_0^\infty e^{-y^2} dy\right] dx$$

$$= \int_0^\infty e^{-x^2} \frac{\sqrt{\pi}}{2}\, dx \qquad \left(\because \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-x^2} dx$$

$$= \frac{\sqrt{\pi}}{2} \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$

(OR)

By changing into polar coordinates, $x = r\cos\theta,\ y = r\sin\theta$

$dxdy = r\,dr\,d\theta$

$\theta \to 0$ to $\pi/2$

$r \to 0$ to ∞

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

put $r^2 = u$

$2r\,dr = du$

$r\,dr = \frac{1}{2}du$

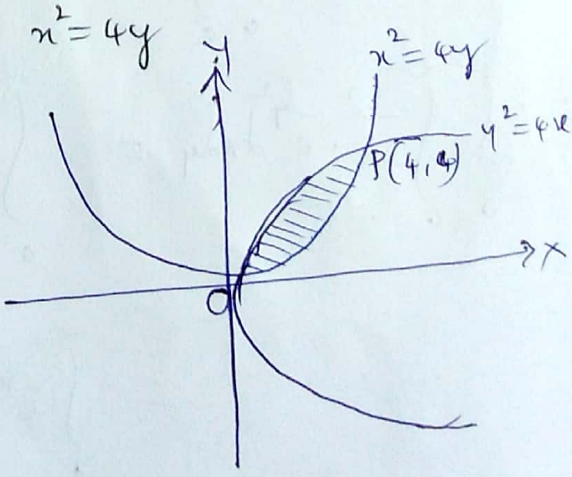
$$= \int_0^{\pi/2} \int_{r=0}^{\infty} \left(\frac{1}{2}\right) e^{-u}\,du\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\frac{e^{-u}}{-1}\right)_{r=0}^{\infty} d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} [0-1]\,d\theta$$

$$= \frac{1}{2}[\theta]_0^{\pi/2} = \frac{\pi}{4}$$

(4) Evaluate i) $\iint_R y\,dx\,dy$ ii) $\iint_R y^2\,dx\,dy$ where R is the region bounded by parabalas $y^2 = 4x$ and $x^2 = 4y$



Given parabolas

$y^2 = 4x$ — ①

$x^2 = 4y$ — ②

Solving ① & ②

$\left(\frac{x^2}{4}\right)^2 = 4x \Rightarrow x^4 = 64x$

$\Rightarrow x(x^3 - 64) = 0$

$\Rightarrow x = 0,\ x = 4$

$y^2 = 4x$

$y^2 = 16 \Rightarrow y = \pm 4$

Thus the two parabolas intersect at the points $(0,0)$ $(4,4)$

i) $\iint_R y\,dx\,dy = \int_{x=0}^{4}\int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y\,dx\,dy = \int_{x=0}^{4}\left(\frac{y^2}{2}\right)_{y=\frac{x^2}{4}}^{2\sqrt{x}} dx$

$$= \int_{x=0}^{4}\left(\frac{\cancel{4}^{2}x}{\cancel{2}} - \frac{1}{2}\,\frac{1}{16}x^4\right)dx$$

$$= \left(\frac{\cancel{2}x^2}{\cancel{2}} - \frac{1}{32}\,\frac{x^5}{5}\right)_{x=0}^{4}$$

$$= 16 - \frac{1}{\cancel{32}_{\cancel{4}}}\,\frac{\cancel{64\times 16}^{2}\ \cancel{128}^{8}\ 4}{5}$$

$$= 16 - \frac{32}{5} = \frac{80-32}{5} = \frac{48}{5}$$

ii) $\iint_R y^2\,dx\,dy =$

(5) Evaluate $\int_0^4\int_0^{x^2} e^{y/x}\,dy\,dx$

$$\int_0^4\int_0^{x^2} e^{y/x}\,dx\,dy = \int_0^4\left[e^{y/x}\cdot x\right]_{y=0}^{x^2} dx$$

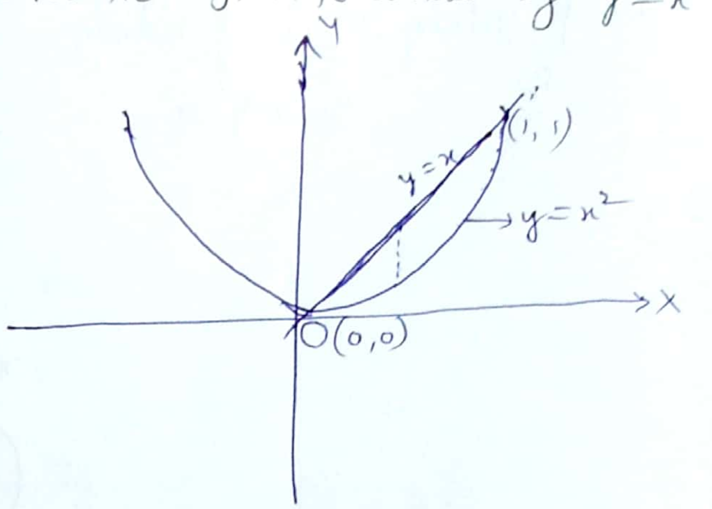
$$= \int_0^4\left(e^{\cancel{x^2}/\cancel{x}}\cdot x - x\right)dx$$

$$= \left(xe^x - e^x - \frac{x^2}{2}\right)_0^4$$

$$= (4e^4 - e^4 - 8) - (0 - 1 - 0)$$

$$= 3e^4 - 7$$

⑥ Evaluate $\iint xy(\cancel{x}+y)\,dx\,dy$ over the region R bounded by $y=x^2$ and $y=x$



Sol:

$y=x^2,\ y=x$

$\Rightarrow x^2=x$

$\Rightarrow x^2-x=0$

$\Rightarrow x(x-1)=0$

$\Rightarrow x=0,1$ & $y=0,1$

The point of intersection of the curves are $(0,0)\ (1,1)$

We have to fix 'x' first, y varies from x^2 to x. & x from 0 to 1

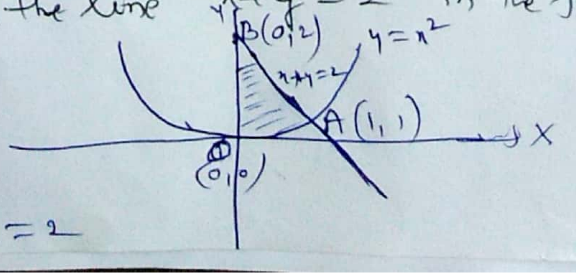
$$\iint xy(x+y)\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{x}(x^2y+xy^2)\,dx\,dy$$

$$= \int_{x=0}^{1}\left(\frac{x^2y^2}{2}+\frac{xy^3}{3}\right)_{y=x^2}^{x}dx$$

$$= \int_{x=0}^{1}\left[\left(\frac{x^2x^2}{2}+\frac{x^4}{3}\right)-\left(\frac{x^6}{2}+\frac{x^7}{3}\right)\right]dx$$

$$= \frac{3}{56}$$

⑦ Evaluate $\iint_R y\,dx\,dy$ where R is the domain bounded by y-axis the curve $y=x^2$ and the line $x+y=2$ in the first quadrant



Sol: Y-axis $\Rightarrow x=0$

So given curves are

$x=0,\ y=x^2,\ x+y=2$

The point of intersection of curves are

$y = x^2$, $x + y = 2$

$x + x^2 = 2$

$x^2 + x - 2 = 0$

$\Rightarrow (x+2)(x-1) = 0$

$\Rightarrow x = -2,\ x = 1$

When $x = 1$, $y = 1$

When $x = 0$ $y = 0$

When $x = 0$ $y = 2$

The given curves intersect at $O(0,0)$ $A(1,1)$ $B(0,2)$

Hence $\iint_F y\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{2-x} y\,dx\,dy = \int_{x=0}^{1}\left(\frac{y^2}{2}\right)_{y=x^2}^{2-x} dx$

$= \int_{x=0}^{1}\left[\frac{(2-x)^2}{2} - \frac{x^4}{2}\right] dx$

$= \left(\frac{(2-x)^3}{(-6)} - \frac{x^5}{10}\right)_{x=0}^{1}$

$= \left(\frac{-1}{6} - \frac{1}{10}\right) - \left(\frac{8}{-6}\right)$

$= -\frac{1}{6} + \frac{8}{6} - \frac{1}{5}$

$= \frac{7}{6} - \frac{1}{10}$

$= \frac{70 - 6}{60} = \frac{\cancel{64}^{16}}{\cancel{60}_{15}} = \frac{16}{15}$

Double integrals in Polar Co-ordinates

To evaluate over the region bounded by the lines $\theta=\theta_1$, $\theta=\theta_2$ and the curves $r=r_1$ $r=r_2$, we first integrate w.r.to r between limits $r=r_1$ and $r=r_2$ keeping θ fixed. The resulting expression is integrated w.r.to θ from θ_1 to θ_2.

① Evaluate $\displaystyle\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$ (2017)

Sol:– $$\int_{\theta=0}^{\pi}\int_{r=0}^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi}\left[\int_0^{a\sin\theta} r\,dr\right]d\theta$$

$$= \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a\sin\theta} d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\sin^2\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\frac{1-\cos2\theta}{2}\,d\theta$$

$$= \frac{a^2}{4}\left[\theta-\frac{\sin2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{4}\left[(\pi-0)-(0-0)\right]$$

$$= \frac{a^2\pi}{4}$$

② Evaluate $\displaystyle\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}\,r\,d\theta\,dr$

Sol– $$\int_0^{\infty}\int_0^{\pi/2} e^{-r^2}r\,d\theta\,dr = \int_0^{\infty} e^{-r^2}r\left[\theta\right]_0^{\pi/2}dr = \frac{\pi}{2}\int_0^{\infty} r\,e^{-r^2}dr$$

$$= -\frac{\pi}{4}\int_{0}^{\infty}(-2r)\,e^{-r^2}\,dr$$

$$= -\frac{\pi}{4}\left[e^{-r^2}\right]_0^{\infty} = -\frac{\pi}{4}[0-1] = \frac{\pi}{4}$$

③ Evaluate $\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sin\theta}\frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

Sol:- $\int_0^{\pi/4}\int_0^{a\sin\theta}\frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta = \int_0^{\pi/4}\left\{\int_0^{a\sin\theta}\frac{r}{\sqrt{a^2-r^2}}\,dr\right\}d\theta$

$$= -\frac{1}{\cancel{2}}\int_0^{\pi/4}\left(\cancel{2}\sqrt{a^2-r^2}\right)_0^{a\sin\theta}d\theta \qquad \because \int\frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}+$$

$$= (-1)\int_0^{\pi/4}\left(\sqrt{a^2-a^2\sin^2\theta}-\sqrt{a^2-0}\right)d\theta$$

$$= (-1)\int_0^{\pi/4}(a\cos\theta - a)\,d\theta$$

$$= (-a)\left[\sin\theta-\theta\right]_0^{\pi/4}$$

$$= (-a)\left[\left(\sin\frac{\pi}{4}-\frac{\pi}{4}\right)-(0-0)\right]$$

$$= (-a)\left[\frac{1}{\sqrt{2}}-\frac{\pi}{4}\right]$$

$$= a\left[\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right]$$

④ Evaluate $\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty}\frac{r}{(r^2+a^2)^2}\,dr\,d\theta$

Sol:- $\int_0^{\pi/2}\int_0^{\infty}\frac{r}{(r^2+a^2)^2}\,dr\,d\theta = \int_0^{\pi/2}\left\{\int_0^{\infty}\frac{r}{(r^2+a^2)^2}\,dr\right\}d\theta$

$$= \frac{1}{2}\int_0^{\pi/2}\left\{\int_0^{\infty}\frac{2r}{(r^2+a^2)^2}\,dr\right\}d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left\{\frac{-1}{(r^2+a^2)}\right\}_0^{\infty} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[0+\frac{1}{a^2}\right]d\theta$$

$$= \frac{1}{2a^2}\int_0^{\pi/2} d\theta$$

$$= \frac{1}{2a^2}\left(\theta\right)_0^{\pi/2}$$

$$= \frac{1}{2a^2}\left[\frac{\pi}{2}-0\right] = \frac{\pi}{4a^2}$$

(5) Evaluate $\displaystyle\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta \, dr\, d\theta$

$$\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r^2\cos\theta \, dr\, d\theta = \int_{\theta=0}^{\pi}\cos\theta \, d\theta\int_{r=0}^{a(1+\cos\theta)} r^2 dr$$

$$= \int_0^{\pi}\cos\theta\left(\frac{r^3}{3}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi}\cos\theta(1+\cos\theta)^3 d\theta$$

Let $I = \dfrac{a^3}{3}\displaystyle\int_0^{\pi}\cos\theta(1+\cos\theta)^3 d\theta$

$$= \frac{a^3}{3}\int_0^{\pi}\cos(\pi-\theta)(1+\cos(\pi-\theta))^3 d\theta \quad \because \int_0^a f(x)dx = \int_0^a f(a-x)$$

$$= -\frac{a^3}{3}\int_0^{\pi}\cos\theta(1-\cos\theta)^3 d\theta = I$$

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$$I+I = \frac{a^3}{3}\int_0^{\pi} \cos\theta \left[(1+\cos\theta)^3 - (1-\cos\theta)^3\right] d\theta$$

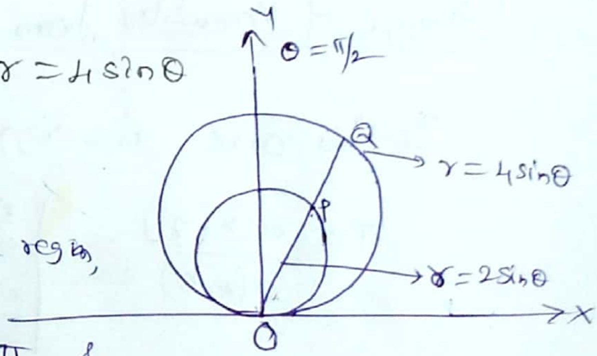
$$= \frac{2a^3}{3}\int_0^{\pi} (3\cos^2\theta + \cos^4\theta)\, d\theta$$

$$= \frac{2a^3}{3} \cdot 2\int_0^{\pi/2} (3\cos^2\theta + \cos^4\theta)\, d\theta$$

$$= \frac{4a^3}{3}\left[3\cdot\frac{\pi}{4} + \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$I = \frac{5\pi a^3}{8}$$

(6) Evaluate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$



Sol:- Here circle cover the whole region,

So θ varies from 0 to π &

r varies from $2\sin\theta$ to $4\sin\theta$.

$$\iint r^3\, dr\, d\theta = \int_{\theta=0}^{\pi}\int_{r=2\sin\theta}^{4\sin\theta} r^3\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \left(\frac{r^4}{4}\right)_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi} (256\sin^4\theta - 16\sin^4\theta)\, d\theta$$

$$= 60\int_0^{\pi} \sin^4\theta\, d\theta$$

$$= 60 \times 2\int_0^{\pi/2} \sin^4\theta\, d\theta = 120 \times \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{45\pi}{2}$$

Change of variables in Double Integral

Transformation of Coordinates

Let $x = f(u,v)$ and $y = g(u,v)$ be the relations between the old variables (x,y) with the new variables (u,v) of the new coordinate system.

$$\iint_R F(x,y)\,dx\,dy = \iint_R F'(f,g)\,|J|\,du\,dv$$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)}$

which is called the Jacobian of the coordinate transformation.

Change of variables from Cartesian to Polar Co-ordinates

Change of Variables from Cartesian to Polar Coordinates

In this case $u = r, v = \theta$ and $x = r\cos\theta$, $y = r\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

Hence

$$\iint_R F(x,y)\,dx\,dy = \iint_R F(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

~~i.e. $\iint_R F(x,y)\,dx\,dy =$~~

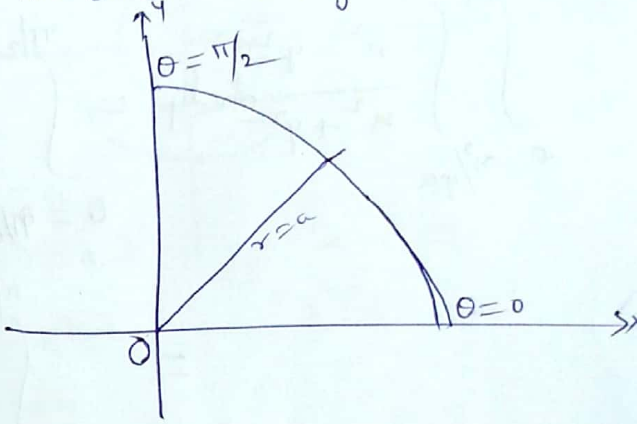
i.e.; $$\iint F(r,\theta)\,dA = \int_{\theta=\theta_1}^{\theta_2} \int_{r=f_1(\theta)}^{f_2(\theta)} F(r,\theta)\,r\,dr\,d\theta$$

① Evaluate the double integral $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dy\, dx$ by changing into polar coordinates.

Sol:- The given ~~Cor~~ region can be written as

$0 \le x \le \sqrt{a^2-y^2}$ & $0 \le y \le a$

ie, R is the region bounded by the circle $x^2+y^2=a^2$ in the first quadrant.



By changing into polar form,

$x = r\cos\theta, \quad y = r\sin\theta$

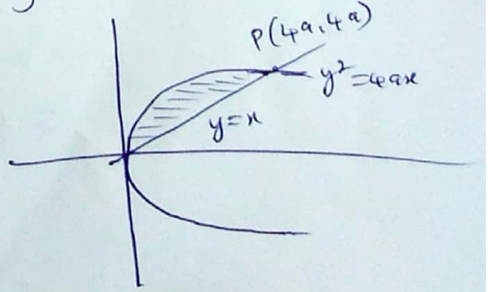
$dx\,dy = r\,dr\,d\theta.$

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2\, r\, dr\, d\theta = \frac{\pi a^4}{8}$$

② Evaluate $\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\, dx\, dy$ by changing into polar coordinates.

Sol:- The region of integration is given by $x = \frac{y^2}{4a}$ ~~&~~, $x = y$ and $y = 0$, $y = 4a$.

The region is bounded by the parabola $y^2 = 4ax$ and the straight line $y = x$.



Let $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

The limits for r are $r = 0$ at 'O' and for P on the parabola

$\cancel{r^2 \sin^2\theta} = y^2 = 4ax$

$$r^2\sin^2\theta = 4a(\cancel{r}\cos\theta) \Rightarrow r = \frac{4a\cos\theta}{\sin^2\theta}$$

For the line $y = x$

$$\tan\theta = 1. \Rightarrow \theta = \pi/4$$

$\therefore$ The limits for θ varies from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

Also $x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta)$ and $x^2 + y^2 = r^2$

$$\therefore \int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{\cancel{r^2}(\cos^2\theta - \sin^2\theta)}{\cancel{r^2}}\, r\,dr\,d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\left(\frac{r^2}{2}\right)_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\left(\frac{\overset{8}{\cancel{16}}a^2\cos^2\theta}{\cancel{2}\sin^4\theta}\right) d\theta$$

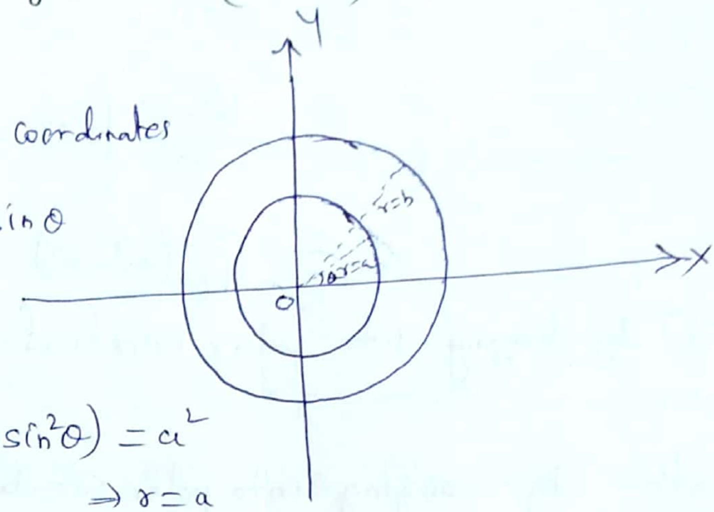
$$= 8a^2\int_{\pi/4}^{\pi/2} (\cot^4\theta - \cot^2\theta)\,d\theta - 8a^2\int_{\pi/4}^{\pi/2}\cot^2\theta\,d\theta$$

$$= 8a^2\left[\frac{\cot^3\theta}{3} - \cot\theta + \theta\right]_{\theta=\pi/4}^{\pi/2} \quad \left(\because \int \cot^4\theta\,d\theta = \frac{\cot^3\theta}{3} - \cot\theta + \theta + c\right)$$

$$= 8a^2\left[\frac{\pi}{2} - \frac{5}{3}\right]$$

③ By changing into polar coordinates, evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annular region between the ~~cercula~~ circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ $(b>a)$

Sol:-



By changing into polar coordinates by taking $x=r\cos\theta$, $y=r\sin\theta$

$$dx\,dy = r\,dr\,d\theta$$

$$x^2+y^2=a^2 \Rightarrow r^2(\cos^2\theta+\sin^2\theta)=a^2$$
$$\Rightarrow r=a$$

$$x^2+y^2=b^2 \Rightarrow r^2(\cos^2\theta+\sin^2\theta)=b^2$$
$$\Rightarrow r=b$$

θ is from 0 to 2π

$$\therefore \iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=a}^{b} \frac{\cancel{r^2}\cos^2\theta\, r^2\sin^2\theta}{\cancel{r^2}(\cos^2\theta+\sin^2\theta)}\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=a}^{b} r^3\cos^2\theta\sin^2\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \cos^2\theta\sin^2\theta \left(\frac{r^4}{4}\right)_{r=a}^{b} d\theta$$

$$= \int_{\theta=0}^{2\pi} \cos^2\theta\sin^2\theta\left(\frac{b^4-a^4}{4}\right)$$

$$= \frac{b^4-a^4}{16}\int_{\theta=0}^{2\pi} (2\sin\theta\cos\theta)^2\,d\theta$$

$$= \frac{b^4-a^4}{16}\int_{\theta=0}^{2\pi} \sin^2 2\theta\,d\theta$$

$$= \frac{b^4-a^4}{32}\int_{\theta=0}^{2\pi}(1-\cos 4\theta)\,d\theta$$

$$= \frac{b^4-a^4}{32}\left[\theta-\frac{\sin 4\theta}{4}\right]_{\theta=0}^{2\pi}$$

$$= \frac{b^4-a^4}{32}\left[(2\pi-0)-(0-0)\right]$$

$$= \frac{\pi}{16}(b^4-a^4)$$

④ By changing into polar coordinates, evaluate $\int_0^\infty\int_0^\infty \frac{x^2}{(x^2+y^2)^{3/2}}\,dx\,dy$.

Sol:- By changing into polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$, $x^2+y^2 = r^2$

The region of integration is in first quadrant of xy plane,

so $r \to 0$ to ∞ and $\theta \to 0$ to $\pi/2$

Hence $\int_0^\infty\int_0^\infty \frac{x^2}{(x^2+y^2)^{3/2}}\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty}\frac{r^2\cos^2\theta}{(r^2)^{3/2}}\,r\,dr\,d\theta$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty}(1+\cos 2\theta)\,dr\,d\theta$$

$$= \frac{1}{2}\int_{r=0}^{\infty}\left[\theta+\frac{\sin 2\theta}{2}\right]_{\theta=0}^{\pi/2}dr$$

$$= \frac{1}{2}\int_{r=0}^{\infty}\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right]dr$$

$$= \frac{1}{2}\cdot\frac{\pi}{2}\int_{r=0}^{\infty}dr = \frac{\pi}{4}\ \cancel{\lim_{P\to\infty}\int_0^P dr} = \infty$$

Change of order of integration

Procedure

1. ⇒ First identify the variables for the limits
2. ⇒ Draw a rough sketch of the given region of integration.
3. ⇒ If we are evaluating the integral w.r.to y first, then take a vertical strip ie; a strip parallel to y-axis. Otherwise, take a horizontal strip ie; a strip parallel to x-axis.
4. ⇒ Now rotate the strip by an angle of 90° in the anti-clockwise direction and identify the starting and ending points of the strip, which will be the lower and upper limits of that variable.
5. ⇒ Identify the limits for other variables for the region of consideration
6. ⇒ Evaluate the double integral with new order of integration.

① Change the order of integration and evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

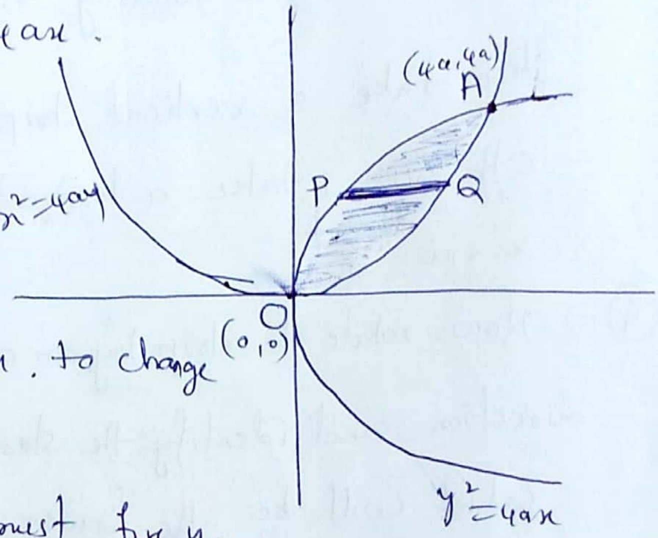
Sol:- In the given integral, for a fixed x, y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$.

and then x varies from 0 to $4a$.

Let us draw the curves $y = \frac{x^2}{4a}$ and $y = 2\sqrt{ax}$

ie; $x^2 = 4ay$ and $y^2 = 4ax$.

The region of integration is the shaded region in figure.



The given integral $\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx$, to change the order of integration, we must fix y first, x varies from $\frac{y^2}{4a}$ to 2$\sqrt{ay}$ then y varies from 0 to $4a$.

Hence the integral

$$\int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a}\left[\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\right]dy$$

$$= \int_{y=0}^{4a}\left[x\right]_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_{y=0}^{4a}\left[\left(2\sqrt{ay}\right) - \left(\frac{y^2}{4a}\right)\right]dy$$

$$= \left[\frac{2\sqrt{a}\,y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_{y=0}^{4a} = \frac{2\sqrt{a}\,4a\sqrt{4a}}{3/2} - \frac{64a^3}{12a}$$

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$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$

2018 $$= \frac{16a^2}{3}$$

*** Imp **

(2) Change the order of integration and hence evaluate the double integral $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$

Sol:- The region of integration is given by $y : x^2 \longrightarrow 2-x$ & $x : 0$ to 1

ie; $y = x^2$, $y = 2-x$ and $x = 0$, $x = 1$

When $x = 0$

$y = 0$ $(0, 0)$

When $x = 1$

$y = 1$

$(1, 1)$

When $x = 0$

$y = 2$ $(0, 2)$

When $x = 1$

$y = 1$

$(1, 1)$

The point of intersection of $y = x^2$, $y = 2-x$ is

$$x^2 = 2-x$$

$$x^2 + x - 2 = 0$$

$$\Rightarrow (x+2)(x-1) = 0$$

$$\Rightarrow x = -2, 1$$

When $x = -2$

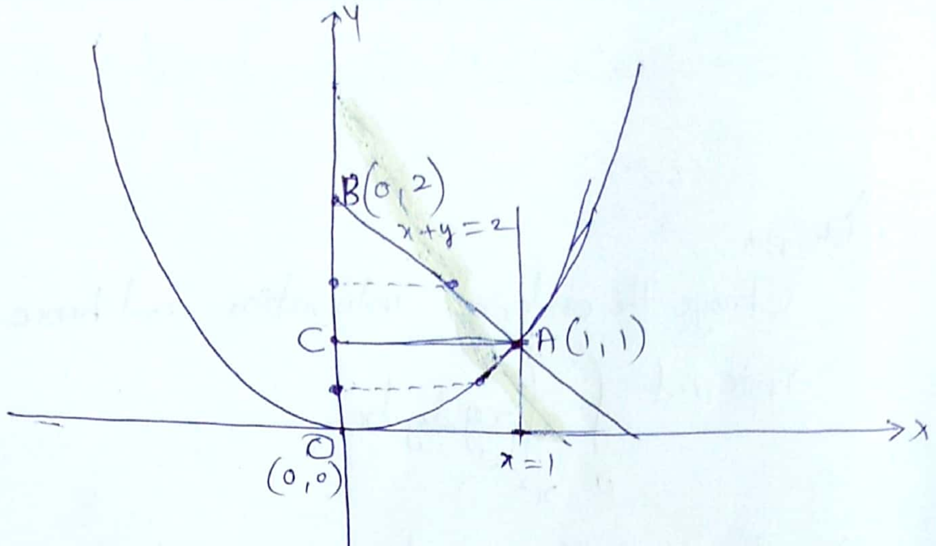
$y = 4$

When $x = 1$

$y = 1$

Hence the point of intersection of the curves are $(-2, 4)$ and $(1, 1)$.

The shaded region is shown in figure.



Suppose we change the order of integration, we have to take two horizontal strips, since during sliding one edge of the strips remains on $x=0$, but the other edge of the strip does not remain on a single curve.

$\therefore$ The regions can be

Area OAB = Area OAC + Area CAB

We shall fix y first, then for the region OACO, x varies from 0 to $\sqrt{y}$ and y varies from 0 to 1.

For the region CABC, for fixed y, x varies from 0 to $2-y$, then y varies from 1 to 2

$$\therefore \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx = \iint_{OACO} xy\,dx\,dy + \iint_{CABC} xy\,dx\,dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy\,dx\,dy$$

$$= \frac{3}{8}$$

Vector Differentiation

* Vector point functions and Scalar point functions
* Gradient, Divergence and curl of a vector
* Directional derivative.
* Tangent plane and normal line
* Vector Identities
* Scalar potential functions, Solenoidal and Irrotational vectors

---

Vector function :- A vector function is a function that assigns a vector to a set of real variables.

The general form is $F = f_1(x_1, x_2 \dots x_n)i + f_2(x_1, x_2 \dots x_n)j + f_3(x_1, x_2 \dots x_n)K$

where $x_1, x_2, x_3 \dots x_n$ are real numbers.

scalar function :- A scalar function is a function that assigns a real number (ie; scalar) to a set of real variables

The general form is $u = u(x_1, x_2 \dots x_n)$

where $x_1, x_2 \dots x_n$ are real numbers

vector point function :- A vector point function is a function that assigns a vector to each point of some region of space. If to each point (x, y, z) of a region R in space there is assigned a vector $F = F(x, y, z)$ then F is called a vector point function

ie; $F = f_1(x, y, z)i + f_2(x, y, z)j + f_3(x, y, z)K$

scalar point function :- A scalar point function is a function that assigns a real number (ie; a scalar) to each point of some region of space.

If to each point (x, y, z) of a region R in space there is assigned a real number $u = \phi(x, y, z)$ then ϕ is called a scalar point function.

Vector differential Operator :- The vector differential operator is denoted with ∇ and it is defined as $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$.

This operator used in defining "gradient", ~~dive~~ "divergence", and "curl" of a vector function.

Gradient of a scalar point function :- Let $\phi(x, y, z)$ be a scalar point function of position defined in some region of space.

Then the vector function $i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$ is known as the gradient of ϕ and is denoted by gradϕ or $\nabla\phi$

i.e; $\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

Properties

1) $\text{grad}(f \pm g) = \text{grad} f \pm \text{grad} g$

2) $\nabla f = \bar{0}$

3) $\text{grad}(fg) = f\,\text{grad}\, g + g\,\text{grad}\, f$

4) $\text{grad}(cf) = c\,\text{grad} f$ if c is a constant

5) $\text{grad}\left(\frac{f}{g}\right) = \frac{g\,\text{grad} f - f\,\text{grad} g}{g^2}$ $(g \neq 0)$

If $\bar{r} = xi + yj + zk$ and $d\bar{r} = (dx)i + (dy)j + (dz)k$ then

Directional derivative :- The directional derivative of a scalar point function ϕ at a point $P(x, y, z)$ in the direction of a unit vector $\bar{e}$ is equal to $\bar{e} \cdot \text{grad}\,\phi = \bar{e} \cdot \nabla\phi$

Note:- If $xi + yj + zk$ be the any vector and it's ~~directional derivative~~ normal unit vector $(\bar{e})$ is given by $\bar{e} = \dfrac{xi + yj + zk}{\sqrt{x^2 + y^2 + z^2}}$.

Gradient of a function of a function

Let $v = f(u)$ where $u = u(x, y, z)$ then

$$\nabla V = \nabla[f(u)] = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) f(u)$$

$$= i\frac{\partial}{\partial x} f(u) + j\frac{\partial}{\partial y} f(u) + k\frac{\partial}{\partial z} f(u)$$

$$= f'(u)\left[i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}\right]$$

$$= f'(u)\,\nabla u$$

$$\therefore \boxed{\nabla f(u) = f'(u)\,\nabla u}$$

(1) If $a = x + y + z$, $b = x^2 + y^2 + z^2$, $c = xy + yz + zx$ then prove that $[\text{grad}\,a, \text{grad}\,b, \text{grad}\,c] = 0$

Sol:-

$$a = x + y + z$$

$$\text{grad}\,a = \nabla a = \left(i\frac{\partial a}{\partial x} + j\frac{\partial a}{\partial y} + k\frac{\partial a}{\partial z}\right) = i + j + k$$

$$\text{grad}\,b = \nabla b = \left(i\frac{\partial b}{\partial x} + j\frac{\partial b}{\partial y} + k\frac{\partial b}{\partial z}\right) = 2xi + 2yj + 2zk$$

$$\text{grad}\,c = \nabla c = \left(i\frac{\partial c}{\partial x} + j\frac{\partial c}{\partial y} + k\frac{\partial c}{\partial z}\right) = (y+z)i + (z+x)j + (x+y)k$$

$$[\text{grad}\,a, \text{grad}\,b, \text{grad}\,c] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 0$$

(2) Prove that $\nabla(r^n) = n r^{n-2} \bar{r}$.

Sol: Let $\bar{r} = xi + yj + zk$ and let $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

then $r^2 = x^2+y^2+z^2$

Diff w.r.to x

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

My $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\Rightarrow \nabla(r^n) = \sum i \frac{\partial}{\partial x}(r^n)$$

$$= \sum i . n r^{n-1} \frac{\partial r}{\partial x}$$

$$= \sum i . n r^{n-1} \frac{x}{r}$$

$$= \sum i . n r^{n-2} x$$

$$= n r^{n-2} . \sum xi$$

$$\boxed{\nabla(r^n) = n r^{n-2} \bar{r}}$$

(3) Find a unit normal vector to the given surface $x^2y + 2xz = 4$ at the point $(2,-2,3)$.

Sol:- Let $f = x^2y + 2xz - 4$.

$$\text{grad} f = \nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = (2xy + 2z)i + x^2 j + 2x k.$$

grad f at $(2,-2,3)$ $= (-8+6)i + 4j + 4k$

$$= -2i + 4j + 4k$$

The unit normal vector along grad f is $= \frac{\nabla f}{|\nabla f|} = \frac{-2i+4j+4k}{\sqrt{4+16+16}}$

(2)

(4) Evaluate the angle between the normals to the surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$

Sol:-

$$\text{grad} f = y i + x j - 2z k$$

Let $n_1 = (\text{grad} f)_{(4,1,2)} = i + 4j - 4k$

$$n_2 = (\text{grad} f)_{(3,3,-3)} = 3i + 3j + 6k$$

Let θ be the angle between the two normals

$$\therefore \cos\theta = \frac{n_1 n_2}{|\bar{n}_1||\bar{n}_2|} = \frac{(i+4j-4k).(3i+3j+6k)}{\sqrt{1+16+16}\sqrt{9+9+36}}$$

$-3+3+6 \cancel{12} + \cancel{12} \; 24 - \cancel{12} + \cancel{12} \; -24$

$6 - 24$

$$\cos\theta = \frac{-9}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{3}\sqrt{11}\sqrt{9}\sqrt{2}\sqrt{3}}$$

$$= \frac{-9}{3.3\sqrt{22}} = \frac{-1}{\sqrt{22}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-1}{\sqrt{22}}\right)$$

(5) Find the directional derivative of $f = xy + yz + zx$ in the direction of vector $i + 2j + 2k$ at the point $(1,2,0)$.

Sol:- Directional derivative of f along the given direction is $= \bar{e}.\nabla f$

$$\nabla f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(xy + yz + zx) = (y+z)i + (z+x)j + (x+y)k$$

If $\bar{e}$ is the unit vector along $i+2j+2k$ then

$$\bar{e} = \frac{i+2j+2k}{\sqrt{1+4+4}} = \frac{1}{3}(i+2j+2k)$$

$\therefore$ Directional derivative $= \bar{e}.\nabla f$

$$= \frac{1}{3}(i+2j+2k)\left((y+z)i + (z+x)j + (x+y)k\right)$$

$$= \frac{1}{3}\left[(y+z) + 2(z+x) + 2(x+y)\right]$$

at $(1,2,0)$

Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P = (1,2,3)$ in the direction of the line PQ where $Q = (5,0,4)$.

Sol: The position vectors of P and Q with respect to the origin are

$\overline{OP} = i + 2j + 3k$ and

$\overline{OQ} = 5i + 4k$

$\therefore \overline{PQ} = \overline{OQ} - \overline{OP} = 4i - 2j + k$

Let $\bar{e}$ be the unit vector in the direction of $\overline{PQ}$ then $\bar{e} = \dfrac{4i - 2j + k}{\sqrt{21}}$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = 2xi - 2yj + 4zk$$

$\therefore$ Directional derivative $= \bar{e} \cdot \nabla f$

$$= \left(\frac{4i - 2j + k}{\sqrt{21}}\right)(2xi - 2yj + 4zk)$$

$$= \frac{1}{\sqrt{21}}(8x + 4y + 4z)$$

at $(1,2,3)$

$$= \frac{1}{\sqrt{21}}(8(1) + 4(2) + 4(3))$$

Directional derivative $= \dfrac{28}{\sqrt{21}}$ //

(7) Find the directional derivative of $1/r$ in the direction of $\bar{r} = xi + yj + zk$ at $(1, 1, 2)$.

Sol:- Given $\bar{r} = xi + yj + zk$ and $r = \sqrt{x^2 + y^2 + z^2}$

$$\therefore \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$\text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = i\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + k\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \frac{-xi}{(x^2+y^2+z^2)^{3/2}} + \frac{-yj}{(x^2+y^2+z^2)^{3/2}} + \frac{-zk}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{-\bar{r}}{r^{3/2}}$$

$\therefore$ Directional derivative $= \bar{e} \cdot \nabla\left(\frac{1}{r}\right)$

$$= \frac{-\bar{r}}{r^{3/2}} \cdot \frac{\bar{r}}{|\bar{r}|}$$

$$= \frac{-(\bar{r})^2}{r^{3/2}\, r}$$

$$= -\frac{(x^2+y^2+z^2)}{r^{3/2}} = \frac{-r^2}{\cancel{r^{5/2}}}$$

$$= -\frac{1}{\sqrt{r}}$$

at $(1,1,2)$

⑧ Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of the normal to the surface $f(x,y,z) = x\log z - y^2$ at $(-1,2,1)$.

Sol:- Given $\phi(x,y,z) = x^2yz + 4xz^2$ at $(1,-2,-1)$

& $f(x,y,z) = x\log z - y^2$ at $(-1,2,1)$

Now $\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$$= (2xyz + 4z^2)\bar{i} + (x^2z)\bar{j} + (x^2y + 8xz)\bar{k}$$

$$(\nabla\phi)_{(1,-2,-1)} = 8\bar{i} - \bar{j} - 10\bar{k}$$

Unit normal to the surface

$f(x,y,z)$ is $\frac{\nabla f}{|\nabla f|} = \frac{\left(\frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}\right)}{|\nabla f|}$

at $(-1,2,1) \rightarrow$ $= \frac{\log z\,\bar{i} + (-2y)\bar{j} + \frac{x}{z}\bar{k}}{\sqrt{16+1}}$

$$= \frac{-4\bar{j} - \bar{k}}{\sqrt{17}}$$

$\therefore$ Directional derivative $= \nabla\phi \cdot \frac{\nabla f}{|\nabla f|}$ $= \frac{(8\bar{i} - \bar{j} - 10\bar{k})(-4\bar{j} - \bar{k})}{\sqrt{17}}$

H.W

1) Find the directional derivative of $\nabla . \nabla \phi$ at the point $(1,-2,1)$ in the direction of the normal to the surface $xy^2z = 3x+z^2$ where $\phi = 2x^3y^2z^4$.

Ans: $1724/\sqrt{21}$

2) Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at $(-1, 1, 2)$

Ans: $a = 5/2$ $b = 1$

3) Find the unit normal vector to the surface $z = x^2+y^2$ at $(-1,-2,5)$

Ans: $= -\frac{1}{\sqrt{21}}(2i+4j+k)$

4) Find the angle between the surface $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$.

Ans: $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

5) If $\bar{a}$ is constant vector then prove that $\text{grad}(\bar{a}.\bar{r}) = \bar{a}$.

proof:- Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ where a_1, a_2, a_3 are constants.

$\bar{a}.\bar{r} = (a_1i + a_2j + a_3\bar{k})(xi+yj+zk) = a_1x + a_2y + a_3z$

$\text{grad}(\bar{a}.\bar{r}) = \nabla(\bar{a}.\bar{r}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$

$= a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$

$\therefore \boxed{\text{grad}(\bar{a}.\bar{r}) = \bar{a}}$

Divergence of a vector

Let $\bar{f}$ be any continuously differentiable vector point function, then $\bar{i}\cdot\frac{\partial \bar{f}}{\partial x} + \bar{j}\cdot\frac{\partial \bar{f}}{\partial y} + \bar{k}\cdot\frac{\partial \bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is written as $\operatorname{div}\bar{f}$.

i.e., $\operatorname{div}\bar{f} = \bar{i}\cdot\frac{\partial \bar{f}}{\partial x} + \bar{j}\cdot\frac{\partial \bar{f}}{\partial y} + \bar{k}\cdot\frac{\partial \bar{f}}{\partial z}$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot \bar{f}$$

$$\operatorname{div}\bar{f} = \nabla\cdot\bar{f}$$ but

properties

1) $\operatorname{div}\bar{f} = \sum \bar{i}\left(\frac{\partial \bar{f}}{\partial x}\right)$

2) $\operatorname{div}(\bar{f}\pm\bar{g}) = \operatorname{div}\bar{f} \pm \operatorname{div}\bar{g}$

3) $(\bar{a}\cdot\nabla)\phi = \sum(\bar{a}\cdot\bar{i})\frac{\partial\phi}{\partial x}$ if ϕ is a scalar & $\bar{f}$ is a vector function

4) $(\bar{a}\cdot\nabla)\bar{f} = \sum(\bar{a}\cdot\bar{i})\frac{\partial\bar{f}}{\partial x}$ " "

Solenoidal vector

A vector point function $\bar{f}$ is said to be solenoidal if $\operatorname{div}\bar{f} = 0$

This equation is also called the equation of continuity (or) Conservation of mass.

Note:- If the vector function $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then

$$\operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

problems

① If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3xyz^2\bar{k}$ find $\operatorname{div}\bar{f}$ at $(1,-1,1)$

Sol:- Given $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$

$$\operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$$

$$\operatorname{div}\bar{f} = y^2 + 2x^2z - 6yz$$

$$(\operatorname{div}\bar{f})_{(1,-1,1)} = 1+2+6 = 9$$

② If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is solenoidal then find p.

Sol:- Let $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$

$= f_1 i + f_2 j + f_3 k$ then

$$\operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 1+1+p$$

$$= 2+p$$

Since $\bar{f}$ is solenoidal, we have $\operatorname{div}\bar{f} = 0$

$$\Rightarrow 2+p = 0$$

$$\Rightarrow \boxed{p = -2}$$

(3) Find $\operatorname{div}\bar{f}$ where $\bar{f} = r^n \bar{r}$. Find n if it is solenoidal?

(or)

Prove that $r^n \bar{r}$ is solenoidal if $n = -3$.

(or)

Prove that $\operatorname{div}(r^n \bar{r}) = (n+3) r^n$. Hence show that $\frac{\bar{r}}{r^3}$ is solenoidal.

Sol:- Given $\bar{f} = r^n \bar{r}$.

where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$

$\Rightarrow r^2 = x^2 + y^2 + z^2$

$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x$

$\Rightarrow x\frac{\partial r}{\partial x} = \frac{x^2}{r}$

$\bar{f} = r^n(x\bar{i} + y\bar{j} + z\bar{k})$

$$\operatorname{div}\bar{f} = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= r^n + x n r^{n-1}\frac{\partial r}{\partial x} + r^n + y n r^{n-1}\frac{\partial r}{\partial y} + r^n + z n r^{n-1}\frac{\partial r}{\partial z}$$

$$= 3r^n + nr^{n-1}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

$$= 3r^n + nr^{n-1}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right]$$

$$= 3r^n + \frac{nr^{n-1}}{r}(r^2)$$

$$= 3r^n + nr^n$$

$$\boxed{\operatorname{div}\bar{f} = (3+n)r^n}$$

If $n = -3$

$\boxed{\operatorname{div}\bar{f} = 0} \Rightarrow$ solenoidal.

4. Show that $\frac{\bar{r}}{r^3}$ is solenoidal

(or)

Evaluate $\nabla \cdot \left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$

Sol:- We have $r = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\bar{r}}{r^3} = \bar{r} \cdot r^{-3} = r^{-3} xi + r^{-3} yj + r^{-3} zk = f_1 i + f_2 j + f_3 k$$

Hence $$\nabla \cdot \left(\frac{\bar{r}}{r^3}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \quad \text{--- (1)}$$

$$f_1 = r^{-3} x$$

$$\frac{\partial f_1}{\partial x} = r^{-3} \cdot 1 + x \cdot (-3) r^{-4} \frac{\partial r}{\partial x} = r^{-3} - 3x^2 r^{-5}$$

~~Similarly $\frac{\partial f_2}{\partial y} =$~~

From (1)

$$\nabla \left(\frac{\bar{r}}{r^3}\right) = \sum \frac{\partial f}{\partial x} = 3r^{-3} - 3r^{-5} \sum x^2$$

$$= 3r^{-3} - 3r^{-5}(x^2+y^2+z^2)$$

$$= 3r^{-3} - 3r^{-5} r^2 \quad (\because r^2 = x^2+y^2+z^2)$$

$$= 3r^{-3} - 3r^{-3}$$

$$\therefore \nabla \left(\frac{\bar{r}}{r^3}\right) = 0$$

$\therefore$ Hence $\frac{\bar{r}}{r^3}$ is solenoidal.

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* Find $\text{div}\left(\frac{\bar{r}}{r}\right)$

(5) Find div $\bar{f}$ when $\bar{f} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$

Sol:- Let $\phi = x^3 + y^3 + z^3 - 3xyz$

$$\text{grad}\,\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

$$\bar{f} = \text{grad}\,\phi = \bar{i}(3x^2 - 3yz) + j(3y^2 - 3xz) + k(3z^2 - 3xy)$$

Now div $\bar{f} = \nabla \cdot \bar{f}$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 6x + 6y + 6z$$

$$= 6(x + y + z) \quad //$$

Curl of a vector

Def:- Let $\bar{f}$ be any differentiable vector point function, then the vector function defined by $\bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z}$ is called Curl of $\bar{f}$ and is denoted by Curl $\bar{f}$ (or) $(\nabla \times \bar{f})$.

$$\therefore \text{Curl}\,\bar{f} = \bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z}$$

$$\boxed{\text{curl}\,\bar{f} = \sum \bar{i} \times \frac{\partial \bar{f}}{\partial x}}$$

properties

1) If $\bar{f}$ is a differiable vector point function given by $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

then $\cancel{\text{curl}\,\bar{f} = i \times \frac{\partial \bar{f}}{\partial x} + j \times \frac{\partial \bar{f}}{\partial y} + k \times \frac{\partial \bar{f}}{\partial z}}$

$$\text{curl}\,\bar{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

2) $\text{curl}(\bar{a} \pm \bar{b}) = \text{curl}\,\bar{a} \pm \text{curl}\,\bar{b}$

Note:- If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ & $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$ then

$$\text{Curl}\,\bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Irrotational vector:- A vector $\bar{f}$ is said to be irrotational vector then $\text{curl}\,\bar{f} = 0$ //

problems

① If $\bar{f} = xy^2 i + 2x^2yz\,j + 3xyz^2 k$ then find $\text{Curl}\,\bar{f}$ at the point $(1,-1,1)$

Sol:- Let $\bar{f} = xy^2 i + 2x^2yz\,j - 3yz^2 k$ then

$$\text{Curl}\,\bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right) - \bar{j}\left(\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right)$$
$$+ \bar{k}\left(\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right)$$

$$\text{curl}\,\bar{f} = \bar{i}(-3z^2 - 2x^2y) + \bar{k}(4xyz - 2xy)$$

at $(1,-1,1)$

$$\text{curl}\,\bar{f} = \bar{i}(-3+2) + \bar{k}(-4+2)$$
$$= -\bar{i} - 2\bar{k} \quad //$$

(2) Find curl $\bar{f}$ where $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$ then

$$\bar{f} = \text{grad}\,\phi = \sum i \frac{\partial \phi}{\partial x}$$

$$= i\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) + j\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz) + k\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$\bar{f} = i(3x^2-3yz) + j(3y^2-3xz) + k(3z^2-3xy)$$

Now

$$\text{Curl}\,\bar{f} = \nabla \times \bar{f}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= i(-3x-(-3x)) - j(-3y+3y) + k(-3z+3z)$$

$$= 0$$

Curl $\bar{f} = 0$

ie; $\bar{f}$ is a irrotational vector

(3) If $\bar{F} = (x+y+1)i + j - (x+y)k$ then show that $\bar{F} \cdot \text{curl}\,\bar{F} = 0$

Sol:- $$\text{Curl}\,\bar{F} = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$= -i + j - k$$

$\therefore \bar{F} \cdot \text{curl} F = \left[(x+y+1) i + j - (x+y) k\right] \cdot \left[-i + j - k\right]$

$= -x - y - 1 + 1 + x + y = 0$ //

(4) Prove that if $\bar{r}$ is the position vector of any point in space, then $r^n \bar{r}$ is irrotational.

(or)

Show that $\text{curl}(r^n \bar{r}) = 0$

proof:-

$\bar{r} = xi + yj + zk \qquad r = |\bar{r}|$

$\Rightarrow r^2 = x^2 + y^2 + z^2 \qquad \Rightarrow 2r \frac{\partial r}{\partial y} = 2y$

$\frac{\partial r}{\partial y} = \frac{y}{r}$

Now $r^n \bar{r} = r^n (xi + yj + zk)$

$\text{Curl}(r^n \bar{r}) = \nabla \times (r^n \bar{r})$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xr^n & yr^n & zr^n \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y} zr^n - \frac{\partial}{\partial z} yr^n\right) - j\left(\frac{\partial}{\partial x} zr^n - \frac{\partial}{\partial z} xr^n\right) + k\left(\frac{\partial}{\partial x} yr^n - \frac{\partial}{\partial y} xr^n\right)$$

$$= i\left(znr^{n-1}\frac{\partial r}{\partial y} - ynr^{n-1}\frac{\partial r}{\partial z}\right) - j\left(zn.r^{n-1}\frac{\partial r}{\partial x} - xnr^{n-1}\frac{\partial r}{\partial z}\right)$$

$$+ k\left(ynr^{n-1}\frac{\partial r}{\partial y} - xn.r^{n-1}\frac{\partial r}{\partial y}\right)$$

$= \cancel{i(znr^{n-1}\frac{y}{r}}$

$= nr^{n-2}\left[(zy - yz) i + (xz - zx) j + (xy - yx) k\right]$

$= 0$ //

⑤ If $\bar{A}$ is irrotational vector, evaluate $\text{div}(A\times\bar{r})$ where $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

Sol:- We have $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

Given $\bar{A}$ is an irrotational vector

i.e. $\nabla\times\bar{A} = 0$

Now ~~div ($\nabla\times$~~

$a.(b\times c) = c.(\bar{a}\times b) - b.(a\times c)$

$$\text{div}(A\times\bar{r}) = \nabla.(\bar{A}\times\bar{r})$$

$$= \bar{r}.(\nabla\times A) - \bar{A}.(\nabla\times\bar{r}) \quad (\because \nabla.(a\times b) = b.(\nabla\times a) - a.(\nabla\times b))$$

$$= \bar{r}.(0) - \bar{A}.(\nabla\times\bar{r})$$

$$= -\bar{A}.(\nabla\times\bar{r})$$

Here $\nabla\times\bar{r} = i\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - j\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + k\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$

$\therefore A.(\nabla\times\bar{r}) = 0$

$$\therefore \boxed{\text{div}(A\times\bar{r}) = 0}$$

⑥ Find constants a, b, c so that the vector $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j}+(4x+cy+2z)\bar{k}$ is irrotational. Also find ϕ such that $\bar{A} = \nabla\phi$

Sol:- Given $\bar{A} = (x+2y+az)\bar{i}+(bx-3y-z)\bar{j}+(4x+cy+2z)\bar{k}$

A is irrotational $\Rightarrow$ Curl $\bar{A} = 0$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$= \bar{i}(c+1) + \bar{j}(a-4) + \bar{k}(b-2) = 0 = 0\bar{i} + 0\bar{j} + 0\bar{k}$

Comparing $\longrightarrow$ $c+1 = 0 \Rightarrow c = -1$

$a = 4, \quad b = 2$

$\therefore \bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k}$

Now $\bar{A} = \nabla\phi$

$\Rightarrow (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k} = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = x+2y+4z \qquad \frac{\partial\phi}{\partial y} = 2x-3y-z$

$\Rightarrow \phi = \frac{x^2}{2} + 2xy + 4xz \qquad \phi = 2xy - \frac{3y^2}{2} - zy$

$\frac{\partial\phi}{\partial z} = 4x-y+2z \Rightarrow \phi = 4xz - yz + z^2$ //

H.W

① If $\bar{a}$ is a constant vector, prove that curl $\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{a}\cdot\bar{r})$

② Find whether the function $\bar{F} = (x^3-y^3)i + (y^2-3x)j + (z^2-xy)k$ is irrotational and hence find scalar potential function.

Operators

① **Vector differential operator :- (∇)**

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$$

ie; $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

ie; $\nabla\phi = \text{grad}\,\phi = \sum i\frac{\partial\phi}{\partial x}$

② Scalar differential operator $(\bar{a}.\nabla)$

$\bar{a}.\nabla = (\bar{a}.\bar{i})\frac{\partial}{\partial x} + (\bar{a}.\bar{j})\frac{\partial}{\partial y} + (\bar{a}.\bar{k})\frac{\partial}{\partial z}$ is defined s.t

ie; i) $(\bar{a}.\nabla)\phi = (\bar{a}.\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}.\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}.\bar{k})\frac{\partial\phi}{\partial z}$

ii) $(\bar{a}.\nabla)\bar{f} = (\bar{a}.\bar{i})\frac{\partial\bar{f}}{\partial x} + (\bar{a}.\bar{j})\frac{\partial\bar{f}}{\partial y} + (\bar{a}.\bar{k})\frac{\partial\bar{f}}{\partial z}$ //

③ Vector differential operator $(\bar{a}\times\nabla)$

$\bar{a}\times\nabla = (\bar{a}\times\bar{i})\frac{\partial}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial}{\partial z}$ is defined s.t

i) $(\bar{a}\times\nabla)\phi = (\bar{a}\times\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial\phi}{\partial z}$

ii) $(\bar{a}\times\nabla).\bar{f} = (\bar{a}\times\bar{i})\frac{\partial\bar{f}}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial\bar{f}}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial\bar{f}}{\partial z}$

iii) $(\bar{a}\times\nabla)\times\bar{f} = (a\times\bar{i})\times\frac{\partial\bar{f}}{\partial x} + (\bar{a}\times\bar{j})\times\frac{\partial\bar{f}}{\partial y} + (\bar{a}\times\bar{k})\times\frac{\partial\bar{f}}{\partial z}$

④ Scalar differential operator $\nabla.$

$\nabla = \bar{i}.\frac{\partial}{\partial x} + \bar{j}.\frac{\partial}{\partial y} + \bar{k}.\frac{\partial}{\partial z}$ is defined s.t

$\nabla.\bar{f} = \bar{i}.\frac{\partial\bar{f}}{\partial x} + \bar{j}.\frac{\partial\bar{f}}{\partial y} + \bar{k}.\frac{\partial\bar{f}}{\partial z} = \text{div}\bar{f}.$

⑤ Vector differential operator $\nabla\times$

$\nabla\times = i\times\frac{\partial}{\partial x} + j\times\frac{\partial}{\partial y} + k\times\frac{\partial}{\partial z}$ is defined s.t

$$\nabla \times f = \bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z} = \text{curl} \bar{f}$$

(6) Laplacian Operator ∇^2

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ is called Laplacian operator.

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \sum \frac{\partial^2 \phi}{\partial x^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi.$$

Note:- (1) $\nabla^2 \phi = \nabla \cdot \nabla \phi = \text{div}(\text{grad} \phi)$

(2) If $\nabla^2 \phi = 0$ then ϕ is said to satisfy Laplacian equation. This ϕ is called a ~~harmoni~~ harmonic function.

Problems

(1) Prove that $\text{div}(\text{grad}\, r^m) = m(m+1) r^{m-2}$

(or)

$$\nabla^2 (r^m) = m(m+1) r^{m-2} \text{ (or) } \nabla^2 (r^n) = n(n+1) r^{n-2}$$

proof:- Let $\bar{r} = xi + yj + zk$ and $r = |\bar{r}| \Rightarrow r^2 = x^2 + y^2 + z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now $\text{grad}\, r^m = \sum i \frac{\partial}{\partial x}(r^m)$

$$= \sum i . m . r^{m-1} \frac{\partial r}{\partial x}$$

$$= \sum i . m r^{m-1} . \left(\frac{x}{r}\right)$$

$$= \sum i m r^{m-2} x$$

Also $\text{div}(\text{grad}\, r^m) = \nabla . \text{grad}\, r^m$

$$= \sum \frac{\partial}{\partial x} m r^{m-2} x$$

$$= m\left[(m-2) r^{m-3} \frac{\partial r}{\partial x} . x + r^{m-2}\right]$$

$$= m\left[\sum (m-2) r^{m-4} . x^2 + r^{m-2}\right]$$

$$= m\left[(m-2)r^{m-4}\sum x^2 + \sum r^{m-2}\right]$$

$$= m\left[(m-2)r^{m-4}(r^2) + 3r^{m-2}\right]$$

$$= m\left[(m-2)r^{m-2} + 3r^{m-2}\right]$$

$$= m\left[(m-2+3)r^{m-2}\right]$$

$$\boxed{\text{div}(\text{grad } r^m) = m(m+1)r^{m-2}}$$

(or)

$$\boxed{\nabla^2(r^m) = m(m+1)r^{m-2}}$$ //

② Prove that i) $(\bar{f}\times\nabla)\cdot r = 0$ ii) $(\bar{f}\times\nabla)\times\bar{r} = -2\bar{f}$

Sol:- i) $(\bar{f}\times\nabla)\cdot r = \sum(\bar{f}\times i)\cdot\frac{\partial\bar{r}}{\partial x}$

$$= \sum(\bar{f}\times i)\cdot i = 0$$

ii) $\bar{f}\times\nabla = (\bar{f}\times i)\frac{\partial}{\partial x} + (\bar{f}\times j)\frac{\partial}{\partial y} + (\bar{f}\times k)\frac{\partial}{\partial z}$

$$(\bar{f}\times\nabla)\times\bar{r} = (\bar{f}\times i)\times\frac{\partial r}{\partial x} + (\bar{f}\times j)\times\frac{\partial r}{\partial y} + (\bar{f}\times k)\times\frac{\partial r}{\partial z}$$

$$= \sum(\bar{f}\times i)\times i \qquad (\because (a\times b)\times c = (a\cdot c)b - (b\cdot c)a$$

$$= \sum[(\bar{f}\cdot i)i - \bar{f}] \;?$$

$$= (\bar{f}\cdot i)i + (\bar{f}\cdot j)j + (\bar{f}\cdot k)k - 3\bar{f}$$

$$= \cancel{3\bar{f}}\;\bar{f} - 3\bar{f} = -2\bar{f}$$

③ If $f = (x^2+y^2+z^2)^{-n}$ then find div grad f and determine n if div grad $f = 0$

Sol:- Let $f = (x^2+y^2+z^2)^{-n}$ and $\bar{r} = xi + yj + zk$

$r = |\bar{r}| \Rightarrow r^2 = x^2 + y^2 + z^2$

$f(r) = (r^2)^{-n} = r^{-2n}$

$f'(r) = -2n\, r^{-2n-1}$

$f''(r) = (-2n)(-2n-1)\, r^{-2n-2} = 2n(2n+1)\, r^{-2n-2}$

$\text{div grad} f = \nabla^2 f(r)$

$= f''(r) + \frac{2}{r} f'(r)$ By Theorem $\left(\because \nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)\right)$

$= (2n)(2n+1)\, r^{-2n-2} - 4n\, r^{-2n-2}$

$= r^{-2n-2}\left[2n(2n+1) - 4n\right]$

$= (2n)(2n-1)\, r^{-2n-2}$

Since $\text{div grad} \bar{f} = 0$

$\Rightarrow 2n(2n-1)\, r^{-2n-2} = 0$

$\Rightarrow 2n(2n-1) = 0$

$\Rightarrow 2n = 0$ or $2n - 1 = 0$

$n = 0$ or $n = 1/2$

H.W

1. Prove that $\nabla \times \left(\frac{\bar{A} \times \bar{r}}{r^n}\right) = \frac{(2-n)\bar{A}}{r^n} + \frac{n(\bar{r}.\bar{A})\bar{r}}{r^{n+2}}$

2. ~~Find~~ Show that $\nabla^2[f(r)] = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = f''(r) + \frac{2}{r} f'(r)$

where $r = |\bar{r}|$.

Vector Identities

① Prove that $\text{curl}(\phi\bar{a}) = (\text{grad}\,\phi)\times\bar{a} + \phi\,\text{curl}\,\bar{a}$

proof:- $\text{Curl}(\phi\bar{a}) = \nabla\times(\phi\bar{a})$

$$= \sum i\times\frac{\partial}{\partial x}(\phi\bar{a})$$

$$= \sum i\times\left[\frac{\partial\phi}{\partial x}\bar{a} + \phi\frac{\partial\bar{a}}{\partial x}\right]$$

$$= \sum\left(i\frac{\partial\phi}{\partial x}\right)\times\bar{a} + \sum\left(i\times\frac{\partial\bar{a}}{\partial x}\right)\phi$$

$$= \cancel{\nabla\times\bar{a}}\ \nabla\phi\times\bar{a} + (\nabla\times\bar{a})\phi$$

$$= (\text{grad}\,\phi)\times\bar{a} + \phi\,\text{Curl}\,\bar{a}$$

② Prove that $\text{grad}(\bar{a}.\bar{b}) = (\bar{b}.\nabla)\bar{a} + (\bar{a}.\nabla)\bar{b} + \bar{b}\times\text{curl}\,\bar{a} + \bar{a}\times\text{curl}\,\bar{b}$

(or)

P.T. $\nabla(\bar{A}.\bar{B}) = (\bar{B}.\nabla)\bar{A} + (\bar{A}.\nabla)\bar{B} + \bar{B}\times(\nabla\times\bar{A}) + \bar{A}\times(\nabla\times\bar{B})$

(or)

P.T. $\text{grad}(\bar{F}.\bar{G}) = \bar{F}\times(\nabla\times\bar{G}) + \bar{G}(\nabla\times\bar{F}) + (\bar{F}.\nabla)\bar{G} + (\bar{G}.\nabla)\bar{F}$

proof:- Consider

$$\bar{a}\times\text{curl}\,\bar{b} = \bar{a}\times(\nabla\times\bar{b}) = \bar{a}\times\sum i\times\frac{\partial\bar{b}}{\partial x}$$

(∵ $a\times(b\times c)$)

$$= \sum\bar{a}\times\left(i\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$= \sum\left[\left(\bar{a}.\frac{\partial\bar{b}}{\partial x}\right)i - (\bar{a}.i)\frac{\partial\bar{b}}{\partial x}\right]$$

$$= \sum i\left\{a.\frac{\partial\bar{b}}{\partial x}\right\} - \left\{\bar{a}\sum i\frac{\partial}{\partial x}\right\}\bar{b}$$

$$\bar{a}\times\text{curl}\,\bar{b} = \sum i\left[\bar{a}.\frac{\partial\bar{b}}{\partial x}\right] - (\bar{a}.\nabla)\bar{b} \quad \text{—— ①}$$

Similarly

$$\bar{b}\times\text{curl}\,\bar{a} = \sum i\left(\bar{b}.\frac{\partial\bar{a}}{\partial x}\right) - (\bar{b}.\nabla)\bar{a} \quad \text{—— ②}$$

$$(1)+(2) \Rightarrow (\bar{a}\times \operatorname{curl}\bar{b})+(\bar{b}\times \operatorname{curl}\bar{a}) = \sum i\left(\bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right)-(\bar{b}\cdot\nabla)\bar{a} + \sum i\left(\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}\right)-(\bar{a}\cdot\nabla)\bar{b}$$

$$\Rightarrow R.H.S = \cancel{(\bar{b}\cdot\nabla)\bar{a}} + \cancel{(\bar{a}\cdot\nabla)\bar{b}} + \sum i\left(\bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right) - \cancel{(\bar{b}\cdot\nabla)\bar{a}} + \sum i\left(\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}\right) - \cancel{(\bar{a}\cdot\nabla)\bar{b}}$$

$$= \sum i\left(\bar{a}\frac{\partial b}{\partial x} + \bar{b}\cdot\frac{\partial a}{\partial x}\right)$$

$$= \sum i\frac{\partial}{\partial x}(\bar{a}\cdot\bar{b})$$

$$= \nabla(\bar{a}\cdot\bar{b})$$

$$= \operatorname{grad}(\bar{a}\cdot b) = L.H.S$$

(3) P.T. $\operatorname{div}(\bar{a}\times\bar{b}) = \bar{b}\cdot\operatorname{curl}\bar{a} - \bar{a}\cdot\operatorname{curl}\bar{b}$.

(or)

$$\nabla\cdot(\bar{a}\times\bar{b}) = \bar{b}\cdot(\nabla\times\bar{a}) - \bar{a}\cdot(\nabla\times\bar{b})$$

proof

$$\operatorname{div}(\bar{a}\times\bar{b}) = \sum i\cdot\frac{\partial}{\partial x}(\bar{a}\times\bar{b})$$

$$= \sum i\cdot\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$= (a\cdot c)b - (a\cdot b)\bar{c}$

$$= \sum i\cdot\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right) + \sum i\left(\bar{a}\times\frac{\partial b}{\partial x}\right)$$

$$= \cancel{\sum\left(i\times\frac{\partial a}{\partial x}\right)\cdot\bar{b} + \sum i\times(a)}$$

$$= \sum\left(i\times\frac{\partial a}{\partial x}\right)\cdot\bar{b} - \sum\left(i\times\frac{\partial b}{\partial x}\right)\cdot\bar{a}$$

$$= (\nabla\times\bar{a})\cdot\bar{b} - (\nabla\times\bar{b})\cdot\bar{a}$$

$$= \bar{b}\cdot\operatorname{curl}\bar{a} - \bar{a}\cdot\operatorname{curl}\bar{b}$$

④ Prove that curl grad $\phi = 0$

proof:- Let ϕ be any scalar point function, then grad $\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

Curl grad ϕ = $\nabla \times$ grad ϕ

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - \bar{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right)$$
$$+ \bar{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) = \bar{0}$$

$\therefore$ curl grad $\phi = 0$

Thus grad ϕ is always irrotational.

Note: Similary we can prove div curl $\bar{f} = 0$

⑤ P.T. $\nabla \times (\nabla \times a) = \nabla(\nabla . \bar{a}) - \nabla^2 \bar{a}$

proof:- $\sum i \times \frac{\partial}{\partial x}(\nabla \times a)$ = $\sum i \times \frac{\partial}{\partial x}\left(\nabla \times \bar{a}\right)$

$$= \sum i \times \frac{\partial}{\partial x}\left[i \times \frac{\partial\bar{a}}{\partial x} + j \times \frac{\partial\bar{a}}{\partial y} + \bar{k} \times \frac{\partial\bar{a}}{\partial z}\right]$$

$$= \sum i \times \left[\bar{i} \times \frac{\partial^2\bar{a}}{\partial x^2} + \bar{j} \times \frac{\partial^2\bar{a}}{\partial x\partial y} + \bar{k} \times \frac{\partial^2\bar{a}}{\partial x\partial z}\right]$$

$$= \sum i \times \left[i \times \frac{\partial^2\bar{a}}{\partial x^2}\right] + i \times \left[j \times \frac{\partial^2\bar{a}}{\partial x\partial y}\right] + i \times \left[k \times \frac{\partial^2\bar{a}}{\partial x\partial z}\right]$$

$$= \sum \left[i . \frac{\partial^2\bar{a}}{\partial x^2}\right] . i - \frac{\partial^2\bar{a}}{\partial x^2} + \left(i . \frac{\partial^2\bar{a}}{\partial x\partial y}\right) j + \left[i . \frac{\partial^2\bar{a}}{\partial x\partial z}\right] \bar{k}$$

$$= \sum i \frac{\partial}{\partial x}\left(i . \frac{\partial\bar{a}}{\partial x}\right) + j\frac{\partial}{\partial y}\left[i . \frac{\partial\bar{a}}{\partial x}\right] + k\frac{\partial}{\partial z}\left[i . \frac{\partial\bar{a}}{\partial x}\right] - \frac{\partial^2\bar{a}}{\partial x^2}$$

$$= \nabla . \left(i . \frac{\partial\bar{a}}{\partial x}\right) - \frac{\partial^2\bar{a}}{\partial x^2}$$

$$= \sum i \times \frac{\partial}{\partial x}(\nabla \times \bar{a}) = \nabla \sum i \cdot \frac{\partial \bar{a}}{\partial x} - \sum \frac{\partial^2 \bar{a}}{\partial x^2}$$

$$= \nabla(\nabla \cdot \bar{a}) - \left[\frac{\partial^2 \bar{a}}{\partial x^2} + \frac{\partial^2 \bar{a}}{\partial y^2} + \frac{\partial^2 \bar{a}}{\partial z^2}\right]$$

$$\nabla \times (\nabla \times \bar{a}) = \nabla(\nabla \cdot a) - \nabla^2 \bar{a}$$

$$\boxed{\text{Curl curl } \bar{a} = \text{grad div } \bar{a} - \nabla^2 \bar{a}}$$

6) Prove that $(\nabla f \times \nabla g)$ is solenoidal.

proof:- w.k.T. $\operatorname{div}(\bar{a} \times \bar{b}) = \bar{b} \operatorname{curl} \bar{a} - \bar{a} \operatorname{curl} \bar{b}$

~~Take~~ by taking $\bar{a} = \nabla f$ & $\bar{b} = \nabla g$

$$\operatorname{div}(\nabla f \times \nabla g) = \nabla g \operatorname{curl}(\nabla f) - \nabla f \cdot \operatorname{curl}(\nabla g)$$

$$= 0 - 0 \qquad (\because \operatorname{curl}(\nabla f) = \operatorname{curl}(\nabla g) = 0)$$

$$\operatorname{div}(\nabla f \times \nabla g) = 0$$

$\therefore$ $\nabla f \times \nabla g$ is solenoidal.

7) Prove that $\nabla\left(\nabla \cdot \frac{\bar{r}}{r}\right) = -\frac{2}{r^3}\bar{r}$.

proof:- $\nabla \cdot \left(\frac{\bar{r}}{r}\right) = \sum i \frac{\partial}{\partial x}\left(\frac{\bar{r}}{r}\right)$

$$= \sum i \cdot \left[\frac{1}{r}\frac{\partial \bar{r}}{\partial x} + \bar{r}\left(-\frac{1}{r^2}\right)\left(\frac{x}{r}\right)\right]$$

$$= \sum i\left[\frac{1}{r} i - \frac{\bar{r}}{r^3} x\right]$$

$$= \frac{1}{r}\sum i \cdot i - \frac{1}{r^3} r^2$$

$$= \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

$$\nabla\left(\nabla \cdot \frac{\bar{r}}{r}\right) = \nabla\left(\frac{2}{r}\right) = \sum i \frac{\partial}{\partial x}\left(\frac{2}{r}\right) = \sum i\left(-\frac{2}{r^2}\right)\left(\frac{x}{r}\right)$$

$$= \frac{-2}{r^3}\sum x i$$

$$= \frac{-2\bar{r}}{r^3}$$

(8) Find $(A.\nabla)\phi$ at $(1,-1,1)$ if $A = 3xyz^2 i + 2xy^3 j - x^2yz k$ and $\phi = 3x^2 - yz$.

Sol:- Given $A = 3xyz^2 i + 2xy^3 j - x^2yz k$ and $\phi = 3x^2 - yz$

W.K.T. $(A.\nabla)\phi = (A.i)\frac{\partial\phi}{\partial x} + (A.j)\frac{\partial\phi}{\partial y} + (A.k)\frac{\partial\phi}{\partial z}$

$= (3xyz^2)(6x) + (2xy^3)(-z) + (-x^2yz)(-y)$

$(A.\nabla)\phi = 18x^2yz^2 - 2xy^3z + x^2y^2z$

$(A.\nabla)\phi$ at $(1,-1,1) = 18(1)(-1)(1) - 2(1)(-1)(1) + 1(-1)(1)$

$= -18 + 2 + 1$

$= -15.$

(9) Evaluate $\nabla.\left[r.\nabla\left(\frac{1}{r^3}\right)\right]$ where $r = \sqrt{x^2+y^2+z^2}$

Sol:- Given $r = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2$

$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$\therefore \nabla\left(\frac{1}{r^3}\right) = grad(r^{-3})$

$= \sum i \frac{\partial}{\partial x}(r^{-3})$

$= \sum i(-3r^{-4})\frac{\partial r}{\partial x}$

$= \sum i(-3r^{-4})\frac{x}{r}$

$\nabla\left(\frac{1}{r^3}\right) = -3r^{-5}(xi+yj+zk)$

$r.\nabla\left(\frac{1}{r^3}\right) = -3r^{-4}(xi+yj+zk) = f_1 i + f_2 j + f_3 k$

$\nabla\left[r.\nabla\left(\frac{1}{r^3}\right)\right] = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \sum \frac{\partial f_1}{\partial x}$

$f_1 = -3r^{-4}x$

$\frac{\partial f_1}{\partial x} = -3r^{-4}.1 + 12r^{-5}x.\frac{\partial r}{\partial x} = -3r^{-4} + 12r^{-5}.x.\frac{x}{r}$

$= -3r^{-4} + 12r^{-6}.x^2$

$$\nabla\cdot\left(r\cdot\nabla\left(\frac{1}{r^3}\right)\right) = \sum\left(-3r^{-4} + 12r^{-6}x^2\right)$$

$$= -3(3r^{-4}) + 12r^{-6}\sum x^2$$

$$= -9r^{-4} + 12r^{-6}r^2$$

$$= -9r^{-4} + 12r^{-4}$$

$$= 3r^{-4}$$

$$\therefore \nabla\left(r\cdot\nabla\left(\frac{1}{r^3}\right)\right) = \frac{3}{r^4} \quad //$$

– Vector Integration –

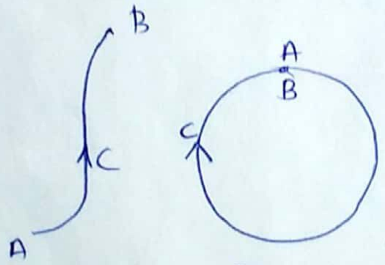
Definite integral :– If $\int \bar{f}(t)dt = \bar{F}(t) + c$ then $\int_a^b \bar{f}(t)dt = \bar{F}(b) - \bar{F}(a)$

This is called the definite integral of $\bar{f}(t)$ between the limits $t = a$ and $t = b$

If $\bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$ then

$$\int_a^b \bar{f}(t)dt = \bar{i}\int_a^b f_1(t)dt + \bar{j}\int_a^b f_2(t)dt + \bar{k}\int_a^b f_3(t)dt$$

Closed Curve :– Let C be ~~the~~ a curve in space. Let A be the initial point and B be the terminal point of the curve C. When the direction along C oriented from A to B is positive then the direction from B to A is called negative direction. If the ~~two~~ two points A and B coinside the curve C is called the closed curve.



Smooth curve :– A curve $\bar{r} = \bar{f}(t)$ is called a smooth curve if $\bar{f}(t)$ is continuously differentiable.

Line integral :– Any integral which is to be evaluated along a curve is called a line integral.

Circulation :– If $\bar{v}$ represents the velocity of a fluid particle and C is a closed curve, then the integral $\oint_C \bar{v}.d\bar{r}$ is called the circulation of $\bar{v}$ round the curve C.

If $\oint_C \bar{v}.d\bar{r} = 0$ then the field $\bar{v}$ is called Conservative.

Work done by a force:— If $\bar{F}$ represents the force vecter acting on a particle moving along an arc AB, then the total work done by $\bar{F}$ during displacement from A to B is given by the line integral

$$\int_A^B \bar{F}\cdot d\bar{r}.$$

Note:— If $\bar{F}$ is Conservative, $\bar{F} = \nabla\phi$ then the work done is independent of the path and vice-versa

ie; $\bar{F} = \nabla\phi$

$\Rightarrow$ Curl $\bar{F} = 0$

problem

① If $\bar{F} = 3xy\bar{i} - y^2\bar{j}$ evaluate $\int_C \bar{F}\cdot d\bar{r}$ where C is the curve $y = 2x^2$ in the xy plane from $(0,0)$ to $(1,2)$

Sol:— The equ of the curve C is $y = 2x^2$

$\Rightarrow dy = 4x\,dx$

Given $\bar{F} = 3xy\bar{i} - y^2\bar{j}$

Since the integration is performed in the xy-plane, therefore $\bar{r} = x\bar{i} + y\bar{j}$ and

$\bar{F}\cdot d\bar{r} = F_1 dx + F_2 dy$ and x varies from 0 to 1

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_C F_1 dx + F_2 dy$$

$$= \int_C 3xy\,dx - y^2 dy$$

$$= \int_0^1 3x(2x^2)\,dx - 4x^4(4x)\,dx$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx$$

$$= \left(\frac{6x^4}{4} - \frac{16x^6}{6}\right)_0^1 = -7/6$$

② Find the workdone by $\overline{F} = (2x-y-z)\bar{i} + (x+y-z)\bar{j} + (3x-2y-5z)\bar{k}$ along a curve c in the xy-plane is given by

i) $x^2+y^2=9$, $z=0$

ii) $x^2+y^2=4$, $z=0$

Sol:- we have $\overline{F} = (2x-y-z)\bar{i} + (x+y-z)\bar{j} + (3x-2y-5z)\bar{k}$

$$\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$$

$$d\bar{r} = dx\bar{i}+dy\bar{j}+dz\bar{k}$$

In the xy-plane, $z=0 \Rightarrow dz=0$

$$\therefore F.d\bar{r} = (2x-y)\,dx + (x+y)\,dy$$

i) Now work done $= \int_C F.d\bar{r}$ where C is the circle $x^2+y^2=9$

$$\text{workdone} = \int_C (2x-y)\,dx + (x+y)\,dy$$

Take $x = 3\cos\theta$, $y = 3\sin\theta$ so that

$dx = -3\sin\theta\, d\theta$, $dy = 3\cos\theta\, d\theta$

then θ varies from 0 to 2π

$$\text{work done} = \int_0^{2\pi} (6\cos\theta - 3\sin\theta)(-3\sin\theta\, d\theta) + (3\cos\theta + 3\sin\theta)(3\cos\theta\, d\theta)$$

$$= \int_0^{2\pi} (-18\cos\theta\sin\theta + 9\sin^2\theta + 9\cos^2\theta + 9\cos\theta\sin\theta)\, d\theta$$

$$= \int_0^{2\pi} (9(\cos^2\theta+\sin^2\theta) - 9\cos\theta\sin\theta)\, d\theta$$

$$= \int_0^{2\pi} (9 - 9\cos\theta\sin\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(18 - 18\sin\theta\cos\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(18 - 9\sin 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[18\theta + \frac{9\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left\{\left[36\pi + \frac{9\cos 4\pi}{2}\right] - \left[18(0) + \frac{9\cos 0}{2}\right]\right\}$$

$$\text{W.D} = \frac{1}{2}\left[36\pi + \frac{9(0)}{2}\right] = 18\pi$$

Similarly

ii) ~~W.D.~~ H.W

③ Find the work done by the force $\bar{F} = (2y+3)\bar{i} + xz\bar{j} + (yz - x)\bar{k}$ when it moves a particle from the point $(0,0,0)$ to $(2,1,1)$ along the curve $x = 2t^2$, $y = t$ and $z = t^3$

Sol:- Here $\bar{F} = (2y+3)\bar{i} + xz\bar{j} + (yz-x)\bar{k}$

$x = 2t^2 \Rightarrow dx = 4t\,dt$

$y = t \Rightarrow dy = dt$

$z = t^3 \Rightarrow dz = 3t^2dt$

t is from 0 to 1

$$\text{W.D} = \int_C \bar{F}\cdot d\bar{r} = \int (2y+3)\,dx + (xz)\,dy + (yz - x)\,dz$$

$$= \int_0^1 (2t+3)\,4t\,dt + (2t^2)(t^3)\,dt + (t^4 - 2t^2)\,3t^2dt$$

$$= \int_0^1 (3t^6 + 2t^5 - 6t^4 + 8t^2 + 12t)\,dt$$

$$= \left[\frac{3t^7}{7} + \frac{2t^6}{6} + \frac{6t^5}{5} + \frac{8t^3}{3} + \frac{2t^2}{2}\right]_0^1$$

$$= \frac{288}{35}.$$

(4) Find the workdone in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + j + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$

Sol:- $W.D = \int_C \bar{F}\cdot d\bar{r}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\bar{F} = 3x^2\bar{i} + \bar{j} + z\bar{k}$

$\bar{F}\cdot d\bar{r} = 3x^2dx + dy + zdz$

$$\int_{(0,0,0)}^{(2,1,3)} \bar{F}\cdot d\bar{r} = \int_{(0,0,0)}^{(2,1,3)} (3x^2dx + dy + dz)$$

$$= \left(x^3 + y + \frac{z^2}{2}\right)_{(0,0,0)}^{(2,1,3)}$$

$$= 8 + 1 + \frac{9}{2} = \frac{27}{2}$$

(5) If $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve C in xy-plane $y = x^3$ from $(1,1)$ to $(2,8)$.

Sol:- Given $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$

Along the curve $y = x^3 \Rightarrow dy = 3x^2dx$

$\therefore \bar{F} = (5x^4 - 6x^2)\bar{i} + (2x^3 - 4x)\bar{j}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + \cancel{dz}$

$= dx\bar{i} + 3x^2dx\,\bar{j}$

$$F.d\bar{r} = \left[(5x^4-6x^2)\bar{i} + (2x^3-4x)\bar{j}\right].\left[dx\bar{i} + 3x^2dx\bar{j}\right]$$

$$= (5x^4-6x^2)dx + (2x^3-4x)\,3x^2dx$$

$$= (6x^5 + 5x^4 - 12x^3 - 6x^2)dx$$

$$\int_C F.d\bar{r} = \int_1^2 (6x^5+5x^4-12x^3-6x^2)dx$$

$$= \left[\frac{6x^6}{6} + \frac{5x^5}{5} - \frac{12x^4}{4} - \frac{6x^3}{3}\right]_1^2$$

$$= 16\left[(4+2-3-1)\right] - \left[1+1-3-2\right] = 32+3 = 35$$

P.T. $\bar{F} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k}$ is Conservative.

H.W

① If $\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$ and C is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t=0$ to $t=1$. Evaluate $\int_C \bar{F}.d\bar{r}$.

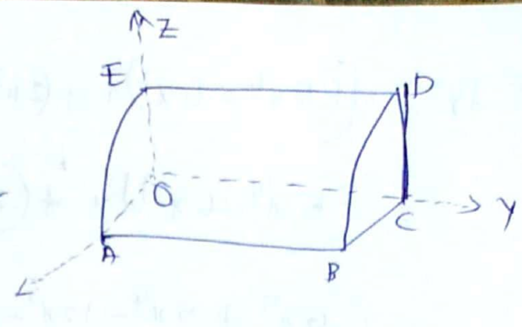
Ans: $\frac{51}{70}$

② Find the workdone by the force $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$ when it moves a particle along the arc of the curve $\bar{r} = \cos t\,\bar{i} + \sin t\,\bar{j} - t\bar{k}$ from $t=0$ to $t = 2\pi$

③ Evaluate $\oint_C (yz\,dx + xz\,dy + xy\,dz)$ over arc of a helix $x = a\cos t$, $y = a\sin t$, $z = kt$ as t varies from 0 to 2π

④ Find the work done in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$

Surface integrals



The surface integral of a vector point function $\bar{F}$ expresses the normal flux through a surface. If $\bar{F}$ represents the velocity vector of a fluid then the surface integral $\int \bar{F}.\bar{n}\,ds$ over a closed surface S represents the rate of flow of fluid through the surface.

Note:- Let R be the projection of S on YZ plane then $\int \bar{F}.\bar{n}\,ds = \iint_R \bar{F}.\bar{n} \frac{dy\,dz}{|\bar{n}.i|}$

Problems

① Evaluate $\int \bar{F}.\bar{n}\,ds$ where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$ and S is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

Sol:- Let $\phi = x^2+y^2-16$ be the surface.

then the normal to the surface S is grad ϕ.

$\therefore$ Normal $(\nabla\phi) = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = 2xi + 2yj + k(0)$

$= 2xi + 2yj$

and unit normal $(\bar{n}) = \frac{2(xi+yj)}{2\sqrt{x^2+y^2}} = \frac{xi+yj}{\sqrt{16}} = \frac{xi+yj}{4}$

Let R be the projection of S on YZ plane then

$\int_S \bar{F}.\bar{n}\,ds = \iint_R \bar{F}.\bar{n}.\frac{dy\,dz}{|\bar{n}.i|}$

$\bar{F}.\bar{n} = (z\bar{i} + x\bar{j} - 3y^2 z\bar{k})\left(\frac{x\bar{i}+y\bar{j}}{4}\right)$

$= \frac{1}{4}(xz + xy)$

$\bar{n}.i = \frac{1}{4}(xi+yj).i = \frac{x}{4}$

~~Given~~

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\frac{dy\,dz}{(\bar{n}\cdot i)}$$

$$= \iint_R \left(\frac{xz+xy}{4}\right)\left(\frac{4}{x}\right)dy\,dz$$

$$= \int_{y=0}^{4}\int_{z=0}^{5}(y+z)\,dy\,dz$$

$$= \int_0^4 y\,dy\int_0^5 dz + \int_0^4 dy\int_0^5 z\,dz$$

$$= \left(\frac{y^2}{2}\right)_0^4 (z)_0^5 + (y)_0^4\left(\frac{z^2}{2}\right)_0^5 = 40+50 = 90$$

$$= 90$$

(2) Evaluate $\iint_S \bar{F}\cdot\bar{n}\,d\bar{s}$ if ~~$\bar{F} = yz\bar{j} + 2y^2\bar{j} + xz^2\bar{k}$~~

$\bar{F} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$ and S is the surface of the cylinder $x^2+y^2=9$ contained in the first octant between the planes $z=0$ and $z=2$.

Sol:- Given $\bar{F} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$

Let $\phi = x^2+y^2-9 = 0$. then ~~$\frac{\partial\phi}{\partial x} = 2x, \frac{\partial\phi}{\partial}$~~

$$\text{grad}\,\phi = \sum \frac{\partial\phi}{\partial x}i = 2(xi+yj)$$

$$\bar{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{2(xi+yj)}{\sqrt{x^2+y^2}} = \frac{\cancel{2}(xi+yj)}{\cancel{2}\sqrt{9}} = \frac{xi+yj}{3}$$

$$\bar{F}\cdot\bar{n} = (yz\,i + 2y^2 j + xz^2 k)\left(\frac{x}{3} i + \frac{y}{3} j\right)$$

$$= \frac{xyz}{3} + \frac{2y^3}{3} = \frac{1}{3}(xyz + 2y^3)$$

Let R be the projection of S on xy plane

then $\int \bar{F}\cdot\bar{n}\, ds = \iint_R \bar{F}\cdot\bar{n} \dfrac{dy\,dz}{|n\cdot i|}$

$$= \iint_R \frac{(xyz + 2y^3)/3}{x/3}\, dy\,dz$$

For the surface ~~in the~~ in the Yz plane $x = 0 \Rightarrow y = 3$,

$z \to 0$ to 2

$$\int \bar{F}\cdot\bar{n}\, ds = \int_{y=0}^{3}\int_{z=0}^{2}\left(yz + \frac{2y^3}{x}\right) dy\,dz$$

$$= \int_{y=0}^{3}\int_{z=0}^{2}\left(yz + \frac{2y^3}{\sqrt{9-y^2}}\right) dy\,dz$$

$$= \int_0^3 \left[2y + \frac{4y^3}{\sqrt{9-y^2}}\right] dy$$

$$= \cancel{(xy)} \left(\frac{\cancel{2}y}{\cancel{2}}\right)_0^3 + 4\int_0^3 \frac{y^3}{\sqrt{9-y^2}}\, dy \quad \text{put } y = 3\sin\theta$$

then $\displaystyle\int_0^3 \frac{y^3}{\sqrt{9-y^2}}\, dy = 18$

$$= 9 + 4(18)$$

$$= \cancel{75} // \; 81$$

(3) Evaluate $\int_S \bar{F}.\bar{n}\, ds$ where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and S is the part of the surface of the plane $2x+3y+6z=12$ located in the first octant

Sol:- Let $\phi = 2x+3y+6z-12$

Normal to the plane is $\nabla\phi$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i(2) + j(3) + k(6)$$

$$\nabla\phi = 2\bar{i} + 3\bar{j} + 6\bar{k}$$

Unit normal vector $(\bar{n}) = \dfrac{2i+3j+6k}{\sqrt{4+9+36}} = \dfrac{2i+3j+6k}{7}$

Let R be the projection of S on xy-plane, then

$$\int_S \bar{F}.\bar{n}\, ds = \iint_R \frac{\bar{F}.\bar{n}\, dx\, dy}{|\bar{n}.\bar{k}|} \quad \text{where } \bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$$

Now $\bar{F}.\bar{n} = (18z\bar{i} - 12j + 3y\bar{k}) . \dfrac{(2i+3j+6k)}{7}$

$$= \frac{36}{7}z - \frac{36}{7} + \frac{18y}{7}$$

$$\bar{n}.\bar{k} = \left(\frac{2i+3j+6k}{7}\right).\bar{k} = \frac{6}{7}$$

Given plane $2x+3y+6z=12$

on xy-plane

$$2x+3y=12 \Rightarrow y = \frac{12-2x}{3}$$

When $y=0 \Rightarrow \dfrac{12-2x}{3} = 0 \Rightarrow 12-2x = 0 \Rightarrow 2x = 12$

$\Rightarrow x = 6$

Now x varies from 0 to 6

y varies from 0 to $\dfrac{12-2x}{3}$

$\therefore$ The surface integral

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \frac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{k}|}\,dx\,dy = \iint_R \frac{\left(\frac{36}{7}z - \frac{36}{7} + \frac{18y}{7}\right)}{\frac{6}{7}}\,dx\,dy$$

$$= \iint_R \left(6z - 6 + 3y\right)dx\,dy$$

$$= \iint_R \left(12 - 2x - 3y - 6 + 3y\right)dx\,dy$$

$$\left(\because 6z = 12 - 2x - 3y\right)$$

$$= \iint_R (6 - 2x)\,dx\,dy$$

$$= 2\int_0^6 \int_0^{\frac{12-2x}{3}} (3 - x)\,dx\,dy$$

$$= 2\int_0^6 \Big[(3-x)y\Big]_0^{\frac{12-2x}{3}} dx$$

$$= 2\int_0^6 (3-x)\left\{\left(\frac{12-2x}{3}\right) - 0\right\}dx$$

$$= \frac{2}{3}\int_0^6 \left(36 - 18x + 2x^2\right)dx$$

$$= \frac{4}{3}\int_0^6 \left(18 - 9x + x^2\right)dx$$

$$= \frac{4}{3}\left[18x - \frac{9x^2}{2} + \frac{x^3}{3}\right]_0^6$$

$$= \frac{4}{3}\left[108 - 162 + 72\right]$$

$$= \frac{\cancel{153}}{\cancel{3}} \times 4 \qquad \frac{4}{\cancel{3}} \times \cancel{18}^{6} = 24$$

③ Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 12x^2y\,\bar{i} - 3yz\,\bar{j} + 2z\,\bar{k}$ and S is the portion of the plane $x+y+z=1$ included in the first octant.

Ans: $\frac{149}{24}$

~~Flux of a~~

Volume integrals *

Let $F(r) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ where F_1, F_2, F_3 are functions of x, y, z. We know that $dv = dx\,dy\,dz$.

The volume integral is given by

$$\int_V F\,dv = \iiint (F_1 i + F_2 j + F_3 k)\,dx\,dy\,dz$$

$$= i\iiint F_1\,dx\,dy\,dz + j\iiint F_2\,dx\,dy\,dz + k\iiint F_3\,dx\,dy\,dz$$

problems

① If $F = (2x^2 - 3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$ then evaluate

i) $\int_V \nabla\cdot F\,dv$ and ii) $\int_V (\nabla\times F)\,dv$ where V is the closed region bounded by $x=0,\ y=0,\ z=0,\ 2x+2y+z=4$

Sol:- i) $\nabla\cdot F = i\cdot\frac{\partial F}{\partial x} + j\cdot\frac{\partial F}{\partial y} + k\cdot\frac{\partial F}{\partial z} = 4x - 2x = 2x$

The limits are: $z=0$ to $z = 4-2x-2y$

$y=0$ to $y = \frac{4-2x}{2} = 2-x$

$x=0$ to $x=2$

$$\therefore \int_V \nabla\cdot\bar{F}\,dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\,dx\,dy\,dz$$

$$= 8 \quad 8/3$$

ii) $$\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2-3z & -2xy & -4x \end{vmatrix} = \bar{j} - 2y\bar{k}$$

$$\int_V \nabla\times F\,dv = \iiint_V (\bar{j} - 2y\bar{k})\,dx\,dy\,dz$$

$$= \int_0^2\int_0^{2-x}\int_0^{4-2x-2y} (j - 2yk)\,dx\,dy\,dz$$

$$= \frac{8}{3}(\bar{j} - \bar{k})$$

H.W

① If $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$ evaluate $\int_V \bar{F}\cdot d\bar{v}$ where V is the region bounded by the surfaces $x=0$, $x=2$, $y=0$, $y=6$, $z=x^2$, $z=4$.

Ans: $\int_V \bar{F}\cdot d\bar{v} = 128\bar{i} - 24\bar{j} + 384\bar{k}$

Vector Integral theorems

There are three important vector integral theorems

i) Gauss Divergence theorem

ii) Green's theorem in plane

iii) Stoke's theorem

These theorems deal with conversion of

i) $\int_S \bar{F}.\bar{n}ds$ into a volume integral where S is a closed surface

ii) $\int_C \bar{F}.d\bar{r}$ into a double integral over a region in a plane where C is a closed curve in the plane and

iii) $\int_S (\text{Curl}\,\bar{F}).\bar{n}ds$ into a line integral around the boundary of an open two sided surface.

Gauss's Divergence theorem

(Transformation between Surface integral and Volume integral)

Let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then

$$\int_V \text{div}\bar{F}dv = \int_S \bar{F}.\bar{n}ds$$ where $\bar{n}$ is the normal vector of surface S.

① Use Divergence theorem to evaluate $\iint_S \bar{F}\cdot d\bar{s}$ where ②

$\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$ and 's' is the surface of the sphere

$x^2+y^2+z^2 = r^2$

Sol:- We have

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3(x^2+y^2+z^2)$$

∴ By Gauss Divergence theorem

$$\nabla\cdot\bar{F}\,dv = \iiint_V \nabla\cdot\bar{F}\,dv = \iiint_V 3(x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2(r^2\sin\theta\, dr\, d\theta\, d\phi)$$

[Changing into spherical polar coordinates $x = r\sin\theta\cos\phi$
$y = r\sin\theta\cdot\sin\phi,\ z = r\cos\theta$]

$$\therefore \iint_S \bar{F}\cdot ds = 3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4\sin\theta\left[\int_{\phi=0}^{2\pi} d\phi\right] dr\, d\theta$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4\sin\theta\,(2\pi-0)\,dr.d\theta =$$

$$= 6\pi\int_{r=0}^{a} r^4\left[\int_0^{\pi}\sin\theta\, d\theta\right] dr$$

$$= 6\pi\int_{r=0}^{a} r^4(-\cos\theta)_0^{\pi}\, dr = -6\pi\int_0^a r^4(\cos\pi - \cos 0)\,dr$$

$$= 12\pi\int_0^a r^4 dr = 12\pi\left[\frac{r^5}{5}\right]_0^5 = \frac{12\pi a^5}{5}$$

Note:- The polar coordinates for the sphere $x^2+y^2+z^2 = r^2$ are

$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$

The limits are $r = 0$ to a, $\theta = 0$ to π,
$\phi = 0$ to 2π. $dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$

② Use Divergence theorem to evaluate $\iint_S \bar{F}.d\bar{s}$ where (3)

$\bar{F} = 4xi - 2y^2j + z^2k$ and 's' is the surface bounded by the region

$x^2+y^2=4$, $z=0$ and $z=3$.

Sol:- we have $\text{div} F = \nabla.F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\left(4xi - 2y^2j + z^2k\right)$

$$= 4 - 4y + 2z$$

∴ By Gauss Divergence theorem

$$\iint_S \bar{F}.ds = \iiint_V \nabla.\bar{F}dv$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{3}(4-4y+2z)\,dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[(4-4y)z + \frac{2z^2}{2}\right]_0^3 dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[12(1-y) + 9\right]dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21-12y)\,dx\,dy$$

$$= \int_{-2}^{2}\left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}21\,dy - 12\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}y\,dy\right]dx$$

$$= \int_{-2}^{2}\left[21\times 2\int_{0}^{\sqrt{4-x^2}}dy - 12(0)\right]dx$$

$$= 42\int_{-2}^{2}(y)_{0}^{\sqrt{4-x^2}}dx$$

$$= 42\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

$$= 42\times 2\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{0}^{2}$$

$$= 84\left[0+2\cdot\frac{\pi}{2}-0\right] = 84\pi$$

③ Verify Gauss Divergence theorem for $\bar{F} = x^3\bar{i}+y^3\bar{j}+z^3\bar{k}$ taken over the cube bounded by $x=0$, $x=a$, $y=0$, $y=a$, $z=0$, $z=a$.

Sol:- $\bar{F} = x^3\bar{i}+y^3\bar{j}+z^3\bar{k}$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(x^3)+\frac{\partial}{\partial y}(y^3)+\frac{\partial}{\partial z}(z^3) = 3x^2+3y^2+3z^2$$

$$\iiint_V \nabla\cdot F\,dv = \iiint_V (3x^2+3y^2+3z^2)\,dx\,dy\,dz$$

$$= 3\int_0^a\int_0^a\left[\frac{x^3}{3}+xy^2+z^2x\right]_0^a dy\,dz$$

$$= 3\int_0^a\int_0^a\left(\frac{a^3}{3}+ay^2+az^2\right)dy\,dz$$

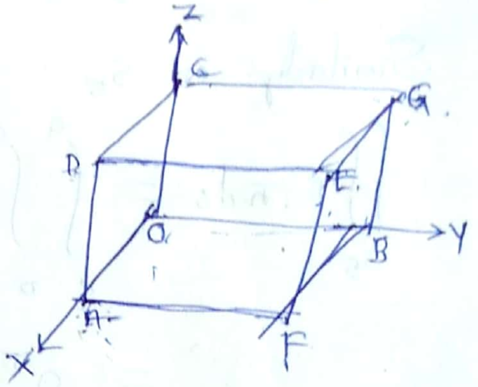
$$= 3\int_0^a\left(\frac{a^3}{3}y+a\frac{y^3}{3}+az^2y\right)_0^a dz$$

$$= 3\int_0^a \left(\frac{a^4}{3} + \frac{a^4}{3} + a^2 z^2\right) dz$$

$$= 3\int_0^a \left(\frac{2a^4}{3} + a^2 z^2\right) dz$$

$$= 3\left[\frac{2}{3}a^4 z + a^2\frac{z^3}{3}\right]_0^a = 3\left[\frac{2}{3}a^5 + \frac{1}{3}a^5\right] = 3a^5$$

To evaluate the surface integral divide the closed surface S of the cube into 6 parts



Surfaces

S_1 : DEFA $\quad$ S_4 : OADC

S_2 : BGCO $\quad$ S_5 : GCDE

S_3 : BGEF $\quad$ S_6 : AFBO

On surface S_1 : DEFA, we have $\bar{n} = \bar{i}$, $x = a$

$$\iint_{S_1} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{a}\int_{y=0}^{a} (a^3\bar{i} + y^3\bar{j} + z^3 k)\cdot i\, dy\, dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a} a^3\, dy\, dz = a^3\int_0^a (y)_0^a\, dz$$

$$= a^4 (z)_0^a = a^5$$

BGCO

On S_2 : AGCO we have $\bar{n} = -\bar{i}$, $x = 0$

$$\iint_{S_2} \bar{F}\cdot\bar{n}\,ds = \int_0^a\int_0^a (y^3 j + z^3 k)(-i)\, dy\, dz = 0$$

On S_3 : AGEF, we have $\bar{n} = j$, $y = a$

$$\iint_{S_3} \bar{F}.\bar{n}\,ds = \int_0^a\int_0^a (x^3\bar{i}+a^3\bar{j}+z^3\bar{k}).\bar{j}\,dx\,dz$$

$$= a^5$$

On S_4, we have $\bar{n} = -\bar{j}$, $y=0$

$$\iint_{S_4} \bar{F}.\bar{n}\,ds = \int_0^a\int_0^a (x^3 i + z^3 k)(-j)\,dx\,dz = 0$$

Similarly on S_5 : $\bar{n} = \bar{k}$, $z = a$

$$\iint_{S_5} \bar{F}.\bar{n}\,ds = \int_0^a\int_0^a (x^3 i + y^3 j + a^3 k).\bar{k}\,dx\,dy$$

$$= a^5$$

On S_6 ; $\bar{n} = -k$, $z = 0$

$$\iint_{S_6} F.\bar{n}\,ds = 0$$

Thus $$\iint_S \bar{F}.\bar{n}\,ds = \iint_{S_1}\bar{F}.\bar{n}\,ds + \iint_{S_2}\bar{F}.\bar{n}\,ds + \int_{S_3}\bar{F}.\bar{n}\,ds$$

$$+ \iint_{S_4}\bar{F}.\bar{n}\,ds + \iint_{S_5}\bar{F}.\bar{n}\,ds + \iint_{S_6}\bar{F}.\bar{n}\,ds$$

$$= a^5 + 0 + a^5 + 0 + a^5 + 0$$

$$= 3a^5$$

Hence $$\iiint_V \nabla.F\,dv = \iint_S \bar{F}.\bar{n}\,ds$$

Hence Gauss Divergence theorem proved.

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Q) Verify Gauss divergence theorem for $\bar{F} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$, over the cube formed by the plane $x=0, x=a, y=0, y=b, z=0, z=c$

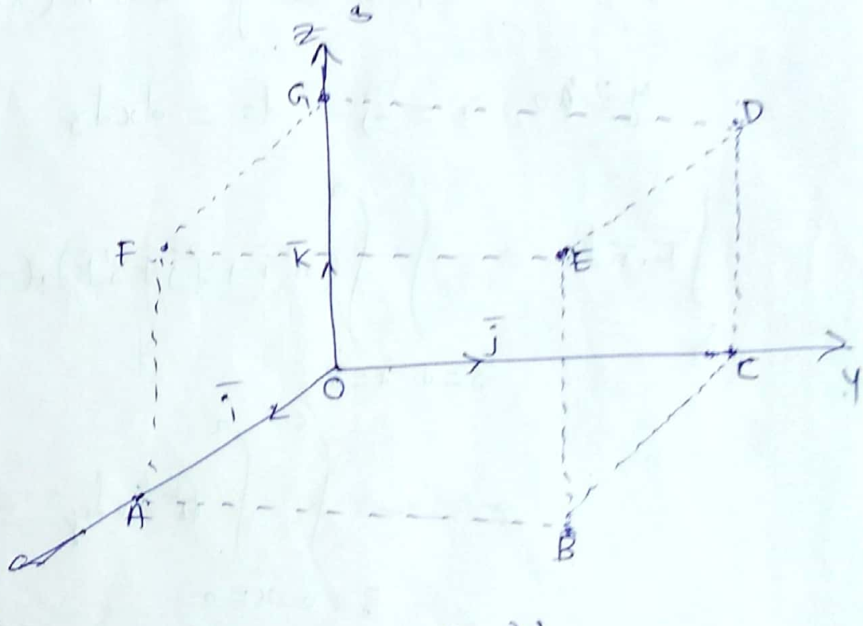
Sol:- By Gauss Divergence theorem

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

$$\nabla\cdot\bar{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(x^2\bar{i} + y^2\bar{j} + z^2\bar{k}\right) = 2(x+y+z)$$

$$R.H.S = \int_V \nabla\cdot\bar{F}\,dv = 2\int_0^c\int_0^b\int_0^a (x+y+z)\,dx\,dy = abc(a+b+c)$$

LHS We will calculate the value of $\int_S \bar{F}\cdot\bar{n}\,ds$ over surfaces of the cube.



Let S_1 be the surface along OABC (XY plane) (opposite)

$z=0$, $\bar{n}=-k$, $ds = dx\,dy$

Now $\int_{S_1} \bar{F}\cdot\bar{n}\,ds = \int_0^b\int_0^a (x^2\bar{i}+y^2\bar{j}+z^2\bar{k})\cdot(\bar{k})\,dx\,dy = -\int_0^b\int_0^a z^2\,dx\,dy = 0$

Let S_2 be the surface along DGFE (~~opposite~~ XY plane)

$z = c$, $\bar{n} = k$ $ds = dx\,dy$

$$\int_{S_2} \bar{F}\cdot\bar{n}\,ds = \int_0^b\int_0^a c^2\,dx\,dy = c^2ab$$

Let S_3 be the surface along BCDE (xz - opposite plane)

$$y = b, \quad \bar{n} = \bar{j} \qquad ds = dx\,dz$$

Now
$$\int_{S_3} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{x=0}^{a} (x^2\bar{i}+y^2\bar{j}+z^2\bar{k})\cdot(\bar{j})\,dx\,dz$$

$$= \int_{z=0}^{c}\int_{x=0}^{a} y^2\,dx\,dz = \int_{z=0}^{c}\int_{x=0}^{a} b^2\,dx\,dz = b^2ca$$

Let S_4 be the surface along OGFA (xz-plane)

$$y = 0, \quad \bar{n} = -\bar{j} \qquad ds = dx\,dz$$

$$\int_{S_4} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{x=0}^{a} (x^2\bar{i}+y^2\bar{j}+z^2\bar{k})\cdot(-\bar{j})\,dx\,dy$$

$$= -\int_{z=0}^{c}\int_{x=0}^{a} b^2\,dx\,dy = 0$$

$$= \cancel{b^2ca}$$

Let S_5 be the surface along ABEF (Yz-plane opposite)

$$x = a, \quad \bar{n} = \bar{i} \qquad ds = dy\,dz$$

$$\int_{S_5} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{y=0}^{b} (x^2 i + y^2 j + z^2 k)\cdot(\bar{i})\,dy\,dz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} x^2\,dy\,dz = \int_{z=0}^{c}\int_{y=0}^{b} a^2\,dy\,dz = a^2bc$$

Let S_6 be the surface along OCDG (YZ plane)

$x=0$, $\bar{n} = -\bar{i}$ $ds = dy\,dz$

$$\int_{S_6} \bar{F}\cdot\bar{n}\,ds = -\int_{y=0}^{b}\int_{z=0}^{c} 0\,dy\,dz = 0$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{S_1}\bar{F}\cdot\bar{n}\,ds + \iint_{S_2}\bar{F}\cdot\bar{n}\,ds + \iint_{S_3}\bar{F}\cdot\bar{n}\,ds + \iint_{S_4}\bar{F}\cdot\bar{n}\,ds$$

$$+ \iint_{S_5}\bar{F}\cdot\bar{n}\,ds + \iint_{S_6}\bar{F}\cdot\bar{n}\,ds$$

$$= c^2ab + 0 + b^2ca + 0 + a^2bc + 0$$

$$\text{L.H.S} = \iint_S \bar{F}\cdot\bar{n}\,ds = abc(c+b+a)$$

$\therefore$ L.H.S = R.H.S

Hence Gauss Divergence theorem is verified.

Green's theorem in a plane

(Transformation between Line integral and Double integral)

If R is a closed region in xy-plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

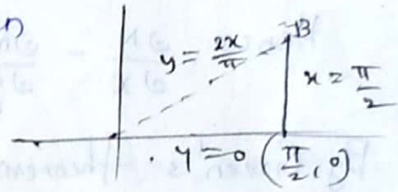
$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where C is traversed in the positive direction.

① Evaluate by Green's theorem $\oint_C (y - \sin x)\,dx + \cos x\,dy$ where C is the triangle ~~endo~~ enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$, $\pi y = 2x$

Sol:- Let $M = y - \sin x$ and $N = \cos x$ then

$\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = -\sin x$



$\therefore$ By Green's theorem $\oint M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$\therefore \int_C (y - \sin x)\,dx + \cos x\,dy = \iint_R (-\sin x - 1)\,dx\,dy$$

$$= -\int_0^{\pi/2}\int_0^{\frac{2x}{\pi}} (1 + \sin x)\,dx\,dy$$

$$= -\frac{2}{\pi}\int_0^{\pi/2} x(\sin x + 1)\,dx$$

$\int uv = u\int v - \int [u' \int v\,dv]$

$$= -\frac{2}{\pi}\left[x(-\cos x + x)\Big|_0^{\pi/2} - \int_0^{\pi/2} 1(-\cos x + x)\,dx \right]$$

$$= -\frac{2}{\pi}\left[x(-\cos x + x) + \sin x - \frac{x^2}{2} \right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[-x\cos x + \frac{x^2}{2} + \sin x\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[\frac{\pi^2}{8}+1\right] = -\left(\frac{\pi}{4}+\frac{2}{\pi}\right) = -\left[\frac{\pi^2+8}{4\pi}\right]$$

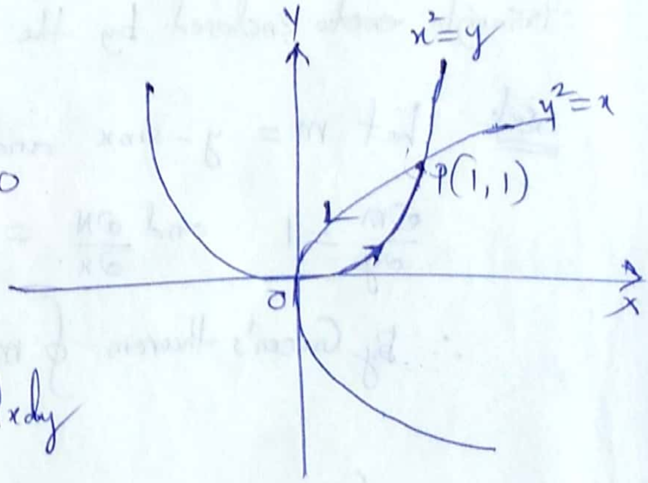
② Evaluate by Green's theorem $\oint (2xy - x^2)\,dx + (x^2+y^2)\,dy$, where 'c' is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$

Sol:- The two parabolas $y^2 = x$ and $y = x^2$ are intersecting at $O(0,0)$ and $P(1,1)$

A (2,0)

Here $M = 2xy - x^2$ and $N = x^2 + y^2$

$$\frac{\partial M}{\partial y} = 2x \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x$$



Hence $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 2x - 2x = 0$

By Green's theorem

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

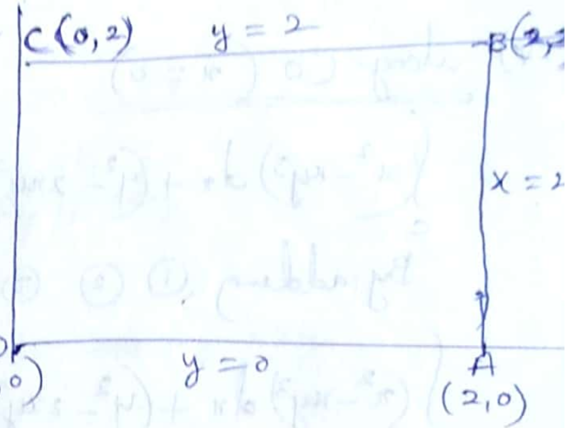
$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (0)\,dx\,dy = 0$$

③ Apply Green's theorem to evaluate $\oint_C (2x^2 - y^2)\,dx + (x^2+y^2)\,dy$, where c is the boundary of the area enclosed by the x-axis and w

③ Verify Green's theorem in the plane for $\int_C (x^2 - xy^3)dx + (y^2 - 2xy)dy$ where C is a square with vertices $(0,0)$ $(2,0)$ $(2,2)$ $(0,2)$.

Sol:- The cartesian form of Green's theorem in the plane is

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



Here $M = x^2 - xy^3$ and $N = y^2 - 2xy$

$$\frac{\partial M}{\partial y} = -3xy^2 \quad \& \quad \frac{\partial N}{\partial x} = -2y$$

To find $\int M dx + N dy$

To evaluate $\int_C (x^2 - xy^3)dx + (y^2 - 2xy)dy$, we shall take C in four different segments

i) along OA $(y=0)$

ii) along AB $(x=2)$

iii) along BC $(y=2)$

iv) along CO $(x=0)$

i) along OA $(y=0)$

$$\int_C (x^2 - xy^3)dx + (y^2 - 2xy)dy = \int_0^2 x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3} \quad \text{—} ①$$

ii) along AB $(x=2)$

$$\int_C (x^2 - xy^3)dx + (y^2 - 2xy)dy = \int_0^2 (y^2 - 4y)dy \quad (\because x = 2,\ dx = 0)$$

$$= \left(\frac{y^3}{3} - 2y^2\right)_0^2 = \frac{8}{3} - 8 = -\frac{16}{3} \quad \text{—} ②$$

iii) along BC $(y=2)$

$$\int_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \int_2^0 (x^2-8x)\,dx \quad (\because y=2,\ dy=0)$$

$$= \left(\frac{x^3}{3} - 4x^2\right)_2^0 = -\left[\frac{8}{3}-16\right] = \frac{40}{3} \quad \text{--- (3)}$$

iv) along CO $(x=0)$

$$\int_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \int_2^0 y^2\,dy = \left(\frac{y^3}{3}\right)_2^0 = -\frac{8}{3} \quad \text{--- (4)}$$

By adding (1) (2) (3) (4), we get

$$\int_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = \frac{24}{3} = 8 \quad \text{--- (5)}$$

To find $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here $x \to 0$ to 2 & $y \to 0$ to 2

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^2\int_0^2 (-2y+3xy^2)\,dx\,dy$$

$$= \int_0^2 \left(-2xy + \frac{3x^2}{2}y^2\right)_0^2 dy = \int_0^2 (-4y+6y^2)\,dy$$

$$= \left(-2y^2 + 2y^3\right)_0^2 = 8 \quad \text{--- (6)}$$

$\therefore$ From (5) & (6)

$$\int_C M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$ Hence the Green's theorem verified.

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~~③ Evaluate by Green's theorem $\oint (y - \sin x)dx + \cos x\, dy$ where C is the triangle enclosed by the lines $y = 0$, $x = \pi$~~

④ Verify Green's theorem for $\int (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the region bounded by $x = 0$, $y = 0$ and $x + y =$

Sol:- By Green's theorem, we have

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$

$M = 3x^2 - 8y^2$ $\quad\quad$ $N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$ $\quad\quad$ $\frac{\partial N}{\partial x} = -6y$

R.H.S $\iint \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \int_{y=0}^{1-x} \int_{x=0}^{1} (-6y + 16y)\, dx dy$

$$= \cancel{10}^{5} \int_{x=0}^{1} \left[\frac{y^2}{\cancel{2}}\right]_{y=0}^{1-x}$$

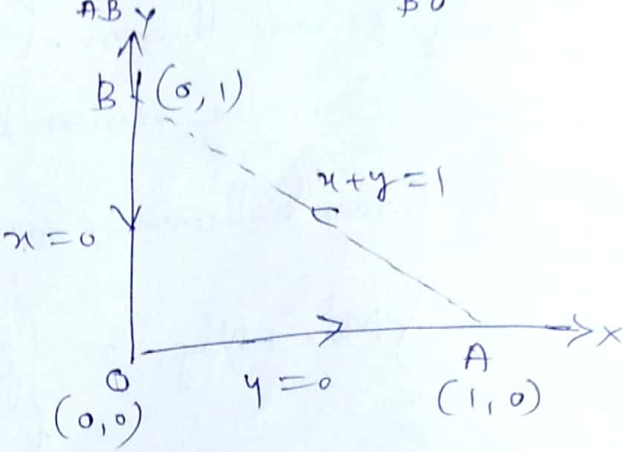
$$= 5 \int_{x=0}^{1} \left[(1 - x)^2\right] dx$$

$$= 5 \left[\frac{(1 - x)^3}{3(-1)}\right]_{x=0}^{1}$$

$$= 5 \left[0 - \left(\frac{-1}{3}\right)\right]$$

$$= \frac{5}{3}$$

Now L.H.S can be obtained by taking a triangle with the directions OA, AB, BO ie; $\oint_C M dx + N dy = \int_{OA} M dx + N dy + \int_{AB} M dx + N dy + \int_{BO} M dx + N dy$ —(1)



Along OA

$y=0, \; dy=0$

$$\int_{OA} M dx + N dy = \int_0^1 3x^2 dx$$

$$= 3\left(\frac{x^3}{3}\right)_0^1$$

$$= 1$$

Along AB

$x+y=1 \qquad \Rightarrow dy = -dx$

$\Rightarrow x = 1-y \qquad y \to 0 \text{ to } 1$

$$\int_{AB} M dx + N dy = \int_0^1 \left[3(y-1)^2 - 8y^2\right](-dy) + \left[4y - 6y(y-1)\right]dy$$

$$= \int_0^1 (11y^2 + 4y - 3)\, dy$$

$$= \left(\frac{11y^3}{3} + \frac{4y^2}{2} - 3y\right)_0^1$$

$$= \frac{11}{3} + 2 - 3$$

$$= \frac{8}{3}$$

Along BO

$x=0 \Rightarrow dx=0 \qquad y \to 1 \text{ to } 0$

$$\int_{BO} Mdx + Ndy = \int_{1}^{0} 4y\,dy = \left(\frac{4y^2}{2}\right)_1^0 = \left(2y^2\right)_1^0 = 0 - 2 = -2$$

$\therefore$ From ①

$$\int_C Mdx + Ndy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$

$\therefore$ L.H.S = R.H.S.

Hence Green's theorem is verified

⑤ Verify Green's theorem for $\bar{F} = (e^x \sin y)\bar{i} + (e^x \cos y)\bar{j}$ and C is the rectangle whose vertices are $(0,0)$ $(1,0)$ $(1, \pi/2)$ $(0, \pi/2)$

(OR)

~~Sol: Here $M = e^x \sin y$ $N = e^x \cos y$~~

Find the circulation of $\vec{F}$ round the curve where $\bar{F} = (e^x \sin y)\bar{i} + (e^x \cos y)\bar{j}$ and C is the rectangle whose vertices are $(0,0)$ $(1,0)$ $(1, \pi/2)$ $(0, \pi/2)$

Sol: Here $M = e^x \sin y$ $\quad N = e^x \cos y$

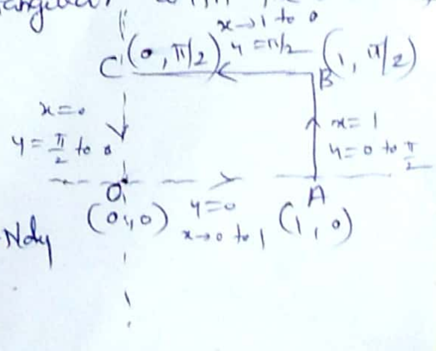
$$\frac{\partial M}{\partial y} = e^x \cos y \qquad \frac{\partial N}{\partial x} = e^x \cos y$$

By Green's theorem

$$\oint Mdx + Ndy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$\text{R.H.S} \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{y=0}^{\pi/2}\int_{x=0}^{1} \left(e^x \cos y - e^x \cos y\right) dxdy = 0$$

Now L.H.S can be obtained by taking a rectangular with the direction OA, AB, BC, CO



$$\oint_C Mdx + Ndy = \int_{OA} Mdx + Ndy + \int_{AB} Mdx + Ndy + \int_{BC} Mdx + Ndy$$

$$+ \int_{CO} Mdx + Ndy \quad \text{—— ①}$$

Along OA : $y=0,\ dy=0$ $x \to 0$ to 1

$$\int_{OA} Mdx+Ndy = \int_0^1 e^x \cancel{\sin y}^{(0)}\,dx \pm \overset{e^x\cos(0)}{\cancel{\sin y}}\cos(0)\int_0^1 \overset{=0}{\cancel{e^x dx = 1.(e-1) = e-1}}$$

Along AB : $x=1,\ dx=0,\ y \to 0$ to $\pi/2$

$$\int_{AB} Mdx+Ndy = \int_{y=0}^{\pi/2}\left(e^{\cancel{x}}\sin y \cancel{dx}^{(0)} + e^{\cancel{x}}\cos y\, dy\right) = e\int_{y=0}^{\pi/2}\cos y\,dy = e(\sin y)_0^{\pi/2} = e(1) = e$$

Along BC : $y=\frac{\pi}{2}$ $dy=0$ $x \to 1$ to 0

$$\int_{BC} Mdx+Ndy = \int_{x=1}^{0}\left(e^x(1) - 0\right)dx = \left(e^x\right)_1^0 = e^0 - e^1 = 1-e$$

Along CO : $x=0$ $dx=0$ $y \to \frac{\pi}{2}$ to 0

$$\int_{CO} Mdx+Ndy = \int_{y=\pi/2}^{0}\left(1.\sin y(0) + \cancel{e}(1)\cos y\right)dy = \left(\sin y\right)_{y=\frac{\pi}{2}}^{0} = 0 - \sin\frac{\pi}{2}$$

$$= -1$$

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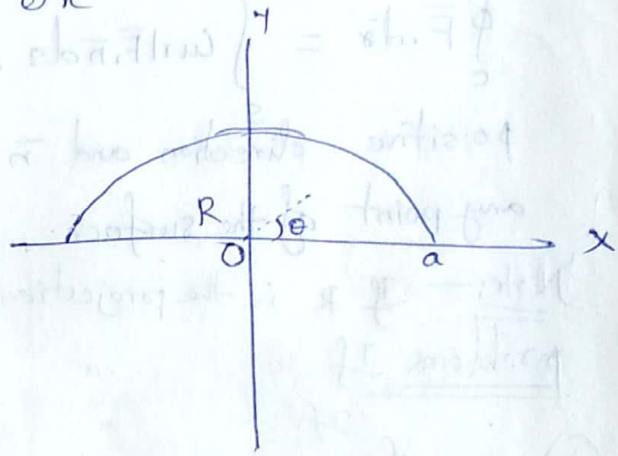
(4) Apply Green's theorem to evaluate $\oint_C (2x^2-y^2)dx + \int(x^2+y^2)dy$ where C is the boundary of the area enclosed by the x-axis and upper half of the circle $x^2+y^2=a^2$.

Sol:- Let $M = 2x^2-y^2$ and $N = x^2+y^2$

$$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = 2x$$



By Green's theorem

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_R (2x + 2y)\, dx\,dy$$

$$= 2\int_0^a \int_0^\pi (r\cos\theta + r\sin\theta)\, r\, d\theta . dr$$

r varies from 0 to a
θ varies from 0 to π

Curl F

$$= 2\int_0^a r^2 dr \int_0^\pi (\cos\theta + \sin\theta) d\theta$$

$$\int_S (\text{Curl} F).\bar{n}.ds \quad = 2\,\frac{a^3}{3}(1+1) = \frac{4a^3}{3}$$

Folium of Descartes $x^3+y^3=3axy$ using

lane for $\oint(3x^2-8y^2)dx + (4y-6xy)dy$

Stoke's theorem

(Transformation between Line integral and surface integral)

Statement

Let S be a open surface bounded by a closed, non intersecting curve C. If $\bar{F}$ is any differentiable vector point function then

$$\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$ where c is transversed in the positive direction and $\bar{n}$ is unit outward drawn normal at any point of the surface.

Note:— If R is the projection of S on xy plane then $\bar{k}\cdot\bar{n}\,ds = dx\,dy$

If " " " yz plane then $\bar{i}\cdot\bar{n}\,ds = dy\,dz$

If " " " zx plane then $\bar{j}\cdot\bar{n}\,ds = dz\,dx$

problems

1. Verify Stoke's theorem for $\bar{F} = -y^3\bar{i} + x^3\bar{j}$, where S is the circular disc $x^2+y^2 \le 1$, $z=0$

Sol:- Given that $\bar{F} = -y^3\bar{i} + x^3\bar{j}$

The boundary C of S is a circle in xy plane.

$x^2+y^2=1$, $z=0$

We use the parametric Co-ordinates $x=\cos\theta$, $y=\sin\theta$

$z=0$, $0 \le \theta \le 2\pi$

$(3x^2+3y^2)$

$dx = -\sin\theta\,d\theta$ $dy = \cos\theta\,d\theta$

By stoke's theorem

$+y^2)\,\bar{k}\cdot\bar{n}\,ds$

$$\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

we have $(\bar{k}\cdot\bar{n})\,ds = dx\,dy$

and R is the region on xy...

$$\oint_C F.dr = \int_C F_1 dx + F_2 dy + F_3 dz$$

$$= \int_C -y^3 dx + x^3 dy$$

$$= \int_0^{2\pi} [-\sin^3\theta(-\sin\theta) + \cos^3\theta\cos\theta] d\theta$$

$$= \int_0^{2\pi} (\cos^4\theta + \sin^4\theta) d\theta$$

$$= \int_0^{2\pi} (1 - 2\sin^2\theta\cos^2\theta) d\theta$$

$$= \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} (\sin 2\theta)^2 d\theta$$

$$= 2\pi - \frac{1}{4}\int_0^{2\pi} (1 - \cos 4\theta) d\theta$$

$$= 2\pi - \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$= 2\pi - \frac{2\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2} \quad \text{——— (1)}$$

$$\text{curl} F = \nabla \times F = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2 + 3y^2)$$

$$\int_S (\text{curl} F).\bar{n}\, ds = 3\int_S (x^2 + y^2)\, \bar{k}.\bar{n}\, ds$$

we have $(\bar{k}.\bar{n}) ds = dx\,dy$
and R is the region on xy-plane

$\therefore \iint_S (\nabla \times \bar{F}).\bar{n}\, ds = 3\iint_R (x^2+y^2)\, dx\, dy$

Put $x = r\cos\phi$ & $y = r\sin\phi$ $dx\,dy = r\,dr\,d\phi$

$r \to 0$ to 1 and $0 \le \phi \le 2\pi$

$$\int (\nabla \times \bar{F}).\bar{n}\, ds = 3\int_{\phi=0}^{2\pi}\int_{r=0}^{1} r^2 . r\, dr\, d\phi = \frac{2\pi}{2} \quad \text{---} (2)$$

From (1) & (2)

$$\oint_C \bar{F}.d\bar{r} = \int (\nabla \times \bar{F}).\bar{n}\, ds$$

Hence the theorem is verified.

(2) Verify stokes theorem for $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by the projection of the xy-plane.

Sol:- The boundary C of S is a circle in xy plane

i.e; $x^2+y^2=1$, $z=0$

The parametric equations are $x = r\cos\theta$, $y = r\sin\theta$

$\theta = 0 \to 2\pi$

$dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$

$$\int_C \bar{F}.d\bar{r} = \int_C F_1 dx + F_2 dy + F_3 dz$$

$$\cancel{= \int y\,dx + (x - 2xz)\,dy - xy\,dz}$$

~~Above xy plane the sphere is $x^2+y^2=a^2$, $z=0$~~

$\oint_C \bar{F}\cdot d\bar{r} = \cancel{\int_C y\,dx + x\,dy}$

put

$$= \int_C (2x-y)\,dx - yz^2\,dy - y^2 z$$

$$= -\int_0^{2\pi} (2\cos\theta - \sin\theta)\sin\theta\,d\theta$$

$$= \int_0^{2\pi} \sin^2\theta - \int_0^{2\pi} \sin 2\theta\,d\theta$$

$$= \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\,d\theta - \int_0^{2\pi} \sin 2\theta\,d\theta$$

$$= \pi$$

Again $\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix} = i(-2yz+2yz) - j(0-0) + \bar{k}(0+1) = \bar{k}$

$$\int_S (\nabla\times F)\cdot\bar{n}\,ds = \int_S \bar{k}\cdot\bar{n}\,ds = \iint_R dx\,dy$$

where R is the projection of S on xy plane and

$\bar{k}\cdot\bar{n}\,ds = dx\,dy$

$$\iint_R dx\,dy = 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} dy\,dx = 4\int_{x=0}^{1} \sqrt{1-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$\therefore$ Hence ~~proved~~ theorem is verified $= 4\left[\frac{1}{2}\sin^{-1}(1)\right] = \pi$

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(3) Evaluate by Stoke's theorem $\int_C (e^x dx + 2y\,dy - dz)$ where c is the curve $x^2+y^2=9$ and $z=2$.

Sol:- Let $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and

$$\bar{F}\cdot d\bar{r} = \bar{F}\cdot(dx\bar{i}+dy\bar{j}+dz\bar{k}) = e^x dx + 2y\,dy - dz$$

then $\bar{F} = e^x\bar{i} + 2y\bar{j} - \bar{k}$

By stokes theorem $\int_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$ —— (1)

Now $\text{curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$

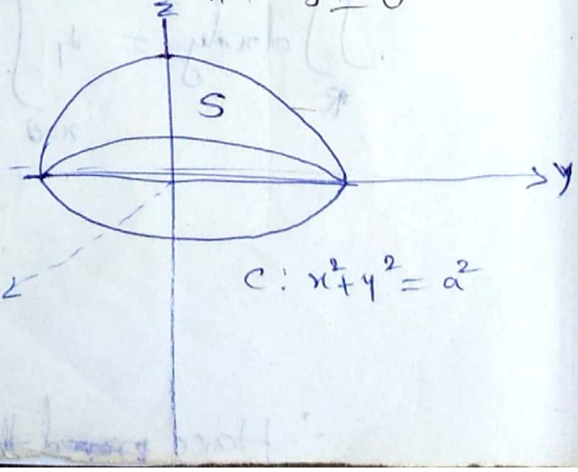
$$= i(0-0) - j(0-0) + k(0-0) = \bar{0}$$

$\therefore \int_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = 0$ by (1)

Hence $\int e^x dx + 2y\,dy - dz = \int_C \bar{F}\cdot d\bar{r} = 0$

(4) Verify Stoke's theorem for $F = y^2 i + yj - 3xk$ and S is the upper half of the sphere $x^2+y^2+z^2=a^2$ and $z\geq 0$

Sol:-



The curve c which is the boundary of the given hemisphere as the basic circle $x^2+y^2=a^2$

By stoke's theorem $\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\bar{F}\cdot \bar{n}\, ds$

$$L.H.S = \oint_C \bar{F}\cdot d\bar{r}$$

$$= \int_C \left(y^2\bar{i} + y\bar{j} - zx\bar{k}\right)\cdot\left(dx\bar{i} + dy\bar{j} + dz\bar{k}\right)$$

$$= \int_C y^2dx + ydy - xzdz$$

$x = a\cos\theta \Rightarrow dx = -a\sin\theta$

$y = a\sin\theta \Rightarrow dx = a\cos\theta d\theta$

Since $z = 0$

$\theta \to 0$ to 2π

$$= \int_C y^2dx + ydy$$

$$= \int_0^{2\pi} a^2\sin^2\theta\,(-a\sin\theta d\theta) + (a\sin\theta)(a\cos\theta)d\theta$$

$$= \int_0^{2\pi} \left[-a^3\sin^3\theta + a^2\sin\theta\cos\theta\right]d\theta$$

$$= -\frac{a^3}{4}\int_0^{2\pi}(3\sin\theta - \sin 3\theta)d\theta + \frac{a^2}{2}\int_0^{2\pi}\sin 2\theta\, d\theta$$

$$= -\frac{a^3}{4}\left[-3\cos\theta + \frac{\cos 3\theta}{3}\right]_0^{2\pi} + \frac{a^2}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= -\frac{a^3}{12}\left(\cos 3\theta - 9\cos\theta\right)_0^{2\pi} - \frac{a^2}{4}\left[\cos 4\pi - \cos 0\right]$$

$$= -\frac{a^3}{12}(0) - \frac{a^2}{4}\left[1 - 1\right] = 0$$

Now

$$\text{curl}F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & y & -zx \end{vmatrix} = zj - 2yk$$

Let $\phi = x^2 + y^2 + z^2 - a^2$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(xi+yj+zk)}{2\sqrt{x^2+y^2+z^2}} = \frac{xi+yj+zk}{a}$$

$$\text{Curl}\,\bar{F}\cdot\bar{n} = (zj-2yk)\cdot\left(\frac{xi+yj+zk}{a}\right) = \frac{yz-2yz}{a} = -\frac{yz}{a}$$

R.H.S. $\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = -\iint \frac{yz}{a}\,\cancel{dxdydz}$ ~~For the~~

~~For the sphere, $x = a\sin\theta\cos\phi$ $y = a\sin\theta\sin\phi$ $z = a\cos\theta$ $dxdydz = $~~

On XY plane $z = 0$

$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = 0$$

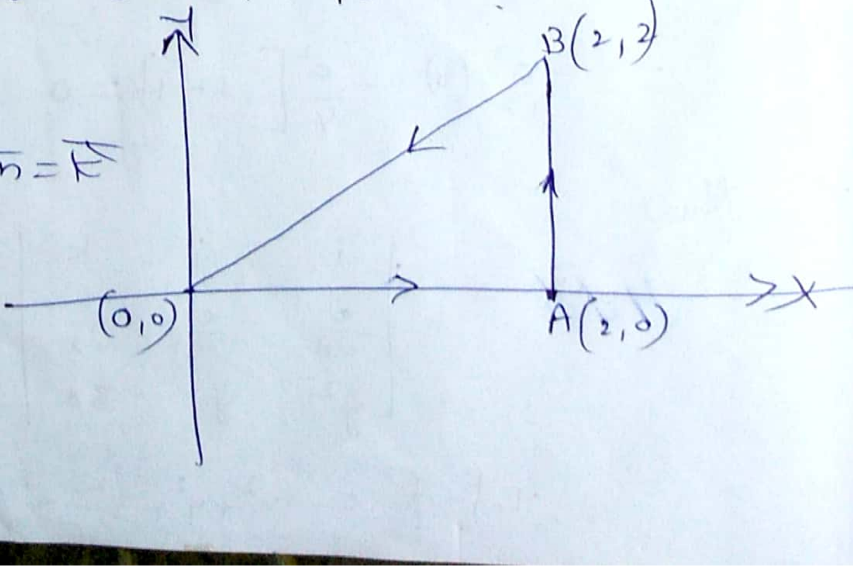
$\therefore$ L.H.S = R.H.S

Hence Stokes theorem is verified

(5) Using stoke's theorem evaluate the integral $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = 2y^2\bar{i} + 3x^2\bar{j} - (2x+z)\bar{k}$ and C is the boundary of the triangle whose vertices are $(0,0,0)$ $(2,0,0)$ $(2,2,0)$

Sol:- Here z-coordinates of each vertex of the triangle is zero, so the triangle lies in XY-plane.

Along XY-plane, $\bar{n} = \bar{k}$



Curl $\bar{F}$ =

$$\text{curl} F = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 & 3x^2 & -2x-3 \end{vmatrix} = 2\bar{j} + (6x-4y)\bar{k}$$

$$\text{curl} F \cdot \bar{n} = \left(2j + (6x-4y)\bar{k}\right) \cdot (\bar{k}) = 6x - 4y$$

Along OA :– $x \to 0$ to 2 $\quad y = 0 \Rightarrow dy = 0$

$$\iint_{OA} \text{curl} F \cdot \bar{n}\, ds = \iint (6x-4y)\,dx\,dy = 0$$

Along AB :– $x = 2$, $y = 0$ to 2

$\Downarrow$
$dx = 0$

$$\iint_{AB} \text{curl} F \cdot \bar{n}\, ds = \iint (6x-4y)\,dx\,dy = 0$$

Along BO :– $y - 0 = 1(x-0) \Rightarrow y = x$

$\Rightarrow dy = dx$

$$\iint_{BO} \text{curl} F \cdot \bar{n}\, ds = \int_{y=0}^{x} \int_{x=2}^{0} (6x - 4y)\,dx\,dy.$$

$$= \int_{x=2}^{0} \left[\left(6xy - \frac{\cancel{4}^{2} y^2}{\cancel{2}}\right)_{y=0}^{x}\right] dx = \int_0^2 (6x^2 - 2x^2)\,dx$$

$$= \int_{x=2}^{0} \left((0-0) - (12x - 8)\right) dx = \int_0^2 4x^2\,dx$$

$$= \left[\frac{4x^3}{3}\right]_0^2$$

$$= -\left[\frac{\cancel{12}^{6} x^2}{\cancel{2}} - 8x\right]_{x=2}^{0}$$

$$= -\left[(0-0) - (24-16)\right] = \cancel{08} \quad \frac{32}{3}$$

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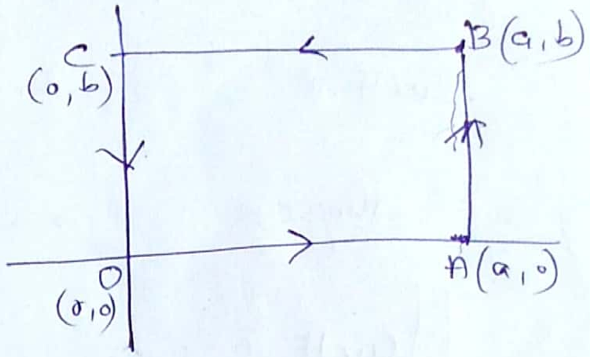
(6) Verify Stoke's theorem for $\bar{F} = (x^2 - y^2)\,\bar{i} + 2xy\,\bar{j}$ over the box bounded by the planes $x=0,\ x=a,\ y=0,\ y=b$

Sol:- $\bar{F} = (x^2 - y^2)\,\bar{i} + 2xy\,\bar{j}$



$$\text{Curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = 4y\bar{k}$$

$$\text{R.H.S} = \int \text{Curl}\,\bar{F}\cdot\bar{n}\,ds = \int 4y\bar{k}\cdot\bar{n}\,ds$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} 4y\,dx\,dy = 2ab^2$$

L.H.S

$$\bar{F}\cdot d\bar{r} = (x^2 - y^2)\,dx + 2xy\,dy$$

Along OA

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_{x=0}^{a} x^2\,dx = \frac{a^3}{3}$$

Along AB

$x=0,\ y = 0$ to b

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_{y=0}^{b} (2xy)\,dy = \left[2a\,\frac{y^2}{2}\right]_0^b = ab^2$$

Along BC

$y = b,\ dy = 0$

$$\int_{BC} \bar{F}\cdot dr = \int_{x=a}^{0} (x^2 - y^2)\,dx = ab^2 - \frac{a^3}{3}$$

Along CO : $x=0,\ dx=0 \quad y \to b$ to 0

$$\int_{C} \bar{F}\cdot d\bar{r} = \int_{y=b}^{0} 2xy\,dy = 0$$

$$\therefore \int \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} = \cancel{\frac{a^3}{3}} + ab^2 + ab^2 - \cancel{\frac{a^3}{3}} = 2ab^2$$

$\therefore$ L.H.S = R.H.S